PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCLXII.

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VOL. LXX.

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THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions*, take this opportunity to acquaint the Public, that it fully appears, as well from the Council-books and Journals of the Society, as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgement of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the

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thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

The Meteorological Journal hitherto kept by the Assistant Secretary at the Apartments of the Royal Society, by order of the President and Council, and published in the Philosophical Transactions, has been discontinued. The Government, on the recommendation of the President and Council, has established at the Royal Observatory at Greenwich, under the superintendence of the Astronomer Royal, a Magnetical and Meteorological Observatory, where observations are made on an extended scale, which are regularly published. These, which correspond with the grand scheme of observations now carrying out in different parts of the globe, supersede the necessity of a continuance of the observations made at the Apartments of the Royal Society, which could not be rendered so perfect as was desirable, on account of the imperfections of the locality and the multiplied duties of the observer.

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APPENDIX.

Presents . . . . . [1]

ERRATA.

Page 163, lines 2, 9, 13 (art. XLVII.), erase the word "non-polar."

— — line 7, *for* because *read* when.

— 588, line 10 from bottom, cancel p at the end of the formula, which should be

$$v\frac{Ct}{p}-\tfrac{1}{3}AJK\left(\frac{273{\cdot}7}{t}\right)^3.$$

— 755, 7 lines from bottom, *for* Plate A. *read* Plate XXVII.

— 864, 8 lines from bottom of text, *for* fig. 2 *read* fig. 3.

— — 6 lines from bottom of text, *for* fig. 3 *read* fig. 2.

Page 869, 13 lines from bottom of text, *for* more *read* less.

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FRANKLIN INSTITUTE

PHILOSOPHICAL TRANSACTIONS.

I. *On the Influence of Temperature on the Electric Conducting Power of Metals.*

By AUGUSTUS MATTHIESSEN, *F.R.S., and* MORITZ VON BOSE.

Received December 5, 1861,—Read January 16, 1862.

THE results obtained by different observers in their researches on the influence of temperature on the electric conducting power of metals do not agree at all together. The differences in their results may be partly owing to their not having tested pure metals, and partly to their not having taken into consideration the fact that, when a wire of a pure metal is heated for the first time to 100° C., an alteration in the conducting power of the wire is observed on its again being cooled; in fact, it is necessary to keep the wire for several days at 100° before its conducting power, on again being cooled, becomes constant.

In the experiments we are about to detail we have taken great care to employ only pure metals, as well as a method and a disposition of the apparatus with which great accuracy could be obtained.

The method employed for the determination of the resistances is fully described in the 'Philosophical Magazine' for February 1857. Fig. 1 shows the disposition of the apparatus. B is the trough in which the wires were heated: these were soldered to two thick copper wires F (4–5 millims. thick), bent as shown in the figure, and ending in the mercury-cups E, which were connected with the apparatus by two other copper wires, F', of the same thickness. C is a piece of board placed in such a manner as to prevent the heat of the trough from radiating on the apparatus. The mercury-cups O are made of small blocks of wood, through which holes are bored just large enough to take the thick wires, and to the bottoms of which blocks amalgamated copper plates are fastened. Now it is clear that if the ends of the thick copper wires are filed flat, and well amalgamated, and the mercury-cups are filled with mercury, this method of connexion may be looked upon as a soldering of the copper plates to the wires, or, in other words, as a perfect connexion; for the wires may be removed as often as required, and on replacing them the same resistance is always observed. The wires F'',

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to which the normal wire (in the glass cylinder G) is soldered, are also 4–5 millims. thick. The reason why such thick wires were chosen was to make any difference in their resistance, caused by the change of temperature in the room or by the heating of the ends in the oil-bath, so small that no correction was necessary. This was proved to be the case by the following experiment:—After having soldered a wire in the trough to the ends of the thick copper wires, and determined its resistance with the normal wire generally used, the wire F' at *e* was heated with the 6-Bunsen burner much above

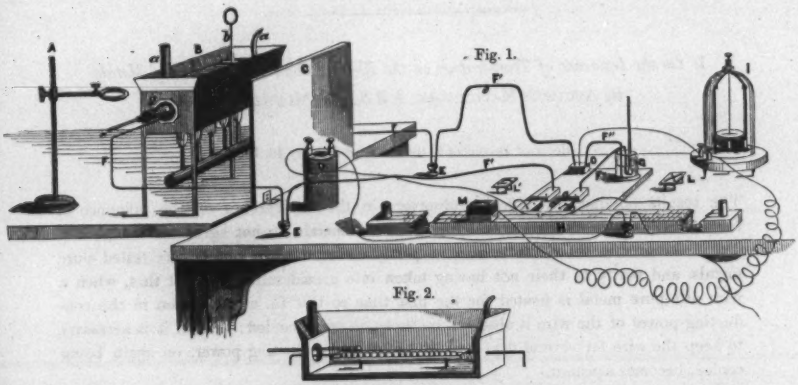

Fig. 1.

Fig. 2.

100° C., and the resistance of the circuit was again determined whilst the wire was at that temperature, when it was found to have increased only 0·08 per cent.; we did not, therefore, consider it necessary to make any correction for the increase of resistance caused by the heating of the ends of the thick wires in the trough. The resistance of the copper wires was determined at the ordinary temperature, and brought into calculation without further correction. Before the commencement of each series, all the ends of the wires dipping into the mercury-cups were carefully re-amalgamated. L, L' are the two commutators fitting into four mercury-cups at *o*.

The wire stretched on the board H is of german silver instead of copper, as was formerly described; its half-length was 4550 millims. The length of the board is about 1500 millims.; the wire, therefore, was wound backwards and forwards several times on the one side; this is not visible in the figure. By using normal wires of different resistances, and by choosing proper lengths of the wire to be tested, it was always possible to begin the observations with the block M within 100 millims. of the middle of the wire. Great care was taken to lift the block M off the wire when it was moved, in order to prevent as much as possible its wearing. It may be mentioned that, although we

generally worked with only one of the commutators, and therefore mostly used the one half of the wire, the zero-point of the wire only varied, during the whole of the experiments, which have taken almost a year to carry out, 3 millims. The zero-point was always determined before each series was begun. The distance the block M was moved when the resistance of a wire was determined, first at 0° and then at 100°, was, for pure metals in a solid state, about 800 millims., or about 8 millims. for 1°. As, however, the movement of the block M of 1 millim. caused a deflection of the needles of the galvanometer I of 20° to 30°, it is evident, with the apparatus employed, that the differences in the resistance of a wire to values less than those corresponding to 0°·1 C. can be accurately determined. Our results, moreover, prove this to be the case, as in many instances the difference between the observed and calculated conducting powers for the whole series do not amount to values equal to 0°·1 to 0°·2 C.

The trough B is a double one, the space between the inner and outer one being 20 millims. The dimensions of the inner trough were 400 millims. long, 80 millims. wide, and 80 millims. deep. Through the ends of both two holes of about 20 millims. wide were made, in which good corks were fitted, and through these passed the thick copper wires F; and also at one end a glass tube *d*, wide enough to allow the thermometer *c* to pass freely. A piece of india-rubber tubing, fitting over the glass tube *d*, and tightly round the thermometer, closed the tube, but allowed the thermometer to be moved either backward or forward with great ease. The tubes *a* are for filling the space between the inner and outer troughs with oil.

The wire to be tested lay in the trough, as shown in fig. 2, on a small glass tray, made by splitting a glass tube longitudinally, thereby preventing any possibility of its touching the trough, and also guarding it from being moved by the stirrer. A second trough, of somewhat smaller dimensions, was also used.

The use of an oil-bath for heating the wires has been objected to by a former observer*; it was therefore necessary to determine experimentally whether there was any real reason for the objection or not. He states that, as oil conducts electricity better on being heated than when cold, the differences between the conducting powers of cold and hot oil will materially affect the values obtained for the resistances of wire which had been determined at different temperatures in that liquid. In order to test the accuracy of this assertion, two copper plates of about 150 millims. diameter were connected, the one with the galvanometer, the other with a single Bunsen's cell; and to complete the circuit, this was connected with the galvanometer. A piece of filtering-paper, moistened with the olive-oil used, was placed between the copper plates, and these were pressed together with a weight. On completing the circuit not the slightest deflection of the needles was observed; the copper plates were then heated to above 100° C., and still no deflection was visible. To show that the connexions were good, a drop of water was put on the oiled paper; and immediately the needles of the galvanometer were sent with great violence to the stops. This proves that although oil may

* Arndsten, Poggendorff's 'Annalen,' vol civ. p. 1.

have a higher conducting power when hot than cold, yet in either case it is so infinitely small, that it cannot influence the results obtained in the manner just described.

Again, it was proved in a former research* that the formula for the correction of conducting power for temperature of a wire, deduced from the observations made in an oil- or air-bath, were exactly the same. Thus the formula obtained for an annealed wire of the gold-silver alloy heated in the oil-bath was

$$\lambda = 15{\cdot}052 - 0{\cdot}01074t + 0{\cdot}00000714t^2,$$

and that for the same wire heated in an air-bath was

$$\lambda = 15{\cdot}059 - 0{\cdot}01077t + 0{\cdot}00000722t^2.$$

As, however, more accurate results may be obtained by experimenting in an oil- than in an air-bath, on account of the wires taking more readily the temperature of the bath, and of their being more rapidly cooled if heated by the current passing through them, we have chosen this manner of heating the wires in preference to the other.

As oil, and more especially oil when hot, attacks most wires to a degree which would render the observations valueless, we were obliged to varnish them. The best varnish for the purpose is a solution of shell-lac in alcohol. For instance, a hard-drawn copper wire, not varnished, loses in conducting power after having been heated in an oil-bath to 100°, but if varnished, increases. To show that varnishing has no effect on the results, we give in Table I. the conducting power of a hard-drawn gold wire, first not varnished, and then varnished. Each result is the mean of two observations.

TABLE I.

Not varnished.				Varnished.			
T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.	
15·30	72·697	72·705	−0·008	13·85	73·120	73·085	+0·035
30·55	68·806	68·879	−0·073	30·95	68·756	68·782	−0·026
48·65	64·659	64·717	−0·058	49·55	64·523	64·520	+0·003
69·55	60·409	60·423	−0·014	68·40	60·636	60·645	−0·009
83·25	57·915	57·906	+0·009	84·55	57·704	57·680	+0·024
99·85	55·151	55·174	−0·023	98·70	55·346	55·352	−0·006
84·55	57·704	57·680	+0·024	84·90	57·645	57·620	+0·025
70·80	60·224	60·184	+0·040	70·25	60·318	60·289	+0·029
50·85	64·239	64·239	0·000	51·20	64·149	64·164	−0·015
30·95	68·746	68·782	−0·036	30·60	68·886	68·866	+0·020
16·80	72·343	72·316	+0·027	17·85	72·111	72·045	+0·066

The formula deduced from the observations, and from which the conducting powers were calculated, was

$$\lambda = 76{\cdot}838 - 0{\cdot}27973t + 0{\cdot}0006285t^2.$$

The thermometers used were:—1. One divided into degrees, each of which was 3·5

* Philosophical Magazine for February 1861.

millims. long. With very little practice the temperature could be read off to 0°·1 C. with accuracy. This thermometer was calibrated by ourselves, and afterwards compared with a normal thermometer from Kew Observatory, for which we were indebted to the kindness of Mr. BALFOUR STEWART. The corrected readings of our thermometer agreed perfectly with those of the Kew thermometer. 2. A normal thermometer from Messrs. NEGRETTI and ZAMBRA, divided into 0°·2 C. This was compared with the Kew thermometer and found to be correct. The boiling- and freezing-points of the thermometers were taken at intervals, and the necessary corrections made.

As the light in the room where the experiments were made came from above, and as the thermometers lay horizontally in the trough, by placing the eye in a position so that the division on the thermometer covered its reflexion on the column of mercury, all error of parallax was avoided. The thermometers were always read off with the help of the magnifying glass A through the oil in the glass tube *d*, so that the whole of the column of mercury had very nearly the temperature of the bath.

The normal wires were made of annealed german silver, and their resistances determined by comparing them with a hard-drawn wire of the gold-silver alloy*. They were soldered to two thick copper wires, varnished, and when used placed in the cylinder G, filled with oil, in which a thermometer hung. The temperature of the oil was taken immediately after each observation, and the conducting power of the normal wire corrected by the use of the formula

$$\lambda = 7{\cdot}803 - 0{\cdot}0034619t + 0{\cdot}0000003951t^2,$$

which was found by the determination of the conducting powers, at different temperatures, of a piece of wire from the same coil as that from which the normal wires were cut. In this paper we have taken as unit the conducting power of a hard-drawn silver wire at 0° C.=100 (that of the hard-drawn gold-silver alloy at 0° being =15·03), in order to be able to compare at sight the present determinations with those made by one of us a short time ago†.

Before beginning a series, as already stated, all the ends of the wires dipping in the mercury-cups were re-amalgamated, and the zero-point of the scale redetermined. The current from the cell D was only allowed to pass through the apparatus for a second or two at a time, for fear of heating the wires, &c.

From 0° to 100° seven intervals were chosen at which observations were made, viz. 12°, 25°, 40°, 55°, 70°, 85°, 100°. With a little practice the flames of the 6-Bunsen burner could be regulated so as to come within a degree or two of the above temperatures. For about five minutes before, and whilst making the observations, the oil in the trough was stirred, one observer being at the trough whilst the other determined the resistances. Four observations at each interval were generally made on heating the wire to 100°, and again four at each interval on cooling (where this was not the case it will be mentioned with the series).

* Philosophical Magazine, February 1861.

† Philosophical Transactions, 1858 and 1860.

To save space, the mean only of the eight observations will be given, as otherwise the number of figures would be very great. Table I. may be taken as a fair example of the results obtained. The formulæ from which the conducting powers have been calculated is

$$\lambda = x + yt + zt^2,$$

where λ is the conducting power at $t°$ C., x the conducting power at 0°, and y and z constants. The values for x, y, and z were deduced from the mean of the observations by the method of least squares.

We will now proceed to the experiments made with each metal, making at the same time a few remarks on their purification, &c., and then see what general laws and conclusions we may draw from the results obtained.

Silver.

Purified by precipitating nitrate of silver with hydrochloric acid, and reducing the washed chloride with pure carbonate of sodium. Wires 1, 2, and 3 were of different preparations. Table II. gives the results obtained with these wires.

TABLE II.

	First wire.		Second wire.		Third wire.	
	Hard drawn.	Annealed.	Hard drawn.	Annealed.	Hard drawn.	Annealed.
Length	1546 millims.	1535 millims.	1753 millims.	1741 millims.	1962 millims.	1953 millims.
Diameter	0·462 millim.	0·462 millim.	0·596 millim.	0·596 millim.	0·448 millim.	0·648 millim.
Conducting power found before heating the hard-drawn wires	97·645 at 15°·4	Reduced to 0°. 103·528	95·112 at 16°·0	Reduced to 0°. 101·149	94·053 at 16°·0	Reduced to 0°. 99·800
Conducting power after being kept at 100° for 1 day	98·138 at 16·2	104·364	96·618 at 15·6	102·585	95·241 at 15·4	100·839
Ditto, for 2 days	98·913 at 15·6	104·951	101·544 at 16·8	108·303*	96·337 at 16·0	102·223
Ditto, for 3 days	99·837 at 16·0	106·091	102·237 at 16·0	108·714	96·671 at 17·6	103·178
Ditto, for 4 days	99·212 at 18·4	106·377	101·427 at 19·2	109·162	97·917 at 15·6	103·747
Ditto, for 5 days	99·586 at 17·4	106·380	101·750 at 18·6	109·262	97·669 at 17·4	104·168
Ditto, for 6 days	.....	.....	.....	.....	97·322 at 18·2	104·100

The means of the conducting powers found for each of the following temperatures were—

* During the day the temperature of the oil increased, by mistake, to 130°.

First wire, hard drawn.				Second wire, hard drawn.				Third wire, hard drawn.			
T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
11°·00	102·238	102·272	−0·034	12°·20	103·927	103·927	0·000	9°·60	100·534	100·546	−0·012
26·17	96·710	96·645	+0·065	23·70	99·520	99·523	−0·003	23·90	95·452	95·437	+0·015
38·25	92·490	92·505	−0·015	41·70	93·224	93·236	−0·012	38·95	90·476	90·507	−0·031
55·40	87·130	87·149	−0·019	56·20	88·703	88·708	−0·005	56·00	85·513	85·478	+0·035
68·85	83·389	83·374	+0·015	68·90	85·142	85·137	+0·005	68·15	82·244	82·252	−0·008
84·00	79·540	79·572	−0·032	85·45	81·078	81·036	+0·042	84·47	78·393	78·391	+0·002
101·30	75·831	75·813	+0·018	99·20	78·073	78·103	−0·030	98·60	75·477	75·484	−0·007
First wire, annealed.				Second wire, annealed.				Third wire, annealed.			
11°·30	103·391	103·404	−0·013	8°·00	106·447	106·426	+0·021	9°·25	102·543	102·461	+0·082
24·25	98·589	98·576	+0·013	24·35	99·968	99·990	−0·022	25·55	96·371	96·495	−0·124
41·85	92·520	92·530	−0·010	38·05	95·051	95·077	−0·026	40·10	91·589	91·630	−0·041
56·45	88·006	87·965	+0·041	55·17	89·554	89·554	0·000	55·17	87·055	87·047	+0·008
67·75	84·670	84·714	−0·044	68·22	85·847	85·803	+0·044	68·55	83·483	83·367	+0·116
83·65	80·562	80·554	+0·008	83·62	81·882	81·888	−0·006	83·57	79·667	79·674	−0·007
98·80	77·046	77·042	+0·004	100·00	78·319	78·331	−0·012	100·00	76·124	76·163	−0·039

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire (hard drawn) . $\lambda = 106{\cdot}651 - 0{\cdot}40948t + 0{\cdot}0010370t^2$.

For first wire (annealed) . . $\lambda = 107{\cdot}880 - 0{\cdot}40698t + 0{\cdot}0009601t^2$.

For second wire (hard drawn) $\lambda = 108{\cdot}928 - 0{\cdot}42389t + 0{\cdot}0011407t^2$.

For second wire (annealed) . $\lambda = 109{\cdot}802 - 0{\cdot}43138t + 0{\cdot}0011667t^2$.

For third wire (hard drawn) . $\lambda = 104{\cdot}209 - 0{\cdot}39124t + 0{\cdot}0010133t^2$.

For third wire (annealed) . $\lambda = 106{\cdot}088 - 0{\cdot}40160t + 0{\cdot}0010235t^2$.

From the above Table it will be seen that, after heating a silver wire to 100° C. for some days, its conducting power is increased almost to the same extent as if it had been annealed, and that wires 1 and 2 were not completely hard drawn. On comparing the difference in the conducting powers produced by annealing the wires, we find for wire 3 it is only 6 per cent., whereas for wire 2 it is almost 10 per cent., taking the conducting power of the hard-drawn silver wire =100. In a former research * this difference was found to be—

		Reduced to 0°.
1. Hard drawn	95·28 at 14°·0	100·47
Annealed	103·98 at 14·8	109·98
2. Hard drawn	95·36 at 14·6	100·78
Annealed	103·33 at 14·6	109·20

* Philosophical Transactions, 1860.

These values have been reduced by using a formula which is the mean of the six deduced from the experiments; for although there is a difference in the formula obtained for the annealed and hard-drawn (or rather partially annealed) wires, yet it is so small that they may be considered the same, more especially as the difference between the one obtained for the different wires is far greater. Taking the mean of the above values, and assuming the influence of temperature on the conducting power of hard-drawn and annealed wires to be the same, we find the following formulæ:—

For hard-drawn wires $\lambda = 100{\cdot}00 - 0{\cdot}38287t + 0{\cdot}0009848t^2$.

For annealed wires $\lambda = 108{\cdot}574 - 0{\cdot}41570t + 0{\cdot}0010624t^2$.

Copper.

Wires 1 and 2 were of the same piece of electrotype copper prepared for us by Dr. H. Müller at Messrs. De la Rue and Co.'s. Wire 3 was cut off a piece of commercial electrotype copper from the same source. Table III. shows the results obtained with these wires.

Table III.

	First wire.		Second wire.		Third wire.	
	Hard drawn.	Annealed.	Hard drawn.	Annealed.	Hard drawn.	Annealed.
Length	2262 millims.	2245·5 millims.	1753 millims.	1738 millims.	1476 millims.	1461 millims.
Diameter	0·691 millim.	0·691 millim.	0·598 millim.	0·598 millim.	0·537 millim.	0·537 millim.
Conducting power found before heating the hard-drawn wires	95·672 at 10°·6	Reduced to 0°. 99·526	94·355 at 15°·0	Reduced to 0°. 100·021	92·568 at 20°·6	Reduced to 0°. 100·327
Conducting power after being kept at 100° for 1 day ...	96·324 at 9·9	99·943	94·965 at 13·2	99·971	93·263 at 19·0	100·461
Ditto, for 2 days...	96·750 at 11·8	101·097	94·880 at 14·2	100·268	93·720 at 18·0	100·563
Ditto, for 3 days...	96·914 at 12·2	101·418	94·501 at 15·9	100·524	93·434 at 19·0	100·645
Ditto, for 4 days...	97·950 at 9·8	101·671	94·153 at 17·2	100·656	93·278 at 19·6	100·708
Ditto, for 5 days...	98·437 at 8·7	101·682	95·570 at 14·4	101·074	92·865 at 20·6	100·649
Ditto, for 6 days...	.....	.....	94·327 at 18·2	101·230	92·738 at 21·1	100·705
Ditto, for 7 days...	.....	.....	96·575 at 12·7	101·469		

The means of the conducting powers found for each of the following temperatures were—

First wire, hard drawn.				Second wire, hard drawn.				Third wire, hard drawn.			
T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
16°86	95·473	95·467	+0·006	19°17	94·359	94·334	+0·025	12°65	95·769	95·739	+0·030
29·88	91·063	91·002	+0·061	30·95	90·187	90·208	−0·021	25·61	91·061	91·076	−0·015
51·03	84·235	84·315	−0·080	48·53	84·518	84·544	−0·026	39·52	86·415	86·456	−0·041
69·52	78·997	79·044	−0·047	69·22	78·640	78·634	+0·006	53·92	82·069	82·090	−0·021
83·77	75·413	75·347	+0·066	83·77	75·015	74·968	+0·047	69·90	77·798	77·741	+0·057
98·60	71·829	71·838	−0·009	99·00	71·532	71·562	−0·030	84·87	74·172	74·142	+0·030
								99·92	70·951	70·987	−0·036

First wire, annealed.				Second wire, annealed.				Third wire, annealed.			
17°00	95·535	95·567	−0·032	18°96	94·987	94·959	+0·028	13°43	96·954	96·934	+0·020
29·63	91·291	91·239	+0·052	31·86	90·424	90·449	−0·025	26·15	92·246	92·260	−0·014
50·22	84·687	84·726	−0·039	52·05	83·974	84·003	−0·029	39·35	87·727	87·753	−0·026
69·60	79·223	79·209	+0·014	70·27	78·836	78·829	+0·007	55·50	82·675	82·722	−0·047
83·42	75·636	75·638	−0·002	83·81	75·428	75·377	+0·051	69·90	78·742	78·686	+0·056
99·39	71·891	71·893	−0·002	99·57	71·757	71·784	−0·027	84·67	75·047	74·988	+0·059
								99·05	71·766	71·816	−0·050

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire (hard drawn) . . . $\lambda = 101{\cdot}645 - 0{\cdot}37963t + 0{\cdot}0007844t^2$.

For first wire (annealed) . . . . $\lambda = 101{\cdot}791 - 0{\cdot}37959t + 0{\cdot}0007921t^2$.

For second wire (hard drawn) . . $\lambda = 101{\cdot}614 - 0{\cdot}39806t + 0{\cdot}0009546t^2$.

For second wire (annealed) . . . $\lambda = 102{\cdot}143 - 0{\cdot}39629t + 0{\cdot}0009179t^2$.

For third wire (hard drawn) . . . $\lambda = 100{\cdot}620 - 0{\cdot}39885t + 0{\cdot}0010236t^2$.

For third wire (annealed). . . . $\lambda = 102{\cdot}243 - 0{\cdot}40850t + 0{\cdot}0010228t^2$.

The observations made with wires 1 and 2 were as follows: two at each interval on heating and two on cooling; again, two on heating and two on cooling, as shown in Table I.

On looking at the above, we observe that wire 1, after having been kept at 100° for several days, increased in conducting power almost to the same extent as if it had been annealed, wire 2 partially so, and wire 3 hardly at all. The annealing took place in a glass tube heated with a 4-Bunsen burner, whilst a current of hydrogen passed through it. Here, again, as in the case of the silver wire, we may assume that the formulæ of the hard-drawn and annealed copper wires are the same. In a former research* pure copper was found to conduct—

		Reduced to 0°.
1.	93·00 at 18°·6	99·877
2.	93·46 at 20·2	100·980
3.	92·02 at 18·4	99·824
4.	92·76 at 19·3	99·886
5.	92·99 at 17·5	99·453

* Philosophical Transactions, 1860.

The difference found between the conducting powers of hard-drawn and annealed wires was —

			Reduced to 0°.
6.	Hard drawn	95·31 at 11°·0	99·435
	Annealed	97·83 at 11·0	102·065
7.	Hard drawn	95·72 at 11·0	99·864
	Annealed	98·02 at 11·0	102·263

These values have been reduced to 0° as follows: take for instance the first, 93·00 at 18°·6. The mean of the six formulæ obtained for copper (see Table XV.) is

$$\lambda = 100 - 0{\cdot}38701t + 0{\cdot}0009009t^2;$$

and calculating the conducting power for 18°·6 by this formula, we find it equal to 93·114.

Now $$\frac{93{\cdot}00}{93{\cdot}114} = 0{\cdot}99877;$$

and if all the terms of the above formula be multiplied by this number, we deduce a formula by which the above value can be reduced. All the reductions given in this paper of former determinations were made in this manner, using the formulæ given in Table XV. The reductions to 0° in the Tables were made in a like manner, the only difference being that the formulæ found for the respective wires were used instead of the mean. Taking the mean of all the values found for copper, and using the mean for the formulæ given in Table XV., we find as the formula for correction of the conducting power for temperature of

A hard-drawn wire $\lambda = 99{\cdot}947 - 0{\cdot}38681t + 0{\cdot}0009004t^2$

An annealed wire $\lambda = 102{\cdot}213 - 0{\cdot}39557t + 0{\cdot}0009208t^2$.

The values given as first term in the formulæ were found as follows: on referring to the paper* from which the conducting powers of copper were taken, it will be seen that each of them is the mean of three determinations. The reduced values therefore of 1 to 5, the mean of 6, hard drawn, and 7, hard drawn, and the mean of the first determinations of the three wires given in Table III., were added together, and the mean taken as the conducting power of a hard-drawn copper wire at 0° C. For the annealed, the per-centage differences of the values of 6, hard drawn and annealed, 7, ditto, and of the first determinations of the three wires in Table III. and the annealed ones, were added together, and the mean added to the value found for the hard-drawn wire (as a per-centage amount). All the formulæ given as end-result with each metal have been constructed in this manner.

Gold.

Purified as described in the Philosophical Transactions, 1860, p. 175. Wires 1, 2, and 3 were of different preparations. The results obtained with these wires are given in Table IV.

* Philosophical Transactions, 1860.

Table IV.

	First wire.		Second wire.		Third wire.	
	Hard drawn.	Annealed.	Hard drawn.	Annealed.	Hard drawn.	Annealed.
Length.....	2214 millims.	2200 millims.	837 millims.		739·5 millims.	742·5 millims.
Diameter	0·759 millim.	0·759 millim.	0·467 millim.		0·434 millim.	0·434 millim.
Conducting power found before heating the hard-drawn wires	73·239 at 13°·2	Reduced to 0°. 76·821	72·550 at 15°·1	Reduced to 0°. 76·561	67·530 at 36°·8	Reduced to 0°. 77·229
Conducting power after being kept at 100° for 1 day ...	72·746 at 15·2	76·854	73·359 at 12·6	76·733	71·868 at 19·4°	77·223
Ditto, for 2 days ...	72·751 at 15·1	76·832	.....	.....	71·854 at 20·1	77·405
Ditto, for 3 days ...	.....	.....	.....	.....	72·191 at 19·0	77·457
Ditto, for 4 days ...	.....	.....	.....	.....	72·396 at 18·0	77·394

The means of the conducting powers found for each of the following temperatures were—

First wire, hard drawn.				Second wire, hard drawn.				Third wire, hard drawn.			
T.	Conducting power. Observed.	Conducting power. Calculated.	Difference.	T.	Conducting power. Observed.	Conducting power. Calculated.	Difference.	T.	Conducting power. Observed.	Conducting power. Calculated.	Difference.
15°·95	72·567	72·536	+0·031	13°·36	73·222	73·212	+0·010	12°·44	73·854	73·841	+0·013
30·76	68·798	68·828	−0·030	24·79	70·329	70·325	+0·004	23·27	70·965	70·975	−0·010
50·06	64·392	64·410	−0·018	40·80	66·515	66·544	−0·029	39·42	67·002	67·013	−0·011
69·75	60·397	60·385	+0·012	55·65	63·306	63·312	−0·006	55·47	63·441	63·448	−0·007
84·31	57·742	57·722	+0·020	69·52	60·528	60·531	−0·003	70·56	60·455	60·435	+0·020
99·27	55·248	55·263	−0·015	84·12	57·965	57·854	+0·051	84·79	57·904	57·893	+0·011
				100·00	55·203	55·232	−0·029	99·00	55·635	55·647	−0·012
First wire, annealed.								Third wire, annealed.			
14°·92	74·020	73·992	+0·028	At the commencement of the determinations this wire was torn away from its place by the stirrer.				14°·10	74·327	74·293	+0·034
30·05	70·039	70·068	−0·029					26·31	71·067	71·095	−0·028
48·87	65·575	65·611	−0·036					40·51	67·582	67·621	−0·039
69·90	61·220	61·191	+0·029					53·72	64·645	64·628	+0·017
82·82	58·811	58·768	+0·043					70·17	61·229	61·220	+0·009
99·62	55·915	55·948	−0·033					85·36	58·422	58·388	+0·034
								99·30	56·029	56·056	−0·027

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire (hard drawn) . . . $\lambda = 76{\cdot}838 - 0{\cdot}27973t + 0{\cdot}0006285t^2$.

For first wire (annealed) . . . . $\lambda = 78{\cdot}161 - 0{\cdot}28935t + 0{\cdot}0006664t^2$.

For second wire (hard drawn) . . $\lambda = 76{\cdot}786 - 0{\cdot}27549t + 0{\cdot}0005995t^2$.

For third wire (hard drawn . . $\lambda = 77{\cdot}343 - 0{\cdot}29043t + 0{\cdot}0007200t^2$.

For third wire (annealed) . . . $\lambda = 78{\cdot}231 - 0{\cdot}28849t + 0{\cdot}0006564t^2$.

The observations made with wire 1 (hard drawn) are given in Table I., those of the same wire (annealed) were made in the same manner.

Here we find no permanent change in conducting power with wire 1, after being kept at 100° for several days, and only a very slight increase with wires 2 and 3. The formulæ for the hard-drawn and annealed wires agree so closely that they may also, as with silver and copper, be considered the same.

In the paper just alluded to, the conducting power of pure gold was found—

		Reduced to 0°.
1.	72·68 at 19°·3	77·966
2.	73·08 at 23·3	79·524
3.	73·27 at 13·8	77·053
4.	73·99 at 15·1	78·178

The difference between hard-drawn and annealed wires was—

			Reduced to 0°.
5.	Hard drawn	74·20 at 14°·8	78·313
	Annealed	75·53 at 15·2	79·833
6.	Hard drawn	73·78 at 15·5	78·067
	Annealed	75·18 at 15·8	79·635

Taking the mean of the values as with copper, the following formulæ were deduced for the correction of conducting power for temperature:—

For hard-drawn wires $\lambda = 77{\cdot}964 - 0{\cdot}28648t + 0{\cdot}0006582t^2$.

For annealed wires $\lambda = 79{\cdot}327 - 0{\cdot}29149t + 0{\cdot}0006697t^2$.

Zinc.

Zinc free of arsenic was purified by distillation. All pressed wires. In Table V. the results obtained are given.

TABLE V.

	First wire.		Second wire.		Third wire.	
Length.....	502·2 millims.		394 millims.		372 millims.	
Diameter.....	0·588 millim.		0·513 millim.		0·519 millim.	
Conducting power found before heating the wires.....	26·744 at 23°·1	Reduced to 0°. 29·093	26·903 at 18°·5	Reduced to 0°. 28·836	26·835 at 18°·0	Reduced to 0°. 28·639
Ditto, after being kept at 100° for 1 day	26·695 at 23·7	29·103	27·081 at 17·5	28·919	26·784 at 18·5	28·636
Ditto, for 2 days...	.....		26·980 at 18·5	28·919	26·885 at 17·4	28·632

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
11°60	27·915	27·902	+0·013	11°20	27·706	27·687	+0·019	11°16	27·518	27·513	+0·005
24·24	26·639	26·653	−0·014	26·12	26·187	26·199	−0·012	25·96	26·088	26·090	−0·002
41·33	25·077	25·086	−0·009	39·55	24·951	24·959	−0·008	40·10	24·812	24·820	−0·008
55·08	23·925	23·926	−0·001	54·18	23·719	23·716	+0·003	56·85	23·423	23·428	−0·005
70·27	22·757	22·747	+0·010	72·32	22·330	22·330	0·000	71·73	22·306	22·295	+0·011
82·01	21·924	21·912	+0·012	85·77	21·407	21·414	−0·007	85·40	21·348	21·339	+0·009
98·07	20·865	20·875	−0·010	100·23	20·540	20·534	+0·006	98·95	20·462	20·472	−0·010

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire . . $\lambda = 29{\cdot}114 - 0{\cdot}10727t + 0{\cdot}0002372t^2$.

For second wire . $\lambda = 28{\cdot}881 - 0{\cdot}10949t + 0{\cdot}0002616t^2$.

For third wire . . $\lambda = 28{\cdot}649 - 0{\cdot}10424t + 0{\cdot}0002182t^2$.

No permanent alteration in the conducting power takes place after heating the wires for several days to 100°.

The value formerly found for the conducting power of zinc (precipitated galvanoplastically, fused and pressed) was—

	Reduced to 0°.
27·39 at 17°·6	29·220.

Treating these values as before, we find the formula for zinc to be

$$\lambda = 29{\cdot}022 - 0{\cdot}10752t + 0{\cdot}0002401t^2.$$

Cadmium.

The metal was purified as described in the Philosophical Transactions, 1860, p. 177. The wires were pressed. Table VI. shows the results.

TABLE VI.

	First wire.	Second wire.	Third wire.
Length.....	625 millims.	559 millims.	439 millims.
Diameter.....	0·641 millim.	0·678 millim.	0·684 millim.

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
8°87	23·327	23·329	−0·002	8°89	23·374	23·400	−0·026	14°60	21·849	21·859	−0·010
20·75	22·351	22·338	+0·013	21·59	22·280	22·270	+0·010	22·05	21·318	21·310	+0·008
34·47	21·241	21·255	−0·014	36·37	21·075	21·059	+0·016	39·65	20·072	20·061	+0·011
49·38	20·138	20·150	−0·012	48·52	20·157	20·146	+0·011	54·45	19·065	19·067	−0·002
63·39	19·188	19·186	+0·002	62·90	19·171	19·162	+0·009	68·10	18·179	18·194	−0·015
77·74	18·292	18·268	+0·024	80·00	18·109	18·131	−0·022	81·20	17·393	17·397	−0·004
93·55	17·325	17·339	−0·014					89·90	16·896	16·888	+0·008

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire . . $\lambda = 24{\cdot}100 - 0{\cdot}088554t + 0{\cdot}0001740t^2$.

For second wire . $\lambda = 24{\cdot}240 - 0{\cdot}096753t + 0{\cdot}0002548t^2$.

For third wire . . $\lambda = 24{\cdot}974 - 0{\cdot}078004t + 0{\cdot}0001147t^2$.

The values obtained for the alteration in the conducting power of these wires after heating them for several days to 100°, have unfortunately been lost. It may, however, be stated that the differences were very small, and that there was a loss in conducting power.

The conducting power of cadmium was found in the paper already referred to—

	Reduced to 0°.
22·10 at 18°·8	23·678.

Deducing the formula for cadmium in the manner before described, we find

$$\lambda = 23{\cdot}725 - 0{\cdot}087476t + 0{\cdot}0001797t^2.$$

Pure cadmium, when heated to about 80°, becomes exceedingly brittle, in fact it may be powdered in a hot mortar with great ease. We should not have been able to carry out the determinations if the wires had not been varnished, as the movement of the oil by the stirrer would have caused them to fall to pieces. It is worthy of remark that this change in the molecular arrangement of the wires does not make itself apparent in the conducting power to any very marked extent.

Tin.

Purified by dissolving commercial tin in nitric acid, and reducing the washed oxide by heating it with lampblack. Pressed wires were used. Table VII. gives the results.

TABLE VII.

	First wire.		Second wire.		Third wire.	
Length.....	279 millims.		375 millims.		315 millims.	
Diameter	0·559 millim.		0·634 millim.		0·729 millim.	
Conducting power found before heating the wires.....	10·970 at 18°·2	Reduced to 0°. 11·710	11·532 at 18°·1	Reduced to 0°. 12·324	12·285 at 18°·2	Reduced to 0°. 13·108
Ditto, after being kept at 100° for 1 day.....	11·124 at 19·4	11·926	11·442 at 19·1	12·273	12·291 at 18·4	13·124
Ditto, for 2 days....	10·852 at 27·0	11·956	11·448 at 18·6	12·257	12·296 at 18·4	13·129
Ditto, for 3 days....	10·835 at 28·0	11·980	11·444 at 18·4	12·264		

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
12°9	11·4110	11·4202	−0·0092	11°80	11·7144	11·7227	−0·0083	16°00	12·649	12·660	−0·011
25·27	10·9320	10·9246	+0·0074	26·32	11·1287	11·1153	+0·0134	26·54	11·944	11·934	+0·010
40·55	10·3570	10·3436	+0·0134	40·04	10·5805	10·5732	+0·0073	39·52	11·408	11·391	+0·017
54·15	9·8498	9·8558	−0·0060	54·02	10·0451	10·0526	−0·0075	56·27	10·717	10·727	−0·010
70·53	9·2980	9·3046	−0·0066	70·02	9·4883	9·4961	−0·0078	70·30	10·189	10·202	−0·013
83·13	8·9033	8·9078	−0·0045	85·02	9·0102	9·0127	−0·0025	85·72	9·654	9·657	−0·003
100·90	8·3937	8·3881	+0·0056	98·50	8·6158	8·6096	+0·0062	97·30	9·279	9·270	+0·009

The formulæ deduced from the observations, and from which the conducting powers were calculated, were—

For first wire . $\lambda = 11{\cdot}9613 - 0{\cdot}042902t + 0{\cdot}00007422t^2$.

For second wire $\lambda = 12{\cdot}2419 - 0{\cdot}044965t + 0{\cdot}00008213t^2$.

For third wire . $\lambda = 13{\cdot}1186 - 0{\cdot}046561t + 0{\cdot}00007206t^2$.

We see from the results that wires 1 and 2 decrease to a small extent in conducting power, whereas wire 3 increases slightly after being heated to 100°.

The conducting power of tin was found—

	Reduced to 0°.
11·45 at 21°·0	12·351;

and calculating the formula of tin as before, we find

$$\lambda = 12{\cdot}366 - 0{\cdot}044554t + 0{\cdot}00007588t^2.$$

Lead.

Purified by reducing by heat the twice recrystallized acetate. Wires 1 and 2 were pressed; wire 3 drawn. No permanent alteration in the conducting power of the wires was observed after they had been kept at 100° for two days. Table VIII. shows the results.

TABLE VIII.

	First wire.	Second wire.	Third wire.
Length	416 millims.	453 millims.	389 millims.
Diameter	0·669 millim.	0·698 millim.	0·959 millim.

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
14°55	7·9365	7·9336	+0·0029	14°50	7·8685	7·8653	+0·0032	12°40	7·9038	7·9022	+0·0016
25·40	7·6129	7·6152	−0·0023	27·50	7·5336	7·5392	−0·0056	26·20	7·4967	7·4968	−0·0001
40·30	7·2036	7·2071	−0·0035	40·37	7·1405	7·1397	+0·0008	39·60	7·1309	7·1324	−0·0015
54·80	6·8423	6·8420	+0·0003	54·80	6·7789	6·7775	+0·0014	54·60	6·7565	6·7585	−0·0020
70·33	6·4881	6·4863	+0·0018	69·63	6·4392	6·4370	+0·0022	69·70	6·4205	6·4187	+0·0018
84·52	6·1964	6·1929	+0·0035	84·80	6·1189	6·1218	−0·0029	84·40	6·1250	6·1229	+0·0021
99·35	5·9159	5·9189	−0·0030	100·10	5·8388	5·8381	+0·0007	98·85	5·8642	5·8658	−0·0016

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire . $\lambda = 8{\cdot}3882 - 0{\cdot}032346t + 0{\cdot}00007540t^2$.
For second wire $\lambda = 8{\cdot}3147 - 0{\cdot}032055t + 0{\cdot}00007307t^2$.
For third wire . $\lambda = 8{\cdot}2925 - 0{\cdot}032468t + 0{\cdot}00008011t^2$.

The value found for the conducting power of lead was

	Reduced to 0°.
7·77 at 17°·3	8·304.

Treating the mean of the values as above, the formula is

$$\lambda = 8{\cdot}318 - 0{\cdot}032237t + 0{\cdot}00007608t^2.$$

Arsenic.

Purified by sublimation. Small bars were cut from a comparatively solid piece and soldered to two copper wires; on account of the extreme brittleness of arsenic, the bars were placed in glass tubes closed at the ends with gypsum, through which the copper wires passed. As these were dried in a water-bath for several days, no permanent alteration of the conducting power of the bars was found after being heated in the oil-bath for two days. The values found for the conducting power of arsenic agree as well as could be expected, considering the bars were made by hand, and the metal somewhat porous. The difficulty of obtaining bars of metal of sufficient length is so great that we have been contented with two series. These are given in Table IX.

TABLE IX.

	First bar.	Second bar.
Length	50·4 millims.	55·5 millims.
Diameter	0·93 millim.	1·01 millim.

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.	
14°·20	5·0203	5·0180	+0·0023	13°·50	4·0051	4·0037	+0·0014
25·30	4·8007	4·8008	−0·0001	24·50	3·8371	3·8450	−0·0079
37·80	4·5710	4·5736	−0·0026	40·15	3·6367	3·6311	+0·0056
55·00	4·2854	4·2906	−0·0052	55·55	3·4447	3·4341	+0·0106
70·00	4·0767	4·0722	+0·0045	69·90	3·2559	3·2628	−0·0069
85·30	3·8810	3·8764	+0·0046	82·50	3·1144	3·1221	−0·0077
101·00	3·7005	3·7041	−0·0036	99·80	2·9485	2·9435	+0·0050

The formulæ deduced from the observations, and from which the conducting powers were calculated, were—

For first bar $\lambda = 5{\cdot}3168 - 0{\cdot}021874t + 0{\cdot}00005848t^2$.
For second bar $\lambda = 4{\cdot}2078 - 0{\cdot}015506t + 0{\cdot}00002843t^2$.

Taking the mean of the conducting powers at 0°, we deduce the formula for the correction of conducting power for temperature to be

$$\lambda = 4{\cdot}7623 - 0{\cdot}018571t + 0{\cdot}00004228t^2.$$

Antimony.

Purified by twice recrystallizing commercially pure tartrate of antimony and potassium, reducing by heat and re-fusing with antimonic acid. As antimony is so very brittle, it was not possible to manipulate with it in form of wire, it was therefore fused in the bowl of a tobacco-pipe, and when liquid allowed to run into the stem. After breaking off the bowl, the ends of the pipe were made so hot that the metal melted, and clean copper wires were pushed into the liquid metal, which on solidifying held them fast. The free ends of the copper wires were then soldered to the thick ones in the trough. Unfortunately in each case the copper wires in the pipe-stem became loose after heating for two or three days, and had to be therefore resoldered, so that no reliable determinations could be made as to the effect of heating to 100° for several days on the conducting power. It may be stated that the three wires lost in conducting power; but to what extent, we are of course not in a position to say. As the diameter of the pipe-stem could not be accurately determined, and as it could not be ascertained whether there were cavities in the wires (caused by contraction on cooling and crystallization) or not, the first observed conducting power was taken equal to 100. Table X. shows the results.

TABLE X.

The means of the conducting powers found for each of the following temperatures were—

First wire.				Second wire.				Third wire.			
T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
10°00	100·000	100·052	−0·052	8°40	100·000	99·999	+0·001	13°80	100·000	99·901	+0·099
26·35	94·062	93·910	+0·152	25·60	93·947	93·850	+0·097	22·30	96·378	96·514	−0·136
40·40	88·982	89·089	−0·107	42·45	88·139	88·329	−0·190	38·65	90·552	90·527	+0·025
54·55	84·633	84·664	−0·031	57·80	83·707	83·731	−0·024	53·50	85·671	85·692	−0·021
70·65	80·126	80·152	−0·026	69·45	80·691	80·517	+0·174	69·65	81·118	81·082	+0·036
83·50	77·071	76·953	+0·118	86·85	76·138	76·159	−0·021	84·45	77·480	77·454	+0·026
99·40	73·430	73·484	−0·054	101·25	72·922	72·953	−0·031	98·80	74·448	74·480	−0·032

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire . $\lambda = 104{\cdot}095 - 0{\cdot}41487t + 0{\cdot}0010755t^2$.

For second wire $\lambda = 103{\cdot}190 - 0{\cdot}38721t + 0{\cdot}0008748t^2$.

For third wire . $\lambda = 105{\cdot}801 - 0{\cdot}44541t + 0{\cdot}0012995t^2$.

The observed conducting powers in this and the foregoing Table do not agree so well with the calculated as the others, on account of the temperature of the bath never being exactly the same as that of the wire; for in the one case the heat had to traverse the glass tube filled with air, in the other the thickness of the pipe-stem, before reaching the metal.

The conducting power of antimony was found equal to

	Reduced to 0°.
4·29 at 18°·7	4·6172

Using this value as before described, we obtain a formula for antimony where

$$\lambda = 4{\cdot}6172 - 0{\cdot}018389t + 0{\cdot}00004788t^2.$$

Bismuth.

Purified by reducing the basic nitrate of bismuth with lampblack. Table XI. gives the results. The wires were pressed.

TABLE XI.

	First wire.		Second wire.		Third wire.	
Length.....	117 millims.		121·4 millims.		42·5 millims.	
Diameter.....	0·596 millim.		0·596 millim.		0·217 millim.	
		Reduced to 0°.		Reduced to 0°.		Reduced to 0°.
Conducting power found before heating the wires.....	1·1787 at 16°·6	1·2517	1·1036 at 18°·8	1·1773	1·2215 at 16°·6	1·2951
Conducting power after being kept at 100° for 1 day ...	1·3599 at 17·6	1·4494	1·3110 at 19·0	1·3995	1·3683 at 17·8	1·4569
Ditto, for 2 days...	1·3595 at 18·2	1·4521	1·3121 at 19·0	1·4006	1·3709 at 17·6	1·4587
Ditto, for 3 days...	1·3624 at 18·0	1·4541	1·3096 at 19·9	1·4023	1·3710 at 17·9	1·4603

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
9°·20	1·4059	1·4058	+0·0001	8°·60	1·3654	1·3641	+0·0013	9°·40	1·4129	1·4128	+0·0001
26·15	1·3226	1·3226	0·0000	24·00	1·2909	1·2935	−0·0026	25·65	1·3329	1·3339	−0·0010
39·50	1·2609	1·2614	−0·0005	38·75	1·2297	1·2287	+0·0010	43·05	1·2551	1·2538	+0·0013
57·25	1·1863	1·1858	+0·0005	55·30	1·1591	1·1593	−0·0002	57·45	1·1913	1·1912	+0·0001
68·95	1·1397	1·1397	0·0000	68·90	1·1058	1·1050	+0·0008	71·60	1·1315	1·1328	−0·0013
84·35	1·0833	1·0833	0·0000	84·00	1·0478	1·0474	+0·0004	88·60	1·0671	1·0666	+0·0005
96·35	1·0428	1·0429	−0·0001	95·90	1·0036	1·0042	−0·0006				

The formulæ deduced from the observations, by which the conducting powers were calculated, were—

For first wire . $\lambda = 1{\cdot}4535 - 0{\cdot}0052883t + 0{\cdot}00001060t^2$.

For second wire $\lambda = 1{\cdot}4049 - 0{\cdot}0047972t + 0{\cdot}000006453t^2$.

For third wire . $\lambda = 1{\cdot}4603 - 0{\cdot}0051286t + 0{\cdot}000007737t^2$.

From the above we see how bismuth increases in conducting power after being kept at 100° for one day. This increment is so rapid that it may be followed for the first two hours from five to five minutes. Wire 1 altered by one day's heating 16 per cent.; wire 2, 19 per cent.; and wire 3, 12 per cent. Wires 1 and 2 were cut from the same piece.

This behaviour explains why the conducting power of bismuth wires varies so much: for in the paper so often here alluded to, the maximum difference between twelve wires was found to be 22 per cent. In pressing the wires the heat applied to the press is never constant; so that, if pressed very warm, wires of high conducting power would probably be the result. The conducting power of bismuth was found equal to

	Reduced to 0°.
1·19 at 13°·8	1·2484

Taking the mean of the values as before, we find the formula for bismuth to be

$$\lambda = 1{\cdot}2454 - 0{\cdot}0043858t + 0{\cdot}000007134t^2.$$

Mercury.

Purified by allowing a solution of subnitrate of mercury to stand over the metal for several weeks, during which time it was often well shaken up with it. The determinations were made in a calibrated thermometer-tube, to the ends of which wide glass tubes (13 to 14 millims. wide) were fused and bent, as shown in fig. 3. Mercury prepared at different times was used for the determinations. For the experiments, the tube was filled with hot mercury, and its resistance was determined when cold. This was twice repeated; and the resistance being found the same each time, it was assumed that the tube filled in this manner did not contain air-bubbles; this is also proved by the close agreement of the formulæ found in the two cases for the variation of the conducting power at higher temperatures; for if in either case air-bubbles had been present, the formulæ must have differed to a much greater extent, as it can scarcely be assumed that in the two cases the bubbles were equal in bulk. The mercury was connected with the apparatus by amalgamated copper wires (4 to 5 millims. thick). Table XII. shows the results obtained.

Fig. 3.

TABLE XII.

Length ...= 269 millims.
Diameter =1·424 millim.

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.	
0°	1·6521	1·6530	−0·0009	0°	1·6529	1·6533	−0·0004
20·55	1·6276	1·6272	+0·0004	20·95	1·6272	1·6268	+0·0004
40·45	1·6003	1·6011	−0·0008	39·92	1·6010	1·6018	−0·0008
59·82	1·5750	1·5746	+0·0004	60·40	1·5741	1·5738	+0·0003
79·78	1·5465	1·5462	+0·0003	80·70	1·5454	1·5450	+0·0004
99·90	1·5162	1·5164	−0·0002	99·30	1·5174	1·5177	−0·0003

The formulæ deduced from the observations, by which the conducting powers were calculated, were—

For the first series . $\lambda = 1{\cdot}6530 - 0{\cdot}0012240t - 0{\cdot}000001434t^2$.
For the second series $\lambda = 1{\cdot}6533 - 0{\cdot}0012370t - 0{\cdot}000001297t^2$.

The value found for the conducting power of mercury was

	Reduced to 0°.
1·63 at 22°·8	1·6588

Taking the mean of the values as before, we find the formula for mercury to be

$$\lambda = 1{\cdot}656 - 0{\cdot}0012326t - 0{\cdot}000001368t^2.$$

Tellurium.

Purified by dissolving the commercial metal in aqua regia, evaporating to dryness with excess of carbonate of sodium, fusing the residue, which was dissolved in water, and nitrate of barium added to precipitate any selenium present. The filtrate was evaporated to dryness with hydrochloric acid in excess, the residue dissolved in water, and precipitated by sulphurous acid.

On account of the low conducting power of tellurium, small bars of about 15 millims. in length and 3–5 millims. in diameter were used for the experiments. Bars I. and II. are of the same preparation. As the bars could not be accurately measured, we have called the first observed conducting power 100 in each case. Table XIII. gives the results.

Table XIII.

	Bar I.			Bar II.			Bar III.		
Conducting power found before heating the bars to 100°.	100	at 16·4	.....	100	at 15·9	.....	100	at 13·6	
Ditto, after being kept at 100° for 1 day	79·145	at 15·4	.....	86·50	at 13·0	.....	83·16	at 12·6	
Ditto, for 2 days	45·449	at 16·0	.....	76·51	at 13·6	.....	69·23	at 14·1	
Ditto, for 3 days	28·378	at 16·0	.....	70·43	at 16·4	.....	61·25	at 16·9	
Ditto, for 4 days	16·129	at 15·0	.....	65·68	at 16·6	.....	54·92	at 17·2	
Ditto, for 5 days	8·068	at 15·2	.....	61·68	at 16·8	.....	50·69	at 17·8	
Ditto, for 6 days	6·989	at 15·0	.....	56·85	at 17·2	.....	46·11	at 16·6	
Ditto, for 7 days	5·781	at 14·2	.....	54·88	at 16·6	.....	42·35	at 16·4	
Ditto, for 8 days	4·830	at 15·5	.....	51·33	at 16·1	.....	38·64	at 15·8	
Ditto, for 9 days	4·621	at 16·8	.....	46·27	at 15·6	.....	35·31	at 16·2	
Ditto, for 10 days	4·302	at 15·3	.....	45·26	at 16·2	.....	33·50	at 16·4	
Ditto, for 11 days	4·181	at 15·0	Reduced to 0°.	42·10	at 16·6	.....	30·97	at 16·8	
Ditto, for 12 days	4·1371	at 16·1	3·7662	41·31	at 17·4	.....	29·98	at 18·2	
Ditto, for 13 days	4·0844	at 14·6	3·7646	39·28	at 16·0	.....	28·21	at 15·6	
Ditto, for 14 days	.....		.....	37·72	at 17·1	.....	26·73	at 16·8	
Ditto, for 15 days	.....		.....	35·35	at 15·4	.....	23·68	at 15·4	
Ditto, for 16 days	.....		.....	32·23	at 15·6	.....	19·43	at 16·0	
Ditto, for 17 days	.....		.....	29·92	at 17·0	.....	16·65	at 17·6	
Ditto, for 18 days	.....		.....	28·11	at 17·6	.....	14·43	at 17·0	
Ditto, for 19 days	.....		.....	26·25	at 16·2	.....	12·59	at 16·4	
Ditto, for 20 days	.....		.....	25·54	at 13·0	.....	11·68	at 14·4	
Ditto, for 21 days	.....		.....	24·12	at 13·4	.....	10·34	at 13·6	
Ditto, for 22 days	.....		.....	23·29	at 12·8	.....	9·32	at 13·6	
Ditto, for 23 days	.....		.....	22·00	at 13·6	.....	8·64	at 14·1	
Ditto, for 24 days	.....		.....	21·45	at 14·1	.....	7·92	at 13·8	
Ditto, for 25 days	.....		.....	20·86	at 14·6	.....	7·36	at 14·6	
Ditto, for 26 days	.....		.....	20·17	at 15·8	.....	6·97	at 14·2	
Ditto, for 27 days	.....		.....	19·74	at 16·0	.....	6·66	at 14·8	
Ditto, for 28 days	.....		.....	19·68	at 13·0	.....	6·52	at 15·8	
Ditto, for 29 days	.....		.....	19·65	at 12·2	Reduced to 0°.	6·35	at 15·8	
Ditto, for 30 days	.....		.....	19·633	at 12·0	20·145	6·12	at 12·6	
Ditto, for 31 days	.....		.....	19·633	at 11·9	20·137	6·04	at 12·0	Reduced to 0°.
Ditto, for 32 days	.....		.....	.....		.....	6·0330	at 11·8	5·6134
Ditto, for 33 days	.....		.....	.....		.....	6·0602	at 12·2	5·6191

The means of the conducting powers for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
10°40	3·9566	3·9575	−0·0009	3°70	19·976	19·972	+0·004	4°20	5·6646	5·6797	−0·0151
25·25	4·5212	4·5240	−0·0028	11·80	19·650	19·660	−0·010	22·80	6·7456	6·7392	+0·0064
38·85	5·3940	5·3846	+0·0094	22·40	19·477	19·466	+0·011	39·40	8·6703	8·5781	+0·0922
55·10	6·9089	6·9060	+0·0029	29·40	19·468	19·473	−0·005	53·90	10·8480	10·9426	−0·0946
70·45	8·8706	8·9019	−0·0313	29·40	19·468	19·496	−0·028	69·50	14·2472	14·3441	−0·0969
83·10	11·0316	11·0018	+0·0298	34·60	19·546	19·544	+0·002	83·60	18·4043	18·2406	+0·1637
99·40	14·3690	14·3764	−0·0074	39·70	19·689	19·653	+0·036	98·80	23·3209	23·3769	−0·0560
				55·30	20·513	20·464	+0·049				
				69·20	21·825	21·942	−0·117				
				86·40	25·096	25·019	+0·077				
				103·60	29·765	29·784	−0·019				

The formulæ deduced from the observations, and from which the conducting powers were calculated, were—

For first bar . . . . . $\lambda = 3{\cdot}7619 + 0{\cdot}011614t + 0{\cdot}0006598t^2 + 0{\cdot}000002994t^3$.

For second bar to 29·4 . . $\lambda = 20{\cdot}162 - 0{\cdot}055338t + 0{\cdot}001085t^2$.

For second bar from 29·4 to 100 $\lambda = 20{\cdot}014 - 0{\cdot}029569t + 0{\cdot}00009390t^2 + 0{\cdot}000019635t^3$.

For third bar , ... $\lambda = 5{\cdot}5752 + 0{\cdot}019274t + 0{\cdot}0013235t^2 + 0{\cdot}000003088t^3$.

From the above Table we learn that tellurium behaves in a very different manner from the other metals; for it will be seen how very much the conducting power decreases after it has been heated to 100° for some days, and how different is the time required before the conducting power of the different bars becomes constant, or, in other words, until the heating of the bars to 100° causes no further permanent alteration in the conducting power. Bar I. required 13 days; bar II. 32; bar III. 33. The first observed conducting power being taken equal to 100, bar I. is reduced to 4, bar II. to 19·6, and bar III. to 6. If we now look at the determinations of the conducting power at different temperatures of the three bars, we are struck at the great want of concordance in the results. With the first series we observe that the conducting power increases rapidly as the temperature rises; with the second it decreases with the rise of temperature to 29°·4, from which point it increases rapidly, as with bar I.; the third behaves as the first.

Bar I. showed no apparent difference in crystalline structure after being heated; it was thought very probable that the crystalline structure might have been altered by heating, and thus caused the enormous change in conducting power. The three bars, when first heated, behaved as metal to 70° or 80°, that is to say, they lost in conducting power up to that temperature, where it then began to increase. The temperature of this turning-point became lower after each day's heating, until, as in bars I. and II., it is below the lowest temperature at which observations were made.

The behaviour, therefore, of tellurium is intermediate between that of the metal and that of the metalloid; for, according to Hittorf*, selenium increases rapidly in conducting power with the temperature. Graphite and gas-coke† behave in the same manner; and Becquerel‡ found that gases when heated conduct better than when cold. From these facts we learn another marked difference in the physical properties of the metals and metalloids, viz. *that the metals lose in conducting power with an increase of temperature, whereas under the same circumstances the metalloids gain.*

In order to be better able to compare the results obtained with the pure metals, we give the following Tables. Table XIV. contains all the formulæ deduced from the observations by the method of least squares, with the conducting power of each metal taken =100 at 0°; Table XV. the mean of the formulæ found for each metal.

* Poggendorff's 'Annalen,' vol. lxxxvi. p. 214.

† Philosophical Transactions, 1858, p. 386.

‡ Ann. de Chim. et de Phys. (iii.) vol. xxxix. p. 388.

TABLE XIV.

Silver	I.	Hard drawn ...	$\lambda=100-0{\cdot}38394\ t+0{\cdot}0009723\ t^2$
		Annealed	$\lambda=100-0{\cdot}37725\ t+0{\cdot}0008900\ t^2$
	II.	Hard drawn ...	$\lambda=100-0{\cdot}38915\ t+0{\cdot}0010472\ t^2$
		Annealed	$\lambda=100-0{\cdot}39287\ t+0{\cdot}0010625\ t^2$
	III.	Hard drawn ...	$\lambda=100-0{\cdot}37544\ t+0{\cdot}0009724\ t^2$
		Annealed	$\lambda=100-0{\cdot}37855\ t+0{\cdot}0009647\ t^2$
Copper	I.	Hard drawn ...	$\lambda=100-0{\cdot}37351\ t+0{\cdot}0007716\ t^2$
		Annealed	$\lambda=100-0{\cdot}37291\ t+0{\cdot}0007781\ t^2$
	II.	Hard drawn ...	$\lambda=100-0{\cdot}39173\ t+0{\cdot}0009394\ t^2$
		Annealed	$\lambda=100-0{\cdot}38797\ t+0{\cdot}0008986\ t^2$
	III.	Hard drawn ...	$\lambda=100-0{\cdot}39639\ t+0{\cdot}0010173\ t^2$
		Annealed	$\lambda=100-0{\cdot}39954\ t+0{\cdot}0010003\ t^2$
Gold	I.	Hard drawn ...	$\lambda=100-0{\cdot}36405\ t+0{\cdot}0008181\ t^2$
		Annealed	$\lambda=100-0{\cdot}37017\ t+0{\cdot}0008526\ t^2$
	II.	Hard drawn ...	$\lambda=100-0{\cdot}35877\ t+0{\cdot}0007807\ t^2$
	III.	Hard drawn ...	$\lambda=100-0{\cdot}37551\ t+0{\cdot}0009309\ t^2$
		Annealed	$\lambda=100-0{\cdot}36877\ t+0{\cdot}0008390\ t^2$
Zinc	I.	.....	$\lambda=100-0{\cdot}36845\ t+0{\cdot}0008147\ t^2$
	II.	.....	$\lambda=100-0{\cdot}37911\ t+0{\cdot}0009058\ t^2$
	III.	.....	$\lambda=100-0{\cdot}36385\ t+0{\cdot}0007618\ t^2$
Cadmium	I.	.....	$\lambda=100-0{\cdot}36745\ t+0{\cdot}0007220\ t^2$
	II.	.....	$\lambda=100-0{\cdot}39915\ t+0{\cdot}0010511\ t^2$
	III.	.....	$\lambda=100-0{\cdot}33953\ t+0{\cdot}0004995\ t^2$
Tin	I.	.....	$\lambda=100-0{\cdot}35867\ t+0{\cdot}0006205\ t^2$
	II.	.....	$\lambda=100-0{\cdot}36730\ t+0{\cdot}0006709\ t^2$
	III.	.....	$\lambda=100-0{\cdot}35492\ t+0{\cdot}0005493\ t^2$
Lead	I.	.....	$\lambda=100-0{\cdot}38561\ t+0{\cdot}0008989\ t^2$
	II.	.....	$\lambda=100-0{\cdot}38553\ t+0{\cdot}0008788\ t^2$
	III.	.....	$\lambda=100-0{\cdot}39153\ t+0{\cdot}0009661\ t^2$
Arsenic	I.	.....	$\lambda=100-0{\cdot}41141\ t+0{\cdot}0011000\ t^2$
	II.	.....	$\lambda=100-0{\cdot}36851\ t+0{\cdot}0006757\ t^2$
Antimony	I.	.....	$\lambda=100-0{\cdot}39855\ t+0{\cdot}0010332\ t^2$
	II.	.....	$\lambda=100-0{\cdot}37524\ t+0{\cdot}0008477\ t^2$
	III.	.....	$\lambda=100-0{\cdot}42099\ t+0{\cdot}0012983\ t^2$
Bismuth	I.	.....	$\lambda=100-0{\cdot}36383\ t+0{\cdot}0007293\ t^2$
	II.	.....	$\lambda=100-0{\cdot}34146\ t+0{\cdot}0004593\ t^2$
	III.	.....	$\lambda=100-0{\cdot}35120\ t+0{\cdot}0005298\ t^2$
Mercury	I.	.....	$\lambda=100-0{\cdot}074047t+0{\cdot}00008672t^2$
	II.	.....	$\lambda=100-0{\cdot}074820t+0{\cdot}00007844t^2$

TABLE XV.

Silver	$\lambda=100-0{\cdot}38287t+0{\cdot}0009848t^2$
Copper.....	$\lambda=100-0{\cdot}38701t+0{\cdot}0009009t^2$
Gold	$\lambda=100-0{\cdot}36745t+0{\cdot}0008443t^2$
Zinc.....	$\lambda=100-0{\cdot}37047t+0{\cdot}0008274t^2$
Cadmium	$\lambda=100-0{\cdot}36871t+0{\cdot}0007575t^2$
Tin	$\lambda=100-0{\cdot}36029t+0{\cdot}0006136t^2$
Lead	$\lambda=100-0{\cdot}38756t+0{\cdot}0009146t^2$
Arsenic	$\lambda=100-0{\cdot}38996t+0{\cdot}0008879t^2$
Antimony	$\lambda=100-0{\cdot}39826t+0{\cdot}0010364t^2$
Bismuth	$\lambda=100-0{\cdot}35216t+0{\cdot}0005728t^2$
Mean of the above ...	$\lambda=100-0{\cdot}37647t+0{\cdot}0008340t^2$

From the last Table we see how closely the values found for the constants y or z agree together; and to show this more clearly, the conducting powers calculated from these formulæ for 0°, 20°, 40°, 60°, 80°, and 100° are given in Table XVI., together with values calculated from the mean of all the formulæ.

TABLE XVI.

T.	Silver.	Copper.	Gold.	Zinc.	Cadmium.	Tin.	Lead.	Arsenic.	Antimony.	Bismuth.	Calculated values from mean of formulæ.	Greatest difference from mean.
0	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00	0·00
20	92·74	92·62	92·99	92·92	92·93	93·04	92·62	92·56	92·45	93·18	92·80	0·38
40	86·26	85·96	86·65	86·50	86·46	86·51	85·96	85·82	85·73	86·83	86·27	0·56
60	80·57	80·01	80·98	80·75	80·60	80·59	80·04	79·80	79·84	80·93	80·41	0·61
80	75·67	74·80	76·01	75·66	75·35	75·10	74·85	74·50	74·77	75·49	75·23	0·78
100	71·56	70·31	71·70	71·23	70·70	70·11	70·39	69·88	70·54	70·51	70·69	1·01

Again, in Table XVII., we give the conducting power of the metals compared with hard-drawn silver wire at 0°=100, first at 0° and then at 100°, and, lastly, taking silver at 100°=100.

TABLE XVII.

	Conducting power		Taking silver =100 at 100°.
	At 0°.	At 100°.	
Silver (hard drawn)	100·00	71·56	100·00
Copper (hard drawn)	99·95	70·27	98·20
Gold (hard drawn)	77·96	55·90	78·11
Zinc	29·02	20·67	28·89
Cadmium	23·72	16·77	23·44
Tin	12·36	8·67	12·12
Lead	8·32	5·86	8·18
Arsenic	4·76	3·33	4·65
Antimony	4·62	3·26	4·55
Bismuth	1·245	0·878	1·227

From these Tables we think we may deduce the law, that *all pure metals in a solid state vary in conducting power between* 0° *and* 100° *to the same extent*, more especially as we find that wires of one and the same metal show almost the same differences as were found between the mean results obtained for the different metals. In Table XVIII. two examples of this are given.

TABLE XVIII.

T.	Copper.		Cadmium.	
	I. annealed.	III. annealed.	II.	III.
0	100·00	100·00	100·00	100·00
20	92·85	92·41	92·44	93·41
40	86·33	85·62	85·72	87·22
60	80·43	79·63	79·84	81·42
80	75·15	74·44	74·79	76·03
100	70·49	70·05	70·60	71·04

In Table XIX. the resistances of the copper wires 1, 2, and 3, and those calculated from the mean of all the formulæ, are given; we do this to show that the resistance of

a wire does not increase in direct ratio to the temperature (as stated by some experimenters in this direction), but, on the contrary, the formula for correction of the resistance of a wire for temperature is

$$r = x + yt + zt^2,$$

and not

$$r = x + yt.$$

TABLE XIX.

First wire, hard drawn.			First wire, annealed.			Second wire, hard drawn.			Second wire, annealed.		
T.	Resistance.	Increase of resistance for 1°.	T.	Resistance.	Increase of resistance for 1°.	T.	Resistance.	Increase of resistance for 1°.	T.	Resistance.	Increase of resistance for 1°.
° 0	98·382	.....	° 0	98·241	.....	° 0	98·412	.....	° 0	97·902	.....
16·86	104·74	0·3771	17·0	104·67	0·3782	19·17	105·98	0·3948	18·96	105·28	0·3891
29·88	109·81	0·3825	29·63	109·54	0·3813	30·95	110·88	0·4028	31·86	110·59	0·3982
51·03	118·72	0·3985	50·22	118·08	0·3950	48·53	118·32	0·4102	52·05	119·08	0·4069
69·52	126·59	0·4057	69·60	126·23	0·4021	69·22	127·16	0·4153	70·27	126·85	0·4119
83·77	132·60	0·4085	83·42	132·21	0·4072	83·77	133·31	0·4166	83·81	132·58	0·4138
98·60	139·22	0·4142	99·37	139·10	0·4112	99·00	139·80	0·4181	99·57	139·36	0·4164

Third wire, hard drawn.			Third wire, annealed.			Resistance calculated from the mean of the six formulæ found for copper.			Resistance calculated from the mean of all the formulæ.		
° 0	99·384	.....	° 0	97·806	.....	° 0	100	.....	° 0	100	.....
12·65	104·42	0·3981	13·45	103·14	0·3966	20	107·97	0·3985	20	107·76	0·3680
25·61	109·82	0·4075	26·15	108·41	0·4055	40	116·33	0·4082	40	115·91	0·3977
39·52	115·72	0·4134	39·35	113·99	0·4113	60	124·98	0·4163	60	124·36	0·4060
53·92	121·85	0·4167	55·50	120·95	0·4170	80	133·69	0·4211	80	132·92	0·4115
69·90	128·54	0·4171	69·90	127·00	0·4176	100	142·22	0·4222	100	141·46	0·4146
84·87	134·82	0·4175	84·67	133·25	0·4186						
99·92	140·94	0·4159	99·05	139·32	0·4190						

The calculations from a formula of four or more terms, as

$$\lambda = x + yt + zt^2 + at^3,$$

agree better with the observed values than that of three. An example of this is shown in Table XX., where the formulæ, deduced from observations made with a hard-drawn wire (of course previously heated to 100° for several days), of three and four terms, with the differences, are given.

TABLE XX.

T.	Conducting power.		Difference.	Conducting power, calculated from formula of four terms.	Difference.
	Observed.	Calculated from formula of three terms.			
10°9	95·169	95·134	+0·035	95·166	+0·003
30·1	88·537	88·588	−0·051	88·534	+0·003
49·5	82·610	82·627	−0·017	82·605	+0·005
69·0	77·320	77·297	+0·023	77·304	−0·014
82·8	73·976	73·926	+0·050	73·966	+0·010
97·9	70·579	70·619	−0·040	70·580	−0·001

The formula of three terms, deduced from the observations, was

$$\lambda = 99{\cdot}137 - 0{\cdot}37675t + 0{\cdot}0008728t^2,$$

and that of four terms

$$\lambda = 99{\cdot}307 - 0{\cdot}39301t + 0{\cdot}0012318t^2 - 0{\cdot}000002193t^3.$$

From the above it will be seen how much better the observed values agree with the formula of four terms. We have, however, contented ourselves with a formula of three terms, as the conducting powers calculated from it agree with those observed to values corresponding to 0°·1 or 0°·2, and as the calculations for a formula of four terms would have increased the labour of the research to a very great extent. But it may be asked how it happens that the formulæ obtained for wires of one and the same metal vary so much, in fact, show differences almost equal to the mean of those deduced for the different metals?

That this is not due to errors of observation we have repeatedly satisfied ourselves; for compare only the formulæ of the hard-drawn (or rather partially annealed) and the annealed wires, and see how well they agree with each other. It appears, however, to be probably due to the molecular arrangement of the wires being different in each case. Take, for instance, the copper wires experimented with: wire 1 increased in conducting power by heating to 100° for several days, almost to the same extent as if it had been annealed, wire 2 partially so, and wire 3 hardly at all; and here it may be mentioned that silver and copper wires become softer and lose their elasticity, whereas gold does not seem to be annealed at all after having been kept at 100° for several days. Again, take cadmium, where we know that the wires become brittle and crystalline at 80°, and we find the formulæ vary more than those of any other metals; and, lastly, look at the results obtained with bismuth and tellurium, and there can be little doubt that the reason why the formulæ of the wires and bars of the same metal do not agree together is that the molecular arrangement is different in each; and that this is the cause of the differences in the formulæ, we may also assume from the fact that, when the wires on being heated do not at all or only to a very slight degree permanently alter in their conducting power, when cooled again, then the formulæ of wires of the same metal agree very closely with each other. Compare, for instance, those of lead, tin, mercury, &c.

The mean of the conducting powers given in the Tables agrees very well with the mean of the former determinations made with wires of metals of different preparation to that of those used for the experiments described in this paper.

The following questions have suggested themselves during the foregoing investigation, the answers to which we reserve for ourselves. It is intended to make them the subjects of short communications, which from time to time will be laid before the Royal Society:—

1. Will a hard-drawn wire become partially annealed by age? and, on the other hand, will an annealed wire become partially hard drawn?

2. Will bismuth or tellurium return to their original conducting power in time, or by exposure to intense cold?

3. Whether by heating tellurium or any of the metals to a higher temperature than 100° we should not arrive at the same result in a much shorter time.

4. What are the thermo-electric properties of bismuth, antimony, tellurium, &c. after being kept at 100° for several days? will they not have altered? It is remarkable that bismuth, which stands at one end of the thermo-electric series, should gain in conducting power after heating for some days, and that antimony and tellurium, at the other end of the series, should lose, the one slightly, the other, with a much higher thermo-electric number, to a very great extent.

5. Will tellurium conduct better in a melted state than the solid?

6. What law do the alloys follow as regards the influence of temperature on their conducting power?

II. *On the Aquiferous and Oviducal System in the Lamellibranchiate Mollusks.*

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Very different explanations have been offered of the means by which certain of the Lamellibranchiata are enabled to distend their muscular foot until the fluid with which it is swollen up causes it to appear all but transparent. These explanations, different as they are both in principle and in detail, admit yet of being reduced under one or other of three heads. Either they postulate the existence of a system of tubes homologous with the tracheæ of insects, and, like them, distinct from the animal's blood-vessels, as necessary for the explanation of the great changes of volume observed to take place in the mollusk's body; or they suppose these alterations of size to be effected by the agency of the blood-vascular system alone; or, thirdly, they hold the effect in question to be due to the joint working of these two systems of tubes.

Agassiz* refers the great distention observable in the foot of the *Natica heros*, of the *Pyrula carica* and *canaliculata*, and the Acephalous *Mactra solidissima*, to water inhaled by orifices more or less numerous, of less or greater calibre, in the muscular foot: these orifices, and the tubes in connexion with them, he speaks of as a water-vascular system, but he holds that they come into more or less direct and constant communication with the true blood-vascular system.

Theodor von Hessling†, who obtained the same result of injecting fully the blood-vascular system, by throwing in fluid from the glandular depression in the foot of the *Unio margaritifera*, as Agassiz did by a similar procedure with the similar depression in the foot of the Gasteropodous *Pyrulæ*, speaks of the system (which on these grounds he holds to be continuous) as but one system, and that a blood-vascular system, with certain orifices patent and communicating with the external medium in which the animal lives. Von Hessling holds also that the distention of the foot may be in part due to water inhaled through the organ of Bojanus, and mingled thus with the blood, as we shall presently describe.

M. Langer‡ holds that the organ of Bojanus is the route by which the water, upon which the change of volume in the animal's body depends, passes into it, and that this water passes into the blood-vessels, and not into any specialized water-vascular system.

* Zeitschrift für wiss. Zoologie. Pt. 7. p. 176, 1855.

† Perlmuscheln und ihre Perlen. Leipzic, 1859, p. 241.

‡ Denkschriften d. Kaiserlich. Akad. Wiss. xii. p. 55, 1856.

M. Lacaze Duthiers has discovered and described* yet another route than that of the organ of Bojanus, by which, in the *Dentalium* and *Pleurobranchus*, water from without can find its way into the interior of vessels carrying blood, and carrying it in these instances towards the heart, and not towards the gills.

Gegenbaur† differs from these authors merely in postulating the existence of orifices of exit as well as of entrance for the water; and these he holds to correspond with the puncta scattered over the foot-surface, and visible in great abundance occasionally along and near its free edge.

Von Rengarten‡ exactly reverses the functions thus supposed to belong to the punctated foot-pores, and the passage through the organ of Bojanus severally.

In a paper read by us§ before the Royal Society, February 3, 1859, we spoke of the water-vascular system as having its outlet in close approximation to the external orifice of the organ of Bojanus; and its inlet we suggested might be indicated by the position of the parasites which are not rarely to be seen studding the foot-surface and marking out the presence of its numerous pores. Gegenbaur, we observe‖, considers that the great liability of the foot to injury from the entrance of foreign bodies into these pores, is an argument for regarding them as exhalant rather than inhalant orifices.

Further investigations, carried on by us subsequently to the reading of that paper, showed us that our views as to the oviducal system in the Lamellibranchiata were founded in error. An exceedingly courteous notice of this mistake by M. Lacaze Duthiers¶ in the 'Proceedings of the Royal Society,' rendered an earlier retractation of this part of our paper unnecessary. Our views, on the other hand, as to the permeation of the bodies of the Lamellibranchiata by a system of vessels distinct from those in which the blood is contained, remain much what they were.

Before stating our views, and the arguments by which we would support them, we would say that the "perivisceral chamber" of the Brachiopoda, as described by Mr. Hancock** in a paper in the 'Philosophical Transactions,' which was published subsequently to the reading of our paper already referred to, holds much the same relation to the circulatory and reproductive and other viscera, as the system which we have called "aquiferous" in the Lamellibranchiata. As Mr. Hancock†† has himself pointed out the close correspondences of the two systems, we will but remark upon one point of discrepancy between them. In the Brachiopods the genitalia are packed into the main stems of the arborescent perivisceral system, in the direct course of the stream, if we may speak of it as a water-vascular system; in the Lamellibranchiata, or, at all events, in the family Unionidæ, the cæca of the generative gland are appended laterally to the

* Ann. des Sciences Naturelles, tom. xi. 1859, p. 253; tom. vii. Proceedings of the Royal Society, vol. x. p. 194.

† Grundzüge der vergleichenden Anatomie, p. 352, 1859.

‡ Diss. Inaug. Dorpat, 1853, cit. Von Hessling, *loc. cit.* p. 236.

§ Proceedings of the Royal Society, vol. ix. no. 34. p. 634. February 3, 1859.

‖ *Loc. cit.* p. 352. ¶ Proceedings of the Royal Society, vol. x. no. 37. p. 193.

** Philosophical Transactions for 1858. Read May 14, 1857. †† *Loc. cit.* p. 844.

divergent twigs of the principal branches of the water-vascular tree; they do not lie in the direct course of the current of the aquiferous canals, and these canals, beyond and outside of them, break up into a very delicate minutely divided system of capillary tubes. What we shall attempt to prove is, that the orifices on either side of the foot in the Unionidæ lead not only to the generative gland, the products of which may be seen to issue forth from them at the spawning-season, but also to a system of tubes widely spread through the entire foot. We do not believe that any direct communication subsists either between the blood-vascular system and this system of tubes, or between either of these systems and the punctated depressions and inlets along the foot-edge. The blood-vessels seem to us to constitute a system of tubes closed, save at one point and at one lacuna. That point and that lacuna is the pericardial space—a cavity into which, besides the blood of the animal, the water in which it lives also finds its way. As the bivalve shell opens, it necessarily dilates this lacuna, and water is thus drawn into it through the compound sac known in the Acephala as the organ of Bojanus. The water then gains access to the interior of the blood-vessels, as we shall proceed to show, and is carried onward within them. From the blood-vessels we suppose it to transude into the system of water-tubes everywhere in apposition with them, and, under normal conditions, to find its exit by these tubes, whilst under such abnormal circumstances as the sudden removal of the creature from the water, the sudden contraction of the muscular foot, causing jets of water to pour forth from the dilated semitransparent mass, may unload the infiltrated organ in a yet more expeditious manner. As to the way by which the water used by the mollusk for distending its foot comes into the body, we are at one with many other writers upon this subject; but we are not aware that our views, as to the method by which the animal disencumbers itself of the ingested fluid, are shared in by other authors.

Our arguments will be principally based upon the results of experiments made in the way of injection. The animals we operated upon were almost exclusively of the family Unionidæ; and, on account of the size of the specimens, as well as for other reasons, we employed chiefly the species *Anodon cygneus* and *Unio margaritifera*. In all our experiments we strove to reproduce, as nearly as possible, the conditions of the animal's natural life: our injections were always performed under water, by which and by other means as much support was given to the animal's body and its several parts as the water and the shell gave to it during life. Means were always adopted for securing that the animal died with its muscular system in a state of relaxation. We found the prussian blue injecting-fluid of Professor Beale's* invention to possess many properties especially recommending it for use in our experiments, but we employed several other fluids as well.

Experiment 1.—If an *Anodon* or *Unio* (size is of little consequence in this experiment, though large size is a convenience in most) be removed from its shell without injuring the somewhat easily injured tissues which limit the secreting-structure of the

* How to work with the Microscope, p. 78, 1857.

organ of Bojanus, and supported in water with its foot downwards in such a manner a to put its pericardial lacuna, and the parts in connexion with it, as nearly as possible into the condition in which they may be supposed to be in in the shell during life, and if an injection be then made into the pericardial lacuna, the following results will be seen to take place. The so-called "reddish-brown organ of Keber" (a plexus of vessels rich in pigmentary deposit, continuous with other vessels not so coloured in the mantle and elsewhere, and bounding the pericardium on either side, and opening into it by several patent orifices at its anterior end) will become filled with the injecting-fluid first; next the gill-vessels, and sometimes together with them, yet not invariably, the systemic veins; and lastly the external orifice of the organ of Bojanus will, on removing the animal from its prone position, be seen pouring out the injection on either side of the animal's foot.

Experiment 2.—A large *Anodon* was injected with a red stiffening-injection from the central branchial vein, a vessel readily injectible, lying as it does in the gill-cavity superiorly between the two innermost laminæ of the gills, in the angle where they become continuous with each other posteriorly to the posterior edge of the foot, with the following results:—The auricle and ventricle were filled to distention, the reddish-brown organ as well, and, besides the reddish-brown organ, the rest of the mantle, up to within a quarter of an inch of its free edge. No fluid, however, had penetrated into the pericardial space. The absence of penetration into the pericardium we have invariably had to record in our numerous injections from the branchial veins, even when the injection is noted as having been so entirely successful as to have passed through the aorta in such abundance as to inject in fine ramuscular divisions the edge of the muscular foot.

The former of these two experiments is so easy of performance, and yet proves so much, that we cannot but express our surprise at nowhere finding any record of its having been made by any of the different experimenters who have employed injections as a method for investigating the economy of mollusks. We have repeated it so frequently with the same results, as to have become quite convinced that the pericardial lacuna communicates, on the one hand, with blood going gillward, and on the other with the water in which the animal lies.

The uniformity with which our repetitions of Experiment 2 have led to the same negative result inclines us to doubt the existence of any direct communication between the aquiferous pericardial lacuna and the branchial veins properly so called. We are the more disposed to accept this conclusion, as in no mollusk whatever which is possessed of branchial vessels, except the *Pleurobranchus**, has the renal organ been shown to conduct the external water into the cavity of vessels homologous, not with the afferent, but with the efferent† branchial vessels of higher organisms.

Though Experiment 2 may seem to prove that the intravascular blood does not set in any very free current outwards into the pericardial space, especially when coupled with the observation that in multitudinous and varied injections of the different systems of blood-

* M. Lacaze Duthiers, Royal Society's Proceedings, *loc. cit.* † Gegenbaur, Grundzüge, p. 367.

vessels we have* never succeeded in filling the pericardium from the blood-vessels, easy though it be, as in Experiment 1, to make the injected fluid take the reverse direction, more direct evidence is yet needed in support of our view of the organ of Bojanus as the channel for an inwardly-setting current of water. The following considerations seem to us to show conclusively that, though Experiment 1 shows that it is possible for intrapericardial fluid to find its way outwards through the renal organ, such is not the direction usually taken by the fluid contained in the complex aquiferous system thus constituted.

1st. If we examine with the microscope the fluid contained in the pericardial space, we shall find it to contain, besides the morphological elements of blood, certain foreign bodies, such as the *Aspidogaster conchicola* and infusoria. Now these creatures must be supposed to have found their way inwards through the organ of Bojanus.

2ndly. The external orifice of the organ of Bojanus may be seen in a living *Anodon* (and, from its lying exposed in the gill-cavity, with yet greater ease in a *Unio margaritifera*) to execute movements of alternate opening and shutting, similar in character to those executed, as has been repeatedly noticed, by the analogous organ in the Pteropoda. These movements are repeated as frequently as once in every ten seconds (or oftener) in the *Unio margaritifera*; and they possess, there can be little doubt, in these as in other mollusks, the power of filling with water the cavities into which they lead.

3rdly. The glandular portion of the compound organ of Bojanus has its opening into the pericardium guarded by a funnel-shaped projection which acts as a valve looking heartwards, and offers resistance consequently to fluid passing outwards from that lacuna.

4thly (Experiment 3). Fluid thrown in by the external orifice of the organ of Bojanus, as it is either artificially, as in the *Anodon*, or naturally, as in the *Unio margaritifera*, exposed in the gill-cavity, finds it way even more easily into the pericardium than fluid thrown, as already described, into the pericardium finds its way into the gill-cavity by the reverse route. This experiment is but an imitation of what we may suppose to take place whensoever the animal by opening its valves dilates its pericardial space. As an immediate consequence of this dilatation, water is ingested into the blood-vascular system, and is forthwith applied to the purpose of distending the foot and protruding it through the opening valves.

Up to this point our views are in accordance with those adopted by several authors, though we are not aware that our method of proof has been employed by any other observers, so far as its detailed application is concerned.

We will now proceed to give our reasons for supposing that another system of tubes comes in aid of the blood-vascular system, and receives from it the fluid which that system has been the means of taking up in the manner described. Our arguments will go to show that water is transferred from the blood-vessels in the foot of the freshwater

* M. Langer's language (Denkschriften d. Akad. Wiss. *loc. cit.* p. 43), in describing his success in such injections, is so qualified, "und sah *doch nicht immer*," as to allow one, without discourtesy, to give less weight to his views on this than on most other points.

mussel to another set of vessels, the main stem of which has the additional function of outlet to the generative gland. As, however, VON HESSLING* holds that the system of pores in the foot plays no inconsiderable part in the work of supplying the distending foot with water, acting in aid of, and in alliance with, the system of the organ of BOJANUS, and, with AGASSIZ and VON RENGARTEN, as already cited, holds this office to be exclusively discharged by this system of pores and inlets, we will begin by stating our reasons for demurring to these views, in which we ourselves at one time participated. It will be necessary to give the details of two sets of experiments, to show how we came to give up an opinion which can plead such high authorities as those we have cited for its defence.

Experiment 4.—A large *Anodon*, having died with its foot in a semidistended state, was injected from the venous sinus which receives the blood from the systemic veins and distributes it to the renal-portal system, with the prussian-blue injection already spoken of. The injection spread over the liver and over the whole of the generative gland, and the exclusively muscular part of the foot, spreading itself in especial richness along the free edge. No pressure which we subjected the foot to, when thus fully injected, caused any of the blue injection, easily and readily though it runs, to issue forth. Subsequently to this, a stiffening injection of red colour was thrown into the foot-mass from the oviducal outlets. This second injection spread itself very richly over the ovary, over the liver, and into the muscular foot, *along the free edge of which it issued in small jets without any pressure being applied.*

We will disregard, for the moment, the bearing which this experiment has upon the distinctness from the blood-vascular system of the system of tubes in the muscular foot, to which the stem opening under the name of oviduct into the mantle-cavity leads, and we will relate the details of another set of experiments, which led us to consider the phenomenon of the jets issuing from the foot-edge as due, in spite of the frequency with which we have seen it recur, to violence done, possibly unavoidably, to the delicate limitary tissues of these aquiferous tubes.

Experiment 5.—A *Unio margaritifera*, which had died with its foot quite relaxed, had the blue injecting-fluid introduced into its aorta, its venous system, and through the oviducal orifices, until the foot, from a state of perfect softness, became tense and swollen up. *On pressure, none of this triply-injected blue fluid could be made to issue forth from the foot-edge; but small hernia-like projections of transparent membrane rose out like bubbles all along the foot-edge.* They contained at first a transparent fluid, but after a little pressure they became filled with the blue injection. The thinness and transparency of these little sacs will account for the rarity of their appearance, and the comparative frequency with which jets of injected fluid have made themselves noticed in the region corresponding to the cæcal endings of tubes which these sacs must be held to represent. The depressions and pores which do exist in the foot of the Lamellibranchiate mollusk we believe to be glandular in character, and destitute of any direct

* Perlmuschels und ihre Perlen, p. 238 *et seq.*

communication with the blood-vessels or other tubes in the animal's body. Now the *Unio margaritifera* stands in the same relation with reference to these foot-pores to the *Anodonta cygnea* as the *Pyrula carica* and *P. canaliculata* do, according to Agassiz, to the *Mactra*; that is to say, the foot of the *Unio* presents us with a gigantic pore, in the shape of a glandular depression of as much as an inch in length and two lines in diameter, whilst that of the *Anodon* is pierced but by microscopic inlets. Von Hessling*, by whom this organ has been very accurately described, believes that injections can be made to pass, without rupture of any limitary membrane, from its cavity into the blood-vessels; and Agassiz holds a similar view with reference to the nearly similar structure in the *Pyrula*. But in the *Unio* just spoken of as so fully injected, as well as in several others similarly treated, though the sides and walls of this glandular depression were very richly injected, none of the injection could by pressure be made to issue out into the water in which the animal was lying. We should be inclined to consider this involution or glandular depression in the foot of the *Unio* as homologous with the foot-gland of the terrestrial Gasteropods; and the communication which has been held to exist between this Lamellibranchiate organ and its vascular system, we should not believe to be more direct than that which subsists between the muciparous foot-gland of the *Limax* and its venous system†.

It is not quite beside the purpose, to remark that the foot of one of the Unionidæ, when thoroughly distended, has a smooth bright appearance, so uniformly spread over the whole surface of its semigelatinous mass as to suggest the idea of the depressions having become everted and thus contributed to increase the size of the infiltrated organism. Though this appearance may not justify such an interpretation, yet it does seem quite inconsistent with the existence of patent pores communicating with the animal's blood-vessels.

We have repeatedly observed that, if a freshwater mussel die with its muscular foot in a state of contraction, no distention of the foot takes place, either by leaving the animal to soak in water till putrefaction sets in, or by artificial injection.

We will now proceed to state our reasons for holding the existence of a water-vascular system distinct from the blood-vessels of the Lamellibranchiata. Siebold‡ states one of the objections urged against the existence of this system of vessels in the following words:—"The existence in these animals of a double system of lacunæ having this interpretation is attended with many difficulties. For then it must be admitted that one of these systems contains only water and the other blood, and it is difficult to understand how two kinds of wall-less canals can traverse the body without passing into each other." It is, however, demonstrable that in the Unionidæ, at all events, an all but perfectly closed system of blood-vessels exists. We have again and again, with various injecting fluids, found that they will pass from the aorta through a capillary system into a

* *Loc. cit.* p. 238. Von Hessling, however, does not mention the occurrence of calcareous concretions impacted in this gland's duct. This we have observed.

† Siebold, Anatomy of Invertebrata, p. 255, note 6, American edition. ‡ Ibid. p. 213.

systemic-venous system, from that into what may be called the renal-portal system of the organ of BOJANUS, and from that into the branchiæ, without any extravasation, or the formation of any lacuna anywhere. The pericardial space is, in the strictest sense of the phrase, a blood-lacuna; but, as already detailed, fluid cannot be made to pass into it from the blood-vessels, though such communication must take place to a certain extent during the life of the animal, and though the reverse direction of current is one easily demonstrable by artificial means, and is doubtless the ordinary one under normal conditions. There are two venous sinuses, however, in the Unionidæ, receiving, one after the other, the systemic-venous blood, and transmitting it into the organ of BOJANUS. The first of these* lies just within the muscular foot, along its superior and posterior edge; it subtends the second, the only one mentioned by authors, and opens into it by an orifice more or less perfectly guarded in different species of Unionidæ. This second sinus lies between the two opposed organs of BOJANUS; and from it the systemic-venous blood passes into the capillaries of the renal-portal system contained in those organs. But neither of these sinuses at all answers the character intended to be expressed by the term lacuna; they are homologous rather with the dilated great veins of certain vertebrata than with the lacunæ which do exist in certain molluscan families. There is the less occasion, however, to labour further at demonstrating the non-lacunar character of the blood-vascular system of the Unionidæ, as VON HESSLING†, in his recent book on the Pearl Mussel, confirms in this point the views previously enunciated by LANGER, adding to them a description of the histological characters of the vessels intervening between the arterial and venous systems in the *Unio*. It may be considered as beyond a doubt that a system of tubes all but entirely non-lacunar exists in these Lamellibranchiata, carrying their blood from the heart through a systemic, a renal, and a branchial system. No pressure that can in fairness be applied will cause any extravasation of fluid thus injected. Such pressure we have repeatedly applied to *Anodons* very fully distended by injection; and though it be not rare for fluid thrown in by the oviducal outlets to find its way out, as already described, by orifices along the foot, we have never found this to take place with the blood-vascular system.

In making use of the method of injections as a means for showing the independence of the several vascular trees in the Lamellibranchiate mollusks, we have sometimes injected the animal from the oviducal orifice alone, sometimes we have injected the same animal with a differently coloured fluid from its venous or from its arterial system, or from both; in a word, our injections have been either single, double, or triple.

* Into this sinus the cæca of the generative gland project somewhat freely from amongst the trabeculæ which run across what we call the roof of the muscular foot, from one side to the other; and it is here, we believe, that in injections from the oviducal outlets extravasation so often takes place into the blood-vessels.

† *Loc. cit.* p. 219. GEGENBAUR, in his 'Grundrüge,' p. 344, note, hints at some doubt still remaining in his mind as to the distinctness of these capillaries from the tissues they lie amongst. His work bears the same date (1859) as VON HESSLING'S and we suppose both to have been published subsequently to the reading of our paper, February 3, 1859.

There is no difficulty in causing an injection to enter the body of any large individual of the family Unionidæ from its oviducal orifice; it is especially easy, however, to effect this in the *Unio margaritifera*, as the orifice is not in them, as in most species of the family, covered by the inner lamina of the inner gill, but, together with the orifice of the organ of Bojanus, lies exposed and uncovered in the gill-cavity, and, besides this, is prolonged out in such a manner as to render the introduction of the syringe-pipe a very easy matter.

Experiment 6.—An injection thrown in by this orifice will spread itself over the whole of the viscera contained within the foot, not confining itself by any means to the ovary, but passing on beyond the area occupied by it or the male generative gland, into the exclusively muscular part of the foot, and distributing itself with especial richness along its free edge. That an injection thrown in by this orifice should thus spread itself would go some way towards showing that in the Lamellibranchiate, as in the Brachiopod mollusk, the ducts through which the generative products are extruded lead elsewhere as well as to the generative gland, were it possible to be sure that no transference of the injected fluid had taken place from tubes confessedly in connexion with the generative gland to another system of vessels—that, namely, which carries the blood. That such a transference does not rarely take place in one part of the blood-vascular system, we have already mentioned*; and hence arose the necessity for double injections, in which the blood-vascular system was (as has been and will again be described) injected and fully distended throughout the entirety of its own ramifications, before any fluid was thrown into the oviducal orifices, and by still mapping out a tree for itself, showed the independence of the system it led to. Single injections, however inferior to double ones, still furnish us with strong arguments for the view we are supporting. A freshwater mussel may have its whole visceral mass perfectly injected, either from the blood-vessels or from the oviducal system; but when thus injected, a practised eye has no difficulty in seeing into which of the two systems the injection has been thrown. The blood-vascular injection is seen to be contained in coarser tubes, and to form a less close network than the aquiferous, which, though confined within fine capillaries, gives, till closely inspected, an appearance almost of uniform diffusion, on account of the closeness of the network it forms.

Secondly, we will give the details of two double injections.

Experiment 7.—A double injection from the venous system and the oviducal in the same *Anodon*. A stiffening size injection of red colour was used for the oviducal or aquiferous system, and the prussian-blue injection, a more easily running fluid, for the venous system, with the following results. The red injection occupies the area corresponding to the generative gland, with coarse as well as with fine twigs, has imparted a faintish blush to the regions occupied by the liver and stomach, but has filled the interior of the exclusively muscular portion of the foot with so close and fine a network as to give it at a distance a uniform red appearance. The blue injection occupies much of the

* Note, p. 36.

foot-mass in common with and interposed between the red, its larger trunks holding the same position relatively to the larger red trunks as the larger systemic veins do to the larger generative ducts, but it has spread itself into the gills, which the red fluid has not.

Experiment 8.—A similar one to the preceding, but that the blood-vascular system was distended with the fluid used in Experiment 7 for the aquiferous, and *vice versâ*. The red fluid was thrown in by the aorta, it filled a large artery running parallel with the cap of the foot, it filled both labial tentacles, and it set, as it stiffened, in bossy masses along the edge of the foot, lastly it returned to the venous sinus and filled it and the organ of BOJANUS,—occupying thus the entire systemic and renal-portal vessels. The blue cold injection was thrown in by the orifices through which the generative products are extruded; and we shall see that it, when thus thrown in, disclosed the existence of a system of vessels distinct from those already so clearly marked out as coextensive with the systemic vessels. It spread itself chiefly over the ovary, but formed a fine plexus along the free edge of the foot beyond the artery described as running parallel with the edge of the foot, and figured as doing so by LANGER*.

This experiment must be thought to go a considerable way towards demonstrating the existence of a system of tubes distinct from, however closely apposed to, the blood-vascular system,—this system having been, in this experiment, filled with a rigid mass, and filled with it most thoroughly, as the injection of the organ of BOJANUS proves, and yet allowing the trees injectible from the aquiferous outlet to coexist side by side with it, even though the fluid they contained was so much more easily displaced than the stiffening size injection.

Thirdly, of triple injections.

The readiness with which injections pass from the arterial into the venous system make the triple injections which we have practised of less physiological value than at first sight might appear to be the case; and consequently we will content ourselves with giving the details of one such injection.

Experiment 9.—A large *Anodon* was injected from the venous sinus with a yellow stiffening injection; after this had been done, a blue-coloured fluid, also with size for its basis, was thrown into the aorta; and thirdly, a red injection of the same character was thrown in by the aquiferous opening. The blue fluid thrown into the arterial system drove the yellow fluid before it out of the systemic veins almost entirely, but it did not follow it into the renal-portal system of the organ of BOJANUS; this organ and the gills remained richly injected with yellow, to the exclusion of both the other colours; the red fluid, finally, which was thrown in by the aquiferous opening, spread itself in couples with the arterial blue over the entire visceral mass, filling alike the areas of digestive and of reproductive organs, and spreading itself with especial richness over the exclusively muscular part of the foot, which it will be recollected is the part of the animal most preeminently distended and distensible by both natural and artificial means.

Lastly, in a large individual of the freshwater-mussel family in which a stiffening

* Denkschriften der K. Akad. Wiss. Wien, viii. Bd. Taf. i. fig. 1.

or other injection has been thrown in by the orifice through which the generative products are extruded, a simple lens is sufficient to show that the tubes thus injected have the generative cæca affixed to them laterally, and pass on continuously into parts of the foot in which no generative cæca are lodged. It is most especially in that part of the muscular foot into which no viscera are packed, and which forms a belt of considerable width beyond and bounding the generative mass, and yet free from any admixture of its constituent elements (as the microscope will show), that we find the capillary network (shown to be in connexion with the oviducal outlet whilst clear of the terminal cæca) of the gland to attain its maximum development. Now this area is the area also of maximum distention in the distended foot. If in an *Unio* which has been injected from the blood-vascular system and from the oviducal, both with differently coloured injecting-fluids, a portion of the injected tissue be taken from this area and placed under one of the higher powers of the microscope, the fluid which has been thrown in by the oviducal orifice will be seen to be contained in tubes as well and sharply defined as those of the capillaries which the other injected fluid will show to be in connexion with the blood-vessels.

Whilst the analogy of the Echinodermata and many Annelids does away with any *à priori* improbability which may have seemed to attach to the possession by these mollusks of the system of tubes the existence of which we have been striving to demonstrate, the homology of the Brachiopoda furnishes us with a strong *à priori* presumption in favour of the correctness of our view. On the other hand, we cannot forbear pointing out the great improbability which must attach to a view which supposes a fluid of such morphological and such chemical characters as is the blood of the freshwater mussel to be diluted as it must be diluted on the hypothesis of the blood-vessels being the agents by which the animal voluntarily distends itself often to thrice its undistended bulk. How do the blood-corpuscles which we may take from the interior of the animal's heart behave when thus mixed with water under the microscope*? But it is not upon considerations such as these that we would lay most weight, but upon the evidence which injections of the several systems furnish to the unassisted eye, and upon the confirmation of that evidence which microscopic inspection furnishes.

* "Reagentien, wie ein *Ueberschuss von Wasser*, verdünnte Essigsäure, lösen bei der ersteren Art (Blutkörperchen) den scheinbar festen Inhalt auf, und lassen den Kern, wie die eingeschlossenen Körnchen, deutlich hervortreten. Ihre häufigen Formveränderungen, z. B. die "spiessigen hirschgeweihähnlichen Fortsätze," welche sie treiben, hängen *von unvermeidlichen Diffusionverhältnissen ab*, welchen sie bei der grossen Wassermenge gegenüber ihrer verhältnissmässig geringen Anzahl ausgesetzt sind. Während A. Ecker dieselben *durch eine Bildung von Vacuolen, in Folge deren Vergrösserung sie einreissen*, zu erklären sucht, halt Lieberkühn diese Zellenbildungen für Amöben mit selbständigen contractilen Bewegungen. In innigem Zusammenhang mit diesen Erscheinungen steht *das leichte Austreten des Zelleninhalts*, welcher bisweilen in hellen und hyalinen Tropfen herumschwimmt, ja *nicht selten geht ein Zerfallen* desselben in zahlreiche kleine Tröpfchen noch innerhalb der Zellen vor sich, welche dadurch ein maulbeerartiges Ansehen bekommen, ebenso vereinigen die ausgetretenen Sarcodetropfen diese Körperchen zu den oben erwähnten Klümpchen und Flöckchen."—Von Hessling, *loc. cit.* pp. 219, 220.

III. *On the Contact of Curves.* By WILLIAM SPOTTISWOODE, *M.A., F.R.S.*

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LET U=0 be the equation to the curve with which the curve V=0 has an "*m*-pointic contact;" in other words, let the curves U and V have *m* consecutive points in common. The degree of contact of which V is capable is equal to the number of independent constants contained in its equation; *i. e.* to the number of terms in its complete expression, less one. Thus if *n* be the degree of V, the degree of contact will be $=\frac{(n+1)(n+2)}{1.2}-1$.

If the curves U and V have only a single point in common, then the only conditions are

$$U=0,\ V=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

If they have two consecutive points in common, then beside (1.) we have also

$$\left.\begin{aligned}\frac{\partial U}{dx}dx+\frac{\partial U}{dy}dy+\frac{\partial U}{dz}dz=0,\\ \frac{\partial V}{dx}dx+\frac{\partial V}{dy}dy+\frac{\partial V}{dz}dz=0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

which, as is well known, lead to the conditions

$$\left.\begin{aligned}&\frac{\partial U}{dx}:\frac{\partial U}{dy}:\frac{\partial U}{dz}\\ =&\frac{\partial V}{dx}:\frac{\partial V}{dy}:\frac{\partial V}{dz};\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

and if V be linear, $=lx+my+nz$, (3.) will suffice to determine the ratios $l:m:n$, and fix the position of the tangent V. The conditions (3.) may be expressed by a single equation thus: if α, β, γ be any arbitrary quantities, then the equation

$$\begin{vmatrix}\alpha & \beta & \gamma\\ \frac{\partial U}{dx} & \frac{\partial U}{dy} & \frac{\partial U}{dz}\\ \frac{\partial V}{dx} & \frac{\partial V}{dy} & \frac{\partial V}{dz}\end{vmatrix}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

considered as identical in α, β, γ, may be regarded as an expression for the required conditions.

The developed form of (4.) is

$$\left(\gamma\frac{\partial U}{dy}-\beta\frac{\partial U}{dz}\right)\frac{dV}{dx}+\left(\alpha\frac{\partial U}{dz}-\gamma\frac{\partial U}{dx}\right)\frac{\partial V}{dy}+\left(\beta\frac{\partial U}{dx}-\alpha\frac{\partial U}{dy}\right)\frac{\partial V}{dz}=0; \quad . \quad . \quad . \quad (5.)$$

so that, writing

$$\left(\gamma\frac{\partial U}{dy}-\beta\frac{\partial U}{dz}\right)\frac{\partial}{dx}+\left(\alpha\frac{\partial U}{dz}-\gamma\frac{\partial U}{dx}\right)\frac{\partial V}{dy}+\left(\beta\frac{\partial U}{dx}-\alpha\frac{\partial U}{dy}\right)\frac{\partial}{dz}=\square, \quad . \quad . \quad (6.)$$

the effect of the operation $\square$ upon V is equivalent to the elimination of dx, dy, dz, from (1.) and their differentials (2.), and is expressed by the equation

$$\square V=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

If the curves have three consecutive points in common, we have, in addition to former conditions,

$$\left.\begin{aligned}&\frac{\partial U}{dx}d^2x+\frac{\partial U}{dy}d^2y+\frac{\partial U}{dz}d^2z+\frac{\partial^2 U}{dx^2}dx^2+\frac{\partial^2 U}{dy^2}dy^2+\frac{\partial^2 U}{dz^2}dz^2+2\left(\frac{\partial^2 U}{dydz}dydz+\frac{\partial^2 U}{dzdx}dzdx+\frac{\partial^2 U}{dxdy}dxdy\right)=0,\\&\frac{\partial V}{dx}d^2x+\frac{\partial V}{dy}d^2y+\frac{\partial V}{dz}d^2z+\frac{\partial^2 V}{dx^2}dx^2+\frac{\partial^2 V}{dy^2}dy^2+\frac{\partial^2 V}{dz^2}dz^2+2\left(\frac{\partial^2 V}{dydz}dydz+\frac{\partial^2 V}{dzdx}dzdx+\frac{\partial^2 V}{dxdy}dxdy\right)=0,\end{aligned}\right\} \quad (8.)$$

from which dx, dy, dz, d^2x, d^2y, d^2z are to be eliminated by the help of (1.) and (2.), or (3.). Instead, however, of differentiating (2.), we may differentiate (3.), or, what is the same thing, (4.); and we shall then have an expression free from d^2x, d^2y, d^2z.

The results of the elimination may, moreover, in this case be grouped into a single expression. Writing for convenience

$$\frac{\partial U}{dy}\frac{\partial}{dz}-\frac{\partial U}{dz}\frac{\partial}{dy}=D_1, \quad \frac{\partial U}{dz}\frac{\partial}{dx}-\frac{\partial U}{dx}\frac{\partial}{dz}=D_2, \quad \frac{\partial U}{dx}\frac{\partial}{dy}-\frac{\partial U}{dy}\frac{\partial}{dx}=D_3, \quad . \quad . \quad (9.)$$

(3.) may be expressed by any two of the equations

$$D_1V=0, \quad D_2V=0, \quad D_3V=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

Differentiating these and combining their differentials with the first of (2.), we may form the quadratic system,

$$\left.\begin{aligned}&D_1^2V=0 \quad D_1D_2V=0 \quad D_1D_3V=0\\&D_2D_1V=0 \quad D_2^2V=0 \quad D_2D_3V=0\\&D_3D_1V=0 \quad D_3D_2V=0 \quad D_3^2V=0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.)$$

all of which are comprised in the one equation

$$\square^2V=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.)$$

And generally, by a similar train of reasoning, the constants of a curve V, having an m-pointic contact with U, will be determined by the equations

$$V=0, \quad \square V=0, \quad .. \quad \square^{m-1}V=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.)$$

and if

$$\left.\begin{aligned}V&=(a, b, .. k)(x, y, z)^m\\&=a\frac{\partial^m V}{da^m}+b\frac{\partial^m V}{db^m}+.. k\frac{\partial^m V}{dk^m},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

then

$$a : b : \ldots k$$

$$= \begin{vmatrix} \frac{\partial V}{da} & \frac{\partial V}{db} & \ldots & \frac{\partial V}{dk} \\ \square\frac{\partial V}{da} & \square\frac{\partial V}{db} & \ldots & \square\frac{\partial V}{dk} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \square^{n-1}\frac{\partial V}{da} & \square^{n-1}\frac{\partial V}{db} & - & \square^{n-1}\frac{\partial V}{dk} \end{vmatrix} \quad \ldots\ldots\ldots (15.)$$

The equations (11.) are, however, not all independent. In the first place, as in (10.) any one of the three equations is a consequence of the other two, so in (12.) any one column or row is a consequence of the other two. Secondly, if any pair of binary combinations of the Ds, *e. g.* D_1D_2V and D_2D_1V, be developed, they will, with the help of (10.) or (3.), be found identical. This reduces the system (12.) to three independent conditions, of which

$$D_1^2V=0, \quad D_1D_2V=0, \quad D_2^2V=0 \quad \ldots\ldots\ldots (16.)$$

is a type; as it should do. These three equations will suffice to determine the three independent constants of V, when it is capable of a 3-pointic contact, and no more, with U.

The actual calculations may be considerably simplified by using, instead of (13.), the following system,

$$V=0, \quad DV=0, \quad D^2V=0$$

(where, as before, D stands for any one of the symbols D_1, D_2, D_3). Of this system (16.) is a consequence.

By way of example, we have for the ordinary tangent

$$V=ax+by+cz=0,$$
$$DV=aDx+bDy+cDz=0,$$

whence

$$a : b : c$$

$$\begin{Vmatrix} x & y & z \\ Dx & Dy & Dz \end{Vmatrix} \quad \ldots\ldots\ldots (17.)$$

$$=yDz-zDy : zDx-xDz : xDy-yDx$$

$$=y\frac{\partial U}{dy}+z\frac{\partial U}{dz} : -x\frac{\partial U}{dy} : -x\frac{\partial U}{dz}$$

$$=\frac{\partial U}{dx} : \frac{\partial U}{dy} : \frac{\partial U}{dz}. \quad \ldots\ldots\ldots (18.)$$

The equation of the circle of curvature may be put in the form

$$hr^3+2(ayz+bzx+cxy)=0;$$

to which are to be added,

$$hD\,r^3+2(aD\,yz+bD\,zx+cD\,xy)=0,$$
$$hD^2r^3+2(aD^2yz+bD^2zx+cD^2xy)=0,$$

whence

$$h : a : b : c$$

$$\left\| \begin{matrix} r^2 & 2yz & 2zx & 2xy \\ \mathrm{D}r^2 & 2\mathrm{D}yz & 2\mathrm{D}zx & 2\mathrm{D}xy \\ \mathrm{D}^2r^2 & 2\mathrm{D}^2yz & 2\mathrm{D}^2zx & 2\mathrm{D}^2xy; \end{matrix} \right\| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.)$$

also

$$\mathrm{D}r^2 = 2\left(z\frac{\partial \mathrm{U}}{dy} - y\frac{\partial \mathrm{U}}{dx}\right),$$

$$\mathrm{D}^2r^2 = 2\left\{\left(\frac{\partial \mathrm{U}}{dy}\right)^2 + \left(\frac{\partial \mathrm{U}}{dx}\right)^2\right\} + 2\left(z\mathrm{D}\frac{\partial \mathrm{U}}{dy} - y\mathrm{D}\frac{\partial \mathrm{U}}{dx}\right),$$

$$\mathrm{D}yz = y\frac{\partial \mathrm{U}}{dy} - z\frac{\partial \mathrm{U}}{dx},$$

$$\mathrm{D}^2yz = -2\frac{\partial \mathrm{U}}{dy}\frac{\partial \mathrm{U}}{dx} + y\mathrm{D}\frac{\partial \mathrm{U}}{dy} - z\mathrm{D}\frac{\partial \mathrm{U}}{dx},$$

$$\mathrm{D}zx = x\frac{\partial \mathrm{U}}{dy} \quad \mathrm{D}^2zx = x\mathrm{D}\frac{\partial \mathrm{U}}{dy},$$

$$\mathrm{D}xy = -x\frac{\partial \mathrm{U}}{dx} \quad \mathrm{D}^2xy = -x\mathrm{D}\frac{\partial \mathrm{U}}{dx}.$$

Moreover the minors formed from the first two rows of the matrix, and from the columns below written, have the following values:—

$$r^2 \;,\; 2yz = x\left\{-(y^2-z^2)\frac{\partial \mathrm{U}}{dx} + xy\frac{\partial \mathrm{U}}{dy} - xz\frac{\partial \mathrm{U}}{dz}\right\},$$

$$r^2 \;,\; 2zx = x\left\{-yx\frac{\partial \mathrm{U}}{dx} - (z^2-x^2)\frac{\partial \mathrm{U}}{dy} + yz\frac{\partial \mathrm{U}}{dz}\right\},$$

$$r^2 \;,\; 2xy = x\left\{zx\frac{\partial \mathrm{U}}{dx} - zy\frac{\partial \mathrm{U}}{dy} - (x^2-y^2)\frac{\partial \mathrm{U}}{dz}\right\},$$

$$2zx,\; 2xy = 2x^2\frac{\partial \mathrm{U}}{dx},$$

$$2xy,\; 2yz = 2xy^2\frac{\partial \mathrm{U}}{dy},$$

$$2yz,\; 2zx = 2xz^2\frac{\partial \mathrm{U}}{dz};$$

whence, finally,

$$\begin{aligned} h : a : b : c \\ = 4\frac{\partial \mathrm{U}}{dx}\frac{\partial \mathrm{U}}{dy}\frac{\partial \mathrm{U}}{dz} + \frac{2xyz}{(n-1)^2}\mathrm{H} \\ : 2\frac{\partial \mathrm{U}}{dx}\left\{\left(\frac{\partial \mathrm{U}}{dy}\right)^2 + \left(\frac{\partial \mathrm{U}}{dz}\right)^2\right\} + (\;x^2-y^2-z^2)\frac{x}{(n-1)^2}\mathrm{H} \\ : 2\frac{\partial \mathrm{U}}{dy}\left\{\left(\frac{\partial \mathrm{U}}{dz}\right)^2 + \left(\frac{\partial \mathrm{U}}{dx}\right)^2\right\} + (-x^2+y^2-z^2)\frac{y}{(n-1)^2}\mathrm{H} \\ : 2\frac{\partial \mathrm{U}}{dz}\left\{\left(\frac{\partial \mathrm{U}}{dx}\right)^2 + \left(\frac{\partial \mathrm{U}}{dy}\right)^2\right\} + (-x^2-y^2+z^2)\frac{z}{(n-1)^2}\mathrm{H}, \quad . \quad . \quad . \quad . \quad (20.) \end{aligned}$$

where H is the Hessian of U.

Proceeding by the same principles, we shall have merely to replace $\square$ by D in (15.), and there will result a system of expressions for the constants of the curve V having an m-pointic contact with a given curve U.

The following method of reduction may sometimes be used with advantage. Let $\xi, \eta, \zeta, \ldots \varphi, \chi, \psi$ stand for the powers and products of the variables forming the various terms of V. Then the coefficients of the curve V having an m-pointic contact with U will be proportional to the determinants of which the following is a type:

$$\begin{vmatrix} \mathrm{D}^{m-1}\xi & \mathrm{D}^{m-1}\eta & \ldots & \mathrm{D}^{m-1}\psi \\ \mathrm{D}^{m-2}\xi & \mathrm{D}^{m-2}\eta & \ldots & \mathrm{D}^{m-2}\psi \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \xi & \eta & \ldots & \psi. \end{vmatrix}$$

But this

$$= -\Sigma \mathrm{D}^{m-2}\xi \begin{vmatrix} \mathrm{D}^{m-1}\eta & \mathrm{D}^{m-1}\zeta & \ldots & \mathrm{D}^{m-1}\psi \\ \mathrm{D}^{m-2}\eta & \mathrm{D}^{m-2}\zeta & \ldots & \mathrm{D}^{m-2}\psi \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \eta & \zeta & \ldots & \psi \end{vmatrix} = -\Sigma \mathrm{D}^{m-2}\xi \mathrm{D} \begin{vmatrix} \mathrm{D}^{m-2}\eta & \mathrm{D}^{m-2}\zeta & \ldots & \mathrm{D}^{m-2}\varphi \\ \mathrm{D}^{m-2}\eta & \mathrm{D}^{m-2}\zeta & \ldots & \mathrm{D}^{m-2}\varphi \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \eta & \zeta & \ldots & \varphi \end{vmatrix}$$

$$= (-)^2 \Sigma \mathrm{D}^{m-2}\xi \; \mathrm{D}\Sigma \; \mathrm{D}^{m-3}\eta \mathrm{D} \begin{vmatrix} \mathrm{D}^{m-3}\zeta & \ldots & \mathrm{D}^{m-3}\chi & \mathrm{D}^{m-3}\psi \\ \mathrm{D}^{m-4}\zeta & \ldots & \mathrm{D}^{m-4}\chi & \mathrm{D}^{m-4}\psi \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \zeta & \ldots & \chi & \psi \end{vmatrix}$$

$$= (-)^2 \Sigma \mathrm{D}^{m-2}\xi \; \Sigma \; \mathrm{D}^{m-3}\eta \; \mathrm{D}^2 \begin{vmatrix} \mathrm{D}^{m-3}\zeta & \ldots & \mathrm{D}^{m-3}\chi & \mathrm{D}^{m-3}\psi \\ \mathrm{D}^{m-4}\zeta & \ldots & \mathrm{D}^{m-4}\chi & \mathrm{D}^{m-4}\psi \\ \cdot & & \cdot & \cdot \\ \zeta & \ldots & \chi & \psi \end{vmatrix}$$

$$= (-)^{m-2} \; \Sigma \mathrm{D}^{m-2}\xi \; \Sigma \mathrm{D}^{m-3}\eta \; \ldots \; \Sigma \mathrm{D}\varphi \; \mathrm{D}^{m-2} \begin{vmatrix} \mathrm{D}\chi & \mathrm{D}\psi \\ \chi & \psi \end{vmatrix} \quad \ldots\ldots\ldots\ldots \quad (21.)$$

In the case of the conic of 5-pointic contact this becomes

$$-\Sigma \mathrm{D}^3 x^3 \Sigma \mathrm{D}^3 y^3 \Sigma \mathrm{D} z^3 \mathrm{D}^3 \begin{vmatrix} 2\mathrm{D}zx & 2\mathrm{D}xy \\ zx & xy \end{vmatrix}$$

I now proceed to the calculation of the conic of 5-pointic contact. This may be effected by any of the three methods indicated above, viz. (1) by means of the symbol $\square$, getting out the factors $(\alpha x + \beta y + \gamma z)^p$; (2) by means of the symbols D; (3) by the reduced formula (21.). As most of the steps have been calculated by two methods, I subjoin some of the leading formulæ and transformations which occur in the process.

Writing for convenience

$$\lambda=\gamma\frac{\partial U}{dy}-\beta\frac{\partial U}{dz},\quad \mu=\alpha\frac{\partial U}{dz}-\gamma\frac{\partial U}{dx},\quad \nu=\beta\frac{\partial U}{dx}-\alpha\frac{\partial U}{dy},$$

and putting

$$V=(abcfgh\rangle(xyz)^2,$$

the general expressions (13.) become

$$0=(abcfgh\rangle(xyz)^2,$$

$$0=(abcfgh\rangle(\lambda\mu\nu\rangle(xyz),$$

$$0=(abcfgh\rangle(\lambda\mu\nu)^2+(abcfgh\rangle\square\lambda\square\mu\square\nu\rangle(xyz),$$

$$0=3(abcfgh\rangle\square\lambda\square\mu\square\nu\rangle(\lambda\mu\nu)+(abcfgh\rangle\square^2\lambda\square^2\mu\square^2\nu\rangle(xyz),$$

$$0=3(abcfgh\rangle\square\lambda\square\mu\square\nu)^2+4(abcfgh\rangle\square^3\lambda\square^3\mu\square^3\nu\rangle(\lambda\mu\nu)+(abcfgh\rangle\square^3\lambda\square^3\mu\square^3\nu\rangle(xyz),$$

whence

$$a:b:c:f:g:h=$$

$$\left\|\begin{array}{cccccc}
3(\square\lambda)^2+4\lambda\square^2\lambda+x\square^3\lambda & . & . & 6\square\mu\square\nu+4(\mu\square^3\nu+\nu\square^3\mu)+y\square^3\nu+z\square^3\mu & . & . \\
3\lambda\square\lambda+x\square^3\lambda & . & . & 3(\mu\square\nu+\nu\square\mu)+y\square^3\nu+z\square^3\mu & . & . \\
\lambda^2+x\square\lambda & . & . & 2\mu\nu+y\square\nu+z\square\mu & . & . \\
x\lambda & . & . & y\mu+z\nu & . & . \\
x^2 & y^2 & z^2 & 2yz & 2zx & 2xy.
\end{array}\right\| \quad (22.)$$

The most convenient method for developing these formulæ is to begin with forming the 15 minors derived from the last two lines; and by means of them, the 20 minors derived from the last three lines; and so on. It may be noticed that in order to pass to the corresponding expressions in D, *e. g.* D_1, it is only necessary to make $\beta=0$, $\gamma=0$, and to divide out the αs. This gives

$$\lambda=0,\quad \mu=\frac{\partial U}{dz},\quad \nu=-\frac{\partial U}{dy}.$$

The results of the calculations are as follow: the third minors, formed from the two lower lines of (22.), and from the columns the lower constituents of which alone are written down, will be

$$y^3,\ z^3=-yz\frac{\partial U}{dx},\ z^3,\ x^3=-zx\frac{\partial U}{dy},\ x^3,\ y^3=-xy\frac{\partial U}{dz},$$

$$2zx,\ 2xy=2x^2\frac{\partial U}{dx},\ 2xy,\ 2yz=2y^2\frac{\partial U}{dy},\ 2yz,\ 2zx=2z^2\frac{\partial U}{dz},$$

$$x^2,\ 2yz=x\left(y\frac{\partial U}{dy}-z\frac{\partial U}{dz}\right),\ x^2,\ 2zx=x^2\frac{\partial U}{dy}\ x^2,\ 2xy=-x^2\frac{\partial U}{dz},$$

$$y^2,\ 2yz=-y^2\frac{\partial U}{dx},\ y^2,\ 2zx=y\left(z\frac{\partial U}{dz}-x\frac{\partial U}{dx}\right)\ y^2,\ 2xy=y^2\frac{\partial U}{dz},$$

$$z^2,\ 2yz=z^2\frac{\partial U}{dx},\ z^2,\ 2zx=-z^2\frac{\partial U}{dy},\ z^2,\ 2xy=z\left(x\frac{\partial U}{dx}-y\frac{\partial U}{dy}\right).$$

Similarly, for the second minors,

$$x^2,\ 2xy,\ 2xz = x^2(n-1)^{-2}\mathrm{H},$$

$$2xy,\ y^2,\ 2yz = y^2(n-1)^{-2}\mathrm{H},$$

$$2xz,\ 2yz,\ z^2 = z^2(n-1)^{-2}\mathrm{H},$$

$$x^2,\ 2yz,\ 2zx = 2\left(\frac{\partial \mathrm{U}}{dy}\right)^2\frac{\partial \mathrm{U}}{dz} - zx^2(n-1)^{-2}\mathrm{H},$$

$$x^2,\ 2xy,\ 2yz = 2\frac{\partial \mathrm{U}}{dy}\left(\frac{\partial \mathrm{U}}{dz}\right)^2 - yx^2(n-1)^{-2}\mathrm{H},$$

$$2xy,\ y^2,\ 2zx = 2\left(\frac{\partial \mathrm{U}}{dx}\right)^2\frac{\partial \mathrm{U}}{dz} - xy^2(n-1)^{-2}\mathrm{H},$$

$$2xz,\ y^2,\ 2yz = 2\frac{\partial \mathrm{U}}{dx}\left(\frac{\partial \mathrm{U}}{dz}\right)^2 - zy^2(n-1)^{-2}\mathrm{H},$$

$$2xy,\ 2yz,\ z^2 = 2\left(\frac{\partial \mathrm{U}}{dx}\right)^2\frac{\partial \mathrm{U}}{dy} - yz^2(n-1)^{-2}\mathrm{H},$$

$$2zx,\ 2xy,\ z^2 = 2\frac{\partial \mathrm{U}}{dx}\left(\frac{\partial \mathrm{U}}{dy}\right)^2 - xz^2(n-1)^{-2}\mathrm{H},$$

$$y^2,\ z^2,\ 2yz = \left(\frac{\partial \mathrm{U}}{dx}\right)^2,$$

$$y^2,\ z^2,\ 2zx = -\left\{\left(\frac{\partial \mathrm{U}}{dx}\right)^2\frac{\partial \mathrm{U}}{dy} + yz^2(n-1)^{-2}\mathrm{H}\right\},$$

$$y^2,\ z^2,\ 2xy = -\left\{\left(\frac{\partial \mathrm{U}}{dx}\right)^2\frac{\partial \mathrm{U}}{dz} + y^2z(n-1)^{-2}\mathrm{H}\right\},$$

$$z^2,\ x^2,\ 2yz = -\left\{\left(\frac{\partial \mathrm{U}}{dy}\right)^2\frac{\partial \mathrm{U}}{dx} + z^2x(n-1)^{-2}\mathrm{H}\right\},$$

$$z^2,\ x^2,\ 2zx = \left(\frac{\partial \mathrm{U}}{dy}\right)^2,$$

$$z^2,\ x^2,\ 2xy = -\left\{\left(\frac{\partial \mathrm{U}}{dy}\right)^2\frac{\partial \mathrm{U}}{dz} + zx^2(n-1)^{-2}\mathrm{H}\right\},$$

$$x^2,\ y^2,\ 2yz = -\left\{\left(\frac{\partial \mathrm{U}}{dz}\right)^2\frac{\partial \mathrm{U}}{dx} + xy^2(n-1)^{-2}\mathrm{H}\right\},$$

$$x^2,\ y^2,\ 2zx = -\left\{\left(\frac{\partial \mathrm{U}}{dz}\right)^2\frac{\partial \mathrm{U}}{dy} + x^2y(n-1)^{-2}\mathrm{H}\right\},$$

$$x^2,\ y^2,\ 2xy = \left(\frac{\partial \mathrm{U}}{dz}\right)^2,$$

$$x^2,\ y^2,\ z^2 = \frac{\partial \mathrm{U}}{dx}\frac{\partial \mathrm{U}}{dy}\frac{\partial \mathrm{U}}{dz} - xyz(n-1)^{-2}\mathrm{H},$$

$$2yz,\ 2zx,\ 2xy = -2\frac{\partial \mathrm{U}}{dx}\frac{\partial \mathrm{U}}{dy}\frac{\partial \mathrm{U}}{dz} - 2xyz(n-1)^{-2}\mathrm{H}.$$

Similarly, for the first minors,

$$x^2,\ y^2,\ z^2,\ 2yz = \frac{\partial U}{dx}\left\{\frac{\partial U}{dx}\left(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\right)-3H\left(\frac{\partial U}{dy}\frac{\partial^2 U}{dzdz}-\frac{\partial U}{dz}\frac{\partial^2 U}{dzdy}\right)\right\},$$

$$x^2,\ y^2,\ z^2,\ 2zx = \frac{\partial U}{dy}\left\{\frac{\partial U}{dy}\left(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\right)-3H\left(\frac{\partial U}{dz}\frac{\partial^2 U}{dydx}-\frac{\partial U}{dx}\frac{\partial^2 U}{dydz}\right)\right\},$$

$$x^2,\ y^2,\ z^2,\ 2xy = \frac{\partial U}{dz}\left\{\frac{\partial U}{dz}\left(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\right)-3H\left(\frac{\partial U}{dx}\frac{\partial^2 U}{dzdy}-\frac{\partial U}{dy}\frac{\partial^2 U}{dzdz}\right)\right\},$$

$$y^2,\ z^2,\ 2zx,\ 2xy = \frac{\partial U}{dx}\left\{\frac{\partial U}{dy}\left(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\right)-3H\left(\frac{\partial U}{dz}\frac{\partial^2 U}{dydx}-\frac{\partial U}{dx}\frac{\partial^2 U}{dydz}\right)\right.$$
$$\left.-\frac{\partial U}{dz}\left(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\right)+3H\left(\frac{\partial U}{dx}\frac{\partial^2 U}{dzdy}-\frac{\partial U}{dy}\frac{\partial^2 U}{dzdx}\right)\right\}-3yz\frac{H^2}{(n-1)^2},$$

$$y^2,\ z^2,\ 2xy,\ 2yz = \frac{\partial U}{dx}\left\{\frac{\partial U}{dx}\left(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\right)-3H\left(\frac{\partial U}{dz}\frac{\partial^2 U}{dz^2}-\frac{\partial U}{dz}\frac{\partial^2 U}{dxdz}\right)\right\},$$

$$y^2,\ z^2,\ 2zx,\ 2yz = \frac{\partial U}{dx}\left\{\frac{\partial U}{dx}\left(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\right)-3H\left(\frac{\partial U}{dx}\frac{\partial^2 U}{dxdy}-\frac{\partial U}{dy}\frac{\partial^2 U}{dx^2}\right)\right\},$$

$$z^2,\ x^2,\ 2xy,\ 2zx = \frac{\partial U}{dy}\left\{\frac{\partial U}{dy}\left(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\right)-3H\left(\frac{\partial U}{dy}\frac{\partial^2 U}{dydz}-\frac{\partial U}{dz}\frac{\partial^2 U}{dy^2}\right)\right\},$$

$$z^2,\ x^2,\ 2xy,\ 2yz = \frac{\partial U}{dy}\left\{\frac{\partial U}{dz}\left(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\right)-3H\left(\frac{\partial U}{dx}\frac{\partial^2 U}{dzdy}-\frac{\partial U}{dy}\frac{\partial^2 U}{dzdx}\right)\right.$$
$$\left.-\frac{\partial U}{dx}\left(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\right)+3H\left(\frac{\partial U}{dy}\frac{\partial^2 U}{dxdz}-\frac{\partial U}{dz}\frac{\partial^2 U}{dxdy}\right)\right\}-3zx\frac{H^2}{(n-1)^2},$$

$$z^2,\ x^2,\ 2yz,\ 2zx = \frac{\partial U}{dy}\left\{\frac{\partial U}{dy}\left(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\right)-3H\left(\frac{\partial U}{dx}\frac{\partial^2 U}{dy^2}-\frac{\partial U}{dy}\frac{\partial^2 U}{dydx}\right)\right\},$$

$$x^2,\ y^2,\ 2zx,\ 2xy = \frac{\partial U}{dz}\left\{\frac{\partial U}{dz}\left(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\right)-3H\left(\frac{\partial U}{dy}\frac{\partial^2 U}{dz^2}-\frac{\partial U}{dz}\frac{\partial^2 U}{dzdy}\right)\right\},$$

$$x^2,\ y^2,\ 2yz,\ 2xy = \frac{\partial U}{dz}\left\{\frac{\partial U}{dz}\left(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\right)-3H\left(\frac{\partial U}{dz}\frac{\partial^2 U}{dzdx}-\frac{\partial U}{dx}\frac{\partial^2 U}{dz^2}\right)\right\},$$

$$x^2,\ y^2,\ 2yz,\ 2zx = \frac{\partial U}{dx}\left\{\frac{\partial U}{dy}\left(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\right)-3H\left(\frac{\partial U}{dy}\frac{\partial^2 U}{dxdz}-\frac{\partial U}{dz}\frac{\partial^2 U}{dxdy}\right)\right.$$
$$\left.-\frac{\partial U}{dy}\left(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\right)+3H\left(\frac{\partial U}{dz}\frac{\partial^2 U}{dydx}-\frac{\partial U}{dx}\frac{\partial^2 U}{dydz}\right)\right\}-3xy\frac{H^2}{(n-1)^2},$$

$$2yz,\ 2zx,\ 2xy,\ x^2 = 2\frac{\partial U}{dy}\frac{\partial U}{dz}\left(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\right)-3H\left\{\left(\frac{\partial U}{dy}\right)^2\frac{\partial^2 U}{dz^2}-\left(\frac{\partial U}{dz}\right)^2\frac{\partial^2 U}{dy^2}\right\},$$

$$2yz,\ 2zx,\ 2xy,\ y^2 = 2\frac{\partial U}{dz}\frac{\partial U}{dx}\left(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\right)-3H\left\{\left(\frac{\partial U}{dz}\right)^2\frac{\partial^2 U}{dx^2}-\left(\frac{\partial U}{dx}\right)^2\frac{\partial^2 U}{dz^2}\right\},$$

$$2yz,\ 2zx,\ 2xy,\ z^2 = 2\frac{\partial U}{dx}\frac{\partial U}{dy}\left(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\right)-3H\left\{\left(\frac{\partial U}{dx}\right)^2\frac{\partial^2 U}{dy^2}-\left(\frac{\partial U}{dy}\right)^2\frac{\partial^2 U}{dx^2}\right\}.$$

Proceeding to the calculation of the final expressions for the coefficients a, b, c, f, g, h of the conic of five-pointed contact, which is given rather more in detail, we have, writing $\frac{dU}{dx}=u$, $\frac{dU}{dy}=v$, $\frac{dU}{dz}=w$,

$$\begin{aligned}
f=x^3, y^3, z^3, 2zx, 2xy &= \mathrm{D}^4x^3(y^3, z^3, 2zx, 2xy)\\
&\quad+\mathrm{D}^4y^3(z^3, 2zx, 2xy, x^3)\\
&\quad+\mathrm{D}^4z^3(2zx, 2xy, x^3, y^3)\\
&\quad+2\mathrm{D}^4zx(2xy, x^3, y^3, z^3)\\
&\quad+2\mathrm{D}^4xy(x^3, y^3, z^3, 2zx)\\
&=-(3(\mathrm{D}w)^3+4w\mathrm{D}^2w-y\mathrm{D}^3w)v\ (v\ \mathrm{D}\ \mathrm{H}-3\mathrm{HD}\ v\ )\\
&\quad+(3(\mathrm{D}v\ )^3+4v\ \mathrm{D}^2v\ +x\mathrm{D}^3v\ )w(w\mathrm{D}\ \mathrm{H}-3\mathrm{HD}\ w)\\
&\quad-(\qquad\qquad x\mathrm{D}^3v\ )w(w\mathrm{D}_2\mathrm{H}-3\mathrm{HD}_2w)\\
&\quad-(\qquad\qquad x\mathrm{D}^3w)v\ (v\ \mathrm{D}_1\mathrm{H}-3\mathrm{HD}_1w)\\
&=3(w^3(\mathrm{D}v)^3-v^3(\mathrm{D}w)^3)\mathrm{DH}+4vw(w\mathrm{D}^3v-v\mathrm{D}^3w)\mathrm{DH}\\
&\quad+(w^3x\mathrm{D}^3v+v^3y\mathrm{D}^3w)\mathrm{DH}-x(w^3\mathrm{D}^3v\mathrm{D}_2\mathrm{H}+v^3\mathrm{D}^3w\mathrm{D}_1\mathrm{H})\\
&\quad+3\mathrm{H}\{3\mathrm{D}v\mathrm{D}w(v\mathrm{D}w-w\mathrm{D}v)+4vw(\mathrm{D}v\mathrm{D}^3w-\mathrm{D}w\mathrm{D}^3v)\\
&\qquad-yv\mathrm{D}v\mathrm{D}^3w-zw\mathrm{D}w\mathrm{D}^3v+xv\mathrm{D}_1v\mathrm{D}^3w+xw\mathrm{D}_1w\mathrm{D}^3v\}\\
&=3(w\mathrm{D}v-v\mathrm{D}w)\mathrm{D}(vw)\mathrm{DH}+4vw\mathrm{D}(w\mathrm{D}v-v\mathrm{D}w)\mathrm{DH}\\
&\quad-9(w\mathrm{D}v-v\mathrm{D}w)\mathrm{D}v\mathrm{D}w\mathrm{H}+12vw(\mathrm{D}v\mathrm{D}^2w-\mathrm{D}w\mathrm{D}^2v)\mathrm{H}\\
&\quad+\mathrm{D}^2v\{w^3(z\mathrm{DH}-x\mathrm{D}_2\mathrm{H})-3\mathrm{H}w(z\mathrm{D}w-x\mathrm{D}_2w)\}\\
&\quad+\mathrm{D}^2w\{v^3(y\mathrm{DH}-x\mathrm{D}_1\mathrm{H})-3\mathrm{H}v(y\mathrm{D}v-x\mathrm{D}_1v)\}\\
&=(n-1)^{-2}x^2\{3\mathrm{D}(vw)\mathrm{HDH}+4vw(\mathrm{DH})^2-9\mathrm{D}v\mathrm{D}w\mathrm{H}^2\}\\
&\quad+12vw\mathrm{H}(\mathrm{D}v\mathrm{D}^2w-\mathrm{D}w\mathrm{D}^2v)\\
&\quad+\ 3vw\mathrm{HD}^2(v\mathrm{D}w-w\mathrm{D}v)\\
&\quad-\ 3vw\mathrm{H}(\mathrm{D}v\mathrm{D}^2w-\mathrm{D}w\mathrm{D}^2v),
\end{aligned}$$

which, omitting the common factor $(n-1)^{-2}x^2$,

$$\begin{aligned}
&=3\mathrm{D}(vw)\mathrm{HDH}+4vw(\mathrm{DH})^2+9(n-1)^{-2}x^2\mathrm{H}^2\frac{\partial^2\mathrm{U}}{dydz}+9vw\mathfrak{A}\mathrm{H}^2\\
&\quad+9\frac{3(n-2)}{n-1}vw\mathfrak{A}\mathrm{H}^2-9\frac{x}{n-1}vw\mathrm{H}\left(\mathfrak{A}\frac{\partial\mathrm{H}}{dx}+\mathfrak{B}\frac{\partial\mathrm{H}}{dy}+\mathfrak{C}\frac{\partial\mathrm{H}}{dz}\right)-3vw\mathrm{HD}^2\mathrm{H}.
\end{aligned}$$

But

$$3D(vw)HDH+6\frac{3(n-2)}{n-1}vw\mathfrak{H}H^3-6\frac{x}{n-1}vwH\left(\mathfrak{A}\frac{\partial H}{dx}+\mathfrak{B}\frac{\partial H}{dy}+\mathfrak{C}\frac{\partial H}{dz}\right)$$

$$=3(vDw+wDv)\left(v\frac{\partial H}{dx}-w\frac{\partial H}{dy}\right)H+6vw\left(Dw\frac{\partial H}{dy}-Dv\frac{\partial H}{dx}\right)H$$

$$=(3v^2Dw+3vwDv-6vwDv)H\frac{\partial H}{dx}-(3vwDw+3w^2Dv-6vwDw)H\frac{\partial H}{dy}$$

$$=3Hv(vDw-wDv)\frac{\partial H}{dx}-3Hw(wDv-vDw)\frac{\partial H}{dy}.$$

$$=-3(n-1)^{-2}x^2H^2\left(w\frac{\partial H}{dy}+v\frac{\partial H}{dx}\right).$$

Hence, excepting the common factor $(n-1)^{-2}x^2$, the total expression for f

$$=9(n-1)^{-2}x^2H^2\frac{\partial^2U}{dydx}-3(n-1)^{-2}x^2H^2\left(\frac{\partial U}{dy}\frac{\partial H}{dx}+\frac{\partial U}{dx}\frac{\partial H}{dy}\right)$$

$$-vw\left\{3H\left(D^2H+\frac{x}{n-1}\left(\mathfrak{A}\frac{\partial H}{dx}+\mathfrak{B}\frac{\partial H}{dy}+\mathfrak{C}\frac{\partial H}{dz}\right)-\frac{3(n-2)}{n-1}H\mathfrak{H}\right)-4(DH)^2\right\}.$$

And comparing this with Mr. CAYLEY's expressions in arts. 14 and 15 of his memoir "On the Conic of Five-pointic Contact," in the Philosophical Transactions for 1859, pp. 376, 377; and bearing in mind that in his formulæ we must make $\lambda=1$, $\mu=0$, $\nu=0$, in order to institute a comparison; and lastly, dividing throughout by $(n-1)^{-2}x^2\Theta H^2$, and multiplying by 2, the expression above written becomes

$$2\frac{\partial^2U}{dydx}-\frac{2}{3}\frac{1}{H}\left(\frac{\partial H}{dy}\frac{\partial U}{dx}+\frac{\partial H}{dx}\frac{\partial U}{dy}\right)-2\Lambda\frac{\partial U}{dy}\frac{\partial U}{dx},$$

which is in fact the coefficient of YZ in his general formula, viz.

$$\left(X\frac{\partial}{dx}+Y\frac{\partial}{dy}+Z\frac{\partial}{dz}\right)^2U-\left(\frac{2}{3}\frac{1}{H}\left(X\frac{\partial H}{dx}+Y\frac{\partial H}{dy}+Z\frac{\partial H}{dz}\right)+\Lambda\left(X\frac{\partial U}{dx}+Y\frac{\partial U}{dy}+Z\frac{\partial U}{dz}\right)\right)$$

$$\times\left(X\frac{\partial U}{dx}+Y\frac{\partial U}{dy}+Z\frac{\partial U}{dz}\right)=0.$$

The identity of the expressions for f, as deduced by the present method, with that deduced by Mr. CAYLEY having been thus demonstrated, it is unnecessary to pursue the calculations further.

The following are the principal subsidiary formulæ used in the foregoing calculations.

$$\delta=\alpha x+\beta y+\gamma z$$

$$y\nu-z\mu=\delta\frac{\partial U}{dx},\quad z\lambda-x\nu=\delta\frac{\partial U}{dy},\quad x\mu-y\lambda=\delta\frac{\partial U}{dz}$$

$$y\square\nu-z\square\mu=\delta\square\frac{\partial U}{dx},\quad z\square\lambda-x\square\nu=\delta\square\frac{\partial U}{dy},\quad x\square\mu-y\square\lambda=\delta\square\frac{\partial U}{dz}$$

$$x\square\frac{\partial U}{dx}+y\square\frac{\partial U}{dy}+z\square\frac{\partial U}{dz}=0.$$

Also writing

$$\square\lambda\frac{\partial U}{dx}+\square\mu\frac{\partial U}{dy}+\square\nu\frac{\partial U}{dz}=\Pi,$$

we have

$$\mu\square\nu-\nu\square\mu=\alpha\Pi,\ \nu\square\lambda-\lambda\square\nu=\beta\Pi,\ \lambda\square\mu-\mu\square\lambda=\gamma\Pi;$$

also

$$-\Pi=\begin{vmatrix} \frac{\partial^2 U}{dx} & \frac{\partial^2 U}{dxdy} & \frac{\partial^2 U}{dxdz} & \frac{\partial U}{dx} & \alpha \\ \frac{\partial^2 U}{dydx} & \frac{\partial^2 U}{dy^2} & \frac{\partial^2 U}{dydz} & \frac{\partial U}{dy} & \beta \\ \frac{\partial^2 U}{dzdx} & \frac{\partial^2 U}{dzdy} & \frac{\partial^2 U}{dz^2} & \frac{\partial U}{dz} & \gamma \\ \frac{\partial U}{dx} & \frac{\partial U}{dy} & \frac{\partial U}{dz} & 0 & 0 \\ \alpha & \beta & \gamma & 0 & 0 \end{vmatrix}=-\frac{x^2}{(n-1)^2}H,$$

where H is the Hessian of U. The corresponding reduced form of this expression, *i. e.* when $\alpha=1$, $\beta=0$, $\gamma=0$, is

$$\left(\frac{\partial U}{dz}\right)^2\frac{\partial^2 U}{dy^2}-2\frac{\partial U}{dz}\frac{\partial U}{dy}\frac{\partial^2 U}{dydx}+\left(\frac{\partial U}{dy}\right)^2\frac{\partial^2 U}{dx^2}=-\frac{x^2}{(n-1)^2}H.$$

And by means of these we may deduce the following:—

$$wDv-vDw=(n-1)^{-2}x^2H,\ wD_1v-vD_1u=(n-1)^{-2}yxH,\ wD_2v-vD_2w=(n-1)^{-2}zxH,$$
$$uDw-wDu=(n-1)^{-2}xyH,\ uD_1w-wD_1u=(n-1)^{-2}y^2H,\ uD_2w-wD_2u=(n-1)^{-2}zyH,$$
$$vDu-uDv=(n-1)^{-2}xzH,\ vD_1u-uD_1v=(n-1)^{-2}yzH,\ vD_2u-uD_2v=(n-1)^{-2}z^2H.$$

Again,

$$yD_2H-zD_1H=3(n-2)\frac{\partial U}{dx}H,\ zDH-xD_2H=3(n-2)\frac{\partial U}{dy}H,\ xD_1H-yDH=3(n-2)\frac{\partial U}{dz}H$$

$$yD_2u-zD_1u=(n-1)\left(\frac{\partial U}{dx}\right)^2,\ zDw-xD_2w=xD_1v-yDv=(n-1)\frac{\partial U}{dy}\frac{\partial U}{dx}.$$

To which may be added,

$$\square\lambda Du+\square\mu Dv+\square\nu Dw=\delta H\lambda,$$
$$\square\lambda D_1u+\square\mu D_1v+\square\nu D_1w=\delta H\mu,$$
$$\square\lambda D_2u+\square\mu D_2v+\square\nu D_2w=\delta H\nu,$$
$$uDv+vD_1v+wD_2v=0,$$

$$D^2u=-(n-1)^{-2}\frac{\partial}{dx}(x^2H)-3Au+3xH,$$

$$D^2v=-(n-1)^{-2}\frac{\partial}{dy}(x^2H)-3Av,$$

$$D^2w=-(n-1)^{-2}\frac{\partial}{dz}(x^2H)-3Aw,$$

$$
\begin{aligned}
u\mathrm{D}^2v &+v\mathrm{D}\mathrm{D}_1v &+w\mathrm{D}\mathrm{D}_2v &= -(n-1)^{-1}vx\mathrm{H},\\
u\mathrm{D}^2w &+v\mathrm{D}\mathrm{D}_1w &+w\mathrm{D}\mathrm{D}_2w &= -(n-1)^{-1}wx\mathrm{H},\\
x\mathrm{D}\mathrm{D}_1u &+y\mathrm{D}\mathrm{D}_1v &+z\mathrm{D}\mathrm{D}_1w &= (n-1)^{-2}xy\mathrm{H},\\
y\mathrm{D}v &-x\mathrm{D}_1v &= x\mathrm{D}_2w-z\mathrm{D}w &= -(n-1)\,vw,\\
v\mathrm{D}\mathrm{D}_2w &-w\mathrm{D}\mathrm{D}_2u &= \mathrm{D}\{(n-1)^{-2}yz\mathrm{H}\} &-(n-1)^{-1}vy\mathrm{H},\\
u\mathrm{D}\mathrm{D}_1u &-u\mathrm{D}\mathrm{D}_1v &= \mathrm{D}\{(n-1)^{-2}yz\mathrm{H}\} &-(n-1)^{-1}wz\mathrm{H},\\
z\mathrm{D}_1u &-y\mathrm{D}_2u &= - &(n-1)^{-2}u^2,
\end{aligned}
$$

$$\mathrm{D}v\mathrm{D}w=-(n-1)^{-2}x^2\mathrm{H}\frac{\partial\mathrm{U}}{dydz}-vw\mathfrak{A},$$

$$\mathrm{D}v\mathrm{D}^2w-\mathrm{D}w\mathrm{D}^2v=(n-1)^{-2}x^2\left\{\mathrm{D}w\frac{\partial\mathrm{H}}{dy}-\mathrm{D}v\frac{\partial\mathrm{H}}{dz}-\mathfrak{A}\mathrm{H}\right\},$$

$$\mathrm{D}w\frac{\partial\mathrm{H}}{dy}-\mathrm{D}v\frac{\partial\mathrm{H}}{dz}=(n-1)^{-1}\left\{3(n-2)\mathfrak{A}\mathrm{H}-x\left(\mathfrak{A}\frac{\partial\mathrm{H}}{dx}+\mathfrak{B}\frac{\partial\mathrm{H}}{dy}+\mathfrak{G}\frac{\partial\mathrm{H}}{dz}\right)\right\},$$

whence

$$\mathrm{D}v\mathrm{D}^2w-\mathrm{D}w\mathrm{D}^2v=(n-1)^{-2}x^2\left\{\left(\frac{3(n-2)}{n-1}-1\right)\mathfrak{A}\mathrm{H}-\frac{x}{n-1}\left(\mathfrak{A}\frac{\partial\mathrm{H}}{dx}+\mathfrak{B}\frac{\partial\mathrm{H}}{dy}+\mathfrak{G}\frac{\partial\mathrm{H}}{dz}\right)\right\},$$

to which many others might be added.

IV. *On Larixinic Acid, a crystallizable volatile principle found in the Bark of the Larch Tree* (Pinus Larix, *Linn.*). *By Dr.* JOHN STENHOUSE, *F.R.S.*

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THE most convenient way of preparing this somewhat singular substance consists in cutting the bark of the larch into small pieces, and then digesting it in water for twenty-four hours at a temperature of about 80° Cent. The solution, which has a deep reddish-brown colour, is then poured off on to a second portion of larch bark and digested as before. The concentrated infusion is then cautiously heated in an open porcelain dish, at the temperature of about 80° Cent., till it is converted into a syrup. A portion of this syrup is then distilled, either in glass or porcelain retorts, or, what is better than either, in a silver alembic. Iron retorts cannot be employed for this purpose, as the acetic acid which is always produced during distillation, by forming acetate of iron, instantly destroys the larixinic acid, by changing it into a deep-purple-coloured liquid. When a silver alembic cannot be procured, a very convenient way of distilling the extract of the larch is to pour it into a large Florence flask, the neck of which is passed obliquely through a cork or bung, which is inserted into a glass condenser. When the flask is cautiously heated on a sand-bath, the larixinic acid comes over with the first portions of the liquid, but becomes more abundant as the distillation proceeds, and usually forms large flat crystals which condense on the sides and neck of the receiver. The liquid which is distilled over, and which contains the greater portion of the larixinic acid, should be poured into small flat basins, and cautiously concentrated at about 60° Cent. When the greater portion of the water has been dissipated, it is advisable, especially in warm weather, to complete the operation by spontaneous evaporation; for unless the concentration of the aqueous solution of larixinic acid is conducted cautiously, the larixinic acid volatilizes along with the vapour of water, and is thereby lost. The highly concentrated solution of larixinic acid obtained in the way just described, on standing, deposits brownish-yellow crystals, which are impure larixinic acid. This is to be pressed between folds of blotting-paper, and again to be crystallized out of a small quantity of water. The larixinic acid may be rendered perfectly pure by subliming it once or twice. This is easily effected by placing the larixinic acid between two watch-glasses, or in any other suitable apparatus, and heating it cautiously on a sand-bath, or even on a water-bath, as the larixinic acid sublimes at the very low temperature of 95° Cent. The larixinic acid is a proximate principle, which exists ready formed in the larch. This is easily proved by distilling even a dilute infusion of the bark, when the liquid which passes over will be found to strike a deep purple colour with a persalt of

iron, which is very persistent. The bark of old larch trees contains very little larixinic acid; but the bark of the small branches, and that of the stems of the larch when not more than from twenty to thirty years of age, contains very considerable quantities of this substance, the concentrated syrup from the portions of bark yielding more larixinic acid than an equal weight of catechu does of oxyphenic acid. Larixinic acid, after it has been purified by sublimation, forms beautifully white crystals, often more than an inch in length, of a brilliant silvery lustre, very much resembling benzoic acid in appearance. They sublime at 93° Cent., and melt at 153° Cent.; but its aqueous solutions volatilize at ordinary temperatures. I am indebted to the kindness of Professor W. H. Miller, of Cambridge, for the subjoined measurements of the crystals of larixinic acid.

"The crystals obtained by sublimation become rough so rapidly when exposed to the air, that very little confidence can be placed in the following results:—

Fig. 1.

"The crystals belong to the oblique system; they usually occur in twins, like the annexed figure. They are extremely thin in a direction perpendicular to b. Denoting by l, m, n faces in the zone $a\,b$, and by r a face in the zone $b\,c$, it appears that tan $b\,l$, tan $b\,m$, tan $b\,n$ are nearly as the numbers 2, 3, 6, and that $a\,b=90°\ 0'$, $b\,c=90°\ 0'$, $a\,c=76°\ 0'$, $b\,l=49°\ 29'$, $b\,m=60°\ 20'$, $b\,n=74°\ 6'$, $b\,r=75°\ 30'$. Cleavage a distinct, c imperfect.

"The crystals of larixinic acid crystallized out of water are very imperfect; the angles must be regarded as very rough approximations. They are deduced, as well as I could deduce them, from a mean of a considerable number of observations by no means agreeing well with each other. The angle between the normals to two faces is taken as the measure of the angle between the faces.

"Oblique:—

100, 101 = 26° 22′
010, 111 = 74° 36′; 001, 101 = 44° 20′.

Fig. 2.

"Forms observed:—

010, 001, 110, 011, 023

Angles.	°	′
110, 001 =	71	58
010, 110 =	69	35
010, 011 =	59	37
010, 023 =	68	39
010, 001 =	90	0

"Cleavage 0 0 1 distinct, and very easily obtained."

The smell of the aqueous solution of larixinic acid is sweetish, like that of a syrup, but the smell of the sublimed acid is very peculiar and slightly empyreumatic. As larixinic acid emits a sensible odour at ordinary temperatures, in this respect it con-

siderably resembles naphthaline and ordinary camphor. The taste of larixinic acid is slightly bitter and astringent. It reddens litmus paper very slightly, but a single drop of potash or ammonia, when added to a solution of a large quantity of larixinic acid, renders it alkaline. Larixinic acid is very soluble in boiling water, but is by no means very soluble in cold water, 87·88 parts of water at 15° C. dissolving 1 part of the acid only; but the solubility of larixinic acid in cold aqueous solutions is greatly increased by the addition of either acids or alkalies. Larixinic acid is deposited from its aqueous solutions in crystals which are very brittle, and often an inch or two in length. It likewise dissolves in cold alcohol, but to a much greater extent in hot alcohol. The crystals deposited from its alcoholic solutions are thicker and more distinctly formed than those from water. It also dissolves but sparingly in ether, and is deposited in crystals of very considerable lustre*. The following are the results of the analysis of the larixinic acid:—

I. 0·221 sublimed acid, dried *in vacuo*, gave 0·4633 carbonic acid and 0·1003 water.

II. 0·1993 sublimed acid, dried *in vacuo*, gave 0·417 carbonic acid and 0·0913 water.

III. 0·2272 larixinic acid, crystallized out of water, gave 0·4756 carbonic acid and 0·1030 water.

	Calculated numbers.	Found.		
		I.	II.	III.
C_{20}	= 57·14	57·13	57·06	57·09
H_{10}	= 4·77	5·04	5·09	5·04
O_{10}	= 38·09	37·83	37·85	37·87

From these results it is evident that the carbon, hydrogen, and oxygen in larixinic acid are in the proportion of $C_2 H_1 O_1$, or some multiple of these numbers, $C_{20} H_{10} O_{10}$ being the numbers we have adopted as the more probable.

When a quantity of larixinic acid was dissolved in a great excess of liquid ammonia, a yellow-coloured solution was produced; when this was evaporated to dryness over sulphuric acid *in vacuo*, the larixinic acid was deposited in crystals which were nearly unaltered. It gave its characteristic reactions with salts of iron, and when boiled with milk of lime gave off no trace of ammonia. The combination which larixinic acid forms with ammonia is therefore so feeble that it is decomposed by the volatility of the ammonia. In this respect, therefore, and in its forming no hydrate, larixinic acid closely resembles both pyrogallic and oxyphenic acids.

When larixinic acid was treated with an excess of aqua potassæ it very readily dissolved, forming a yellowish solution. When dried over sulphuric acid *in vacuo*, the potash combination formed long flattish crystals having considerable lustre, but of a reddish-brown colour. These crystals, when pressed between folds of blotting-paper to free them from excess of potash and recrystallized *in vacuo*, yielded crystals which were more deeply coloured than the first. This potash combination is so very feeble that

* The crystals of larixinic acid catch fire readily, and burn with a bright flame, leaving no residue.

it is decomposed by carbonic acid. It contained a considerable quantity of potash; but I have not been able to obtain it of a constant composition.

A solution of larixinic acid gives no precipitate, either with lime-water or with saccharate of lime. The behaviour of larixinic acid with baryta is extremely singular and characteristic. When a solution of caustic baryta is added to a concentrated aqueous solution of larixinic acid, the latter being in excess, a bulky, semitransparent, gelatinous precipitate immediately falls, and if the solutions are concentrated fills the whole vessel. This precipitate, which considerably resembles hydrated alumina, is but slightly soluble in cold water; but it dissolves very readily in boiling water, from which it is again deposited on the cooling of the liquid. This baryta compound is readily decomposed by carbonic acid. When thrown on a filter and washed, the air being carefully excluded, it was dried *in vacuo* over sulphuric acid, and was then found to contain, as the mean of two experiments, 34·92 per cent. of baryta.

A solution of larixinic acid yields no precipitate with either basic or neutral acetate of lead, neither is it precipitated by nitrate or ammonio-nitrate of silver; but when its solution in the latter salt is boiled, the silver is reduced in a pulverulent state. Larixinic acid forms no precipitate with perchloride of platinum, even on the application of heat. It does not contain any nitrogen. It does not reduce oxide of copper when tried by Trommer's test. It dissolves in concentrated sulphuric acid, but no conjugate combination is produced, as was ascertained by neutralizing with carbonate of baryta, larixinic acid being obtained unchanged. When larixinic acid is boiled with a mixture of hydrochloric acid and chlorate of potassa, it is decomposed, but without the formation of chloranile. When it is boiled with a solution of hypochlorite of lime, no coloration is produced. It is readily attacked by nitric acid, especially when assisted by heat; nitrous fumes are given off, and oxalic acid is the only fixed product. It is also readily attacked by bromine, especially when assisted by heat. Abundant vapours of hydrobromic acid are given off, the larixinic acid being entirely destroyed and converted into an uncrystallizable resin. The salts of copper produce an emerald-green colour in solutions of larixinic acid, but no precipitate. Chloride of manganese produces neither coloration nor precipitation. Protosulphate of iron strikes a brownish-red colour with solutions of larixinic acid, which acquire a brighter red colour on standing, resembling meconate of iron. Perchloride and persulphate of iron produce a beautiful purple dahlia-colour, which is very persistent, and stands dilution well. Its reactions with salts of iron are very characteristic of larixinic acid, which forms an excellent reagent for the detection of salts of iron, even in very minute quantity. In this way the presence of iron, in tolerably pure sulphate of copper, can readily be detected by the purple coloration produced. Larixinic acid does not affect neutral protonitrate of mercury in the cold; and on the application of heat no mercury is reduced.

Larixinic acid appears to be peculiar to the larch tree; at least I have not been able to find a trace of it in the bark of the spruce fir (*Abies excelsa*), or in that of the Scotch fir (*Pinus sylvestris*). Larixinic acid evidently belongs to that small group of substances

of which pyrogallic acid and pyrocatechine, the oxyphenic acid of GERHARDT, are the only other members yet known. Larixinic acid is much less easily oxidizable than oxyphenic acid, which again is less easily oxidated than pyrogallic acid. Larixinic acid volatilizes at a much lower temperature than either of these two substances, from which it also differs in being a ready-formed proximate principle, and not an educt.

Addendum.—In consequence of the extremely feeble, if not somewhat doubtful, acid properties of the so-called larixinic acid, perhaps the name *Larixine* would be more appropriate; but in that case the name of pyrogallic acid should be altered to *pyrogalline*, and that of oxyphenic acid to *pyrocatechine*, the name originally given to it by ZWENGER.—J. S.

V. *On the Absorption and Radiation of Heat by Gaseous Matter.*—Second Memoir. *By* JOHN TYNDALL, *F.R.S., Member of the Academies and Societies of Holland, Geneva, Göttingen, Zürich, Halle, Marburg, Breslau, Upsala, la Société Philomathique of Paris, &c.; Professor of Natural Philosophy in the Royal Institution.*

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§ 1. *Instruments.*

THE apparatus made use of in this inquiry is the same in principle as that employed in my last investigation*. It grew up in the following way:—A tube was first procured to receive the gases through which radiant heat was to be transmitted, but it was necessary to close the ends of this tube by a substance pervious to all kinds of heat, obscure as well as luminous. Rock-salt fulfils this condition, and accordingly plates of the substance an inch in thickness, so as to be able to endure considerable pressure, were resorted to. In the earliest experiments a cube of boiling water was placed before one end of this tube, and a thermo-electric pile connected with a galvanometer at the other; it was found that if the needle pointed to any particular degree when the tube was exhausted, it pointed to the same degree when the tube was filled with air. Thus tested, the presence of dry air, oxygen, nitrogen, or hydrogen had no sensible influence on the radiant heat passing through the tube.

In some of these trials the needle stood at 80°, in some at 20°, and in others at intermediate positions. I reasoned thus:—The quantity of heat which produces the deflection of 20° is exceedingly small, and hence a minute fraction of this quantity, even if absorbed, might well escape detection. On the other hand, the quantity of heat which produces the deflection of 80° is comparatively large, but then it would require a large absorption to move the needle even half a degree in this position. A deflection of 20° is represented by the number 20, but a deflection of 80° is represented by the number 710. While pointing to 80, therefore, an absorption capable of producing a deflection of 15 or 20 degrees on the lower part of the scale, would hardly produce a sensible motion of the needle. The problem then was, to work with a copious radiation, and at the same time to preserve the needle in a position where it would be sensitive to the slightest fluctuations in the absolute amount of heat falling upon the pile.

This problem was finally solved by converting the pile into a differential thermometer. Its second face was exposed, and a second source of heat was placed in front of that face. A moveable screen was interposed between the two, by the motion of which the same amount of heat could be caused to fall upon the posterior surface of the pile as

* Philosophical Transactions, 1861.

that received from the experimental tube by its anterior surface. When this was effected, no matter how high the previous deflection might be, it was completely neutralized, and the needle descended to zero.

Supposing this equality to have been established when the tube was exhausted, it is manifest that any gas, capable of absorbing even an extremely small proportion of radiant heat, would, if introduced into the tube, destroy the equilibrium of both sources. The second source of heat would now predominate, and a deflection of the galvanometer needle would be the consequence. The magnitude of this deflection would depend on the quantity of heat cut off by the gas, and properly reduced it became a strict measure of the absorption.

But in these experiments my first source of heat stood at some distance from the anterior end of the tube, and the heat, previously to entering the latter, had to cross a space of air which was not the subject of examination. This air-space I wished to abolish, so as to allow the calorific rays to enter the gas with all the qualities which they possessed at the moment of emission. I first thought of soldering the end of the experimental tube direct to the radiating surface, thus allowing the air to come into direct contact with the source. But it immediately occurred to me that the introduction of cool air into the tube would lower the temperature of the source, and that I could never know how far the indication of my galvanometer under such circumstances could be regarded as a true effect of absorption; hence I abandoned the idea of bringing the gases into contact with the radiating surface.

Instead of this arrangement an independent tube, 8 inches long, and of the same diameter as the experimental tube, was soldered on to the radiating plate. By means of a screw joint, the free end of this tube was connected air-tight with the experimental tube. Thus a chamber, from which the air could be removed, was introduced between the first plate of salt and the radiating surface. Two objects were thus secured; firstly, my source of heat was withdrawn from the action of irregular currents of air; and, secondly, the radiant heat entered the tube unchanged in quality save the infinitesimal change due to its passage through the diathermic salt.

To save the trouble and expense of a new Plate, I will ask permission to reprint in this memoir the Plate made use of in my last; a verbal reference will in most cases be sufficient to indicate the changes recently introduced. S S′ (Plate I.) it will be remembered represented the experimental tube, which was then made of brass polished within. Such a tube could not be used for any gases or vapours capable of attacking brass; and though I combated this difficulty, to some extent, by blackening the tube within, I could never feel at ease regarding the action of the gases upon the blackening substance. Many gases, moreover, present great difficulties on account of their affinity for atmospheric moisture. Hydrobromic and hydrochloric acid, for example, form dense fumes in the air, and however carefully they might have been dried, I should have been reluctant to base any inference on their deportment without actually having them under my eyes during experiment.

The brass tube, then, which stretched from S to S′ in the figure is now replaced by one of glass, 2 feet 9 inches long, and 2·4 inches in diameter. The source of heat in my last-published inquiry was the cube of hot water C; but glass being far inferior to brass in reflecting power, I was unable with this source to bring out with due force the vast differences existing between various kinds of gaseous matter. I therefore had a copper hood constructed, and united by brazing with a tube 8 inches long, which was destined to form the vacuous chamber in front of the first plate of rock-salt. To heat the copper plate, a lamp formed on the principle of Bunsen's burner was made use of. The gas passed upwards by four hollow columns, each perforated for the admission of air. The mixture of air and gas escaped from these columns into a chamber shaped like the frustum of a cone, and over this chamber was placed a shade of thin sheet-iron, the top of which was narrowed to a slit one-eighth of an inch wide and 2 inches long. From this slit the mixture of gas and air issued, and formed upon ignition a sheet of flame. This was caused to glide along the back of the copper plate before referred to, which was thereby raised to a temperature of about 270° C. To preserve this source constant was one of the greatest difficulties of the investigation; for the slightest agitation of the surrounding air, or the slightest flickering of the flame itself, was sufficient to disturb the steadiness of the galvanometer and to render experiments in delicate cases impossible. The flame was surrounded by screens of pasteboard, these being again encompassed by towels, through the meshes of which the flame was fed; a suitable chimney produced a gentle draught and carried off the products of combustion; the rhythmic jumping of the flame itself was destroyed by screens of wire-gauze; in short, six weeks' practice was required to master all the difficulties of this portion of the apparatus. The "compensating cube" C′, the double screen H, and the thermo-electric pile P remain as before. They are exposed in the figure, but during the experiments they were surrounded by a close hoarding, all the chinks of which were stuffed with tow, so as to protect the cube and pile from the disturbing action of the air. To protect the anterior plate of rock-salt from the heat which might have been conducted to it from the source, the front chamber passed as before through a vessel V in which a current of cold water, constantly renewed, was caused to circulate.

§ 2. *Experiments.*

On two points I wished to set my mind at rest previous to starting on my vacation tour this year. These were the absorption of chlorine gas and of ozone. On the 16th, 17th, and 18th of June, I experimented on these two substances, and satisfied myself that chlorine was far outstripped by many colourless gases, and that ozone had a power of absorption very much greater than common oxygen.

The work was resumed on the 12th of September, and my first care was to examine whether my published experiments on moist and dry air stood the test of repetition. Professor Magnus had experimented on dry air and on air saturated with moisture, and found that the presence of the moisture had no influence on the absorption. I, on the

contrary, had previously found, and stated, that dry air had only a small fraction of the absorptive energy of the same air when even partially saturated. I commenced my researches in September with a few experiments on this subject.

Half an atmosphere of the undried air of the laboratory admitted directly into the tube cut off an amount of heat which produced a deflection of 30 degrees.

My drying apparatus at this time consisted of a U-tube filled with fragments of pumice-stone wetted with sulphuric acid. Associated with this was a similar tube filled with like fragments, but moistened with caustic potash solution, to remove the carbonic acid of the air.

The air of the laboratory passed through both these tubes in succession, till a tension of 15 inches was attained, gave a deflection of 26 degrees.

This result surprised me, showing, as it seemed to do, a very close agreement between dry and moist air. On examining the drying-tubes, however, I found that by a mistake of arrangement the air had entered the sulphuric acid tube first, and passed straight from the potash into the experimental tube; thus partially reloading itself with moisture after it had been dried.

The air was now sent through both tubes, commencing with the potash—the deflection fell instantly to less than 5 degrees. Hence this experiment showed the absorption due to the moisture and carbonic acid of the air to be more than six times greater than that of the atmosphere itself. It will presently be seen that the difference here stated falls far short of the truth.

The potash and sulphuric acid were now abandoned, and the air was dried by passing it through a U-tube filled with fragments of chloride of calcium, which had lain in the tube for some months. The deflection produced by air thus dried was 40 degrees; that is to say, 10 degrees more than that produced by the undried air.

This result, and many others of a similar nature, were due to the imperfection of the chloride of calcium. I think chemists ought to be very cautious in the use of this substance as a drying agent. When pure and newly fused it may answer for this purpose, but when old it yields an impalpable powder, which proved in the highest degree perplexing to me in my first experiments. It is generally found, I believe, that a drying-tube of sulphuric acid gains more in weight than one of chloride of calcium, and from this it has been inferred that the quantity of moisture taken up by the former is greater than that taken up by the latter. The difference, however, may really be due to the mechanical carrying away of a portion of the chloride by the current of air.

On the 13th of September these experiments were resumed. The dry air then gave a deflection of less than 2 degrees; the air direct from the laboratory caused, in one experiment, the needle to move from 20 degrees on one side of zero to 28 on the other. In a second experiment the undried air caused the needle to move from 18° on one side of zero to 32° on the other.

Experiments made on the 17th entirely corroborated this result. Three successive trials made the action of the undried air of the laboratory 29°, 31°, and 30° respectively;

the deflection produced by the dried air being less than a single degree. On this day, therefore, the action of the aqueous vapour of the air was at least thirty times that of the air itself.

Almost every week-day during the last four months experiments similar to the above have been executed, and in no case have I observed a deviation from the result that the absorptive action of the aqueous vapour of the air is quite enormous in comparison with that of the air itself. Further on I will give an array of figures illustrating this point.

As I became more and more master of my apparatus, and more acquainted with the precautions necessary in delicate cases, the absorption of air and the elementary gases diminished more and more. I was induced to abandon the use of pumice-stone as well as of chloride of calcium, and to construct my drying apparatus in the following way. The internal portion of a massive block of pure glass was pounded to small fragments in a mortar; these were boiled in pure nitric acid, and afterwards washed several times with distilled water so as to remove all trace of the acid. They were then dried, afterwards moistened with pure sulphuric acid, and introduced by means of a funnel into a U-tube. The funnel was necessary to preserve the neck of the tube from all contact with the acid, the least action of which upon the corks used to close the tube was sufficient to entirely vitiate the experiments. At the top of each arm of the U-tube a quantity of dry fragments of glass was placed, so that if any dust or particles from the cork or sealing-wax chanced to reach the interior they fell upon the dry glass.

Similar precautions were taken with the caustic potash tube. Pure white marble was pounded to fragments and subjected to the action of a dilute acid, which removed the outer surface of the fragments. These were afterwards washed in distilled water and dried, then moistened with pure caustic potash, and introduced into the U-tube in the manner already described. It was sometimes necessary to perform this operation daily, and never on any occasion have I used tubes to dry a feeble gas which had been previously used to dry a powerful one.

In the present communication I shall have to touch upon many subjects which for want of time I have been unable to develope. The following is an example of these. Choosing a day of suitable temperature and moisture—a day on which the human breath shows no signs of precipitation—the action of the substances expired from the lungs may be most strictly determined by our apparatus. By breathing directly into the experimental tube, the action produced by the sum of the products of respiration might be accurately measured; by breathing through the sulphuric acid tube, the moisture of the breath would be withdrawn, and the difference between the action then observed and the former action would give that of the carbonic acid. In this way the products of respiration might be estimated singly, and the influence of various kinds of food and drink, or of physical exertion, on the respiration might be investigated in a manner hitherto unthought of.

I have to record the following experiments only in connexion with this subject. Placing a suitable tube between my lips, I filled my lungs with air; a stopcock which

was interposed between me and the experimental tube being partially opened, I breathed through it slowly into the latter until the mercury gauge of the pump was depressed 15 inches. I had, at the time, two assistants, C. A. and R. C., and they subsequently breathed into the experimental tube the same quantity as myself. In the following Table the absorption produced by the breath of each is stated.

Action of the Products of Respiration on Radiant Heat.

Initials of person's name.	Absorption.
J. T.	62
J. T.	62
R. C.	66
R. C.	68
J. T. again	59
J. T.	59
R. C.	63
C. A.	62
J. T.	60·5

The absorption of dry air on the day that these results were obtained was found to be 1. *The same dry air inhaled, underwent a chemical change which augmented its absorptive energy at least* 60 *times.* I give this as a minor limit, and will not say how much I regard it as falling short of the truth.

The day afterwards the following results were obtained, the same amount as before being exhaled:—

Initials.	Absorption
J. T.	56
R. C.	62
J. T.	56
R. C.	59

In all cases R. C., who is the smallest and least robust man of the three, appeared to have the advantage. I will only add a few results obtained on the 6th of October, the quantity of air expired on the occasion depressing the mercurial column 5 inches.

Initials.	Absorption.
J. T.	33·5
R. C.	35
R. C. After half a glass of Trinity Audit Ale	41
Again	35
After a teaspoonful of brandy	35
After chewing and swallowing a small quantity of onion	40

After taking the ale and brandy my assistant washed his mouth and gargled his throat several times with cold water. I give these results merely as illustrative of one of the

numerous applications of the apparatus. In all the experiments the tube remained perfectly transparent throughout, and, on pumping, the needle in each case returned accurately to zero.

§ 3.

In my last paper I brought the fact somewhat prominently forward that the elementary bodies which I had then examined were far less hostile to the passage of the longer undulations than the compound ones; and I founded at the time certain theoretic considerations on this fact*. I was desirous this year to extend the experiments to one or two of the coloured gases and vapours, and on the 20th of September resumed my experiments on chlorine. This gas is itself highly coloured, and of a specific gravity of 2·45; one of its compounds, hydrochloric acid, is quite transparent, and of specific gravity of only 1·26. Does the act of combination with hydrogen which renders chlorine gas more transparent to light render it also more transparent to heat? Chlorine prepared from hydrochloric acid and peroxide of manganese, and dried by passing it through sulphuric acid, was admitted into the tube till it depressed the mercury gauge 21 inches; the absorption of the gas was expressed by the number 44.

Hydrochloric acid was admitted till it depressed the gauge 19 inches; the absorption was 68. This experiment indicates that transparency to light and opacity to heat accompany the same act of chemical union.

The following results were afterwards obtained. I may remark that a subsidiary gauge was used, so as to prevent the destructive gases from entering the air-pump.

	Absorption.
Chlorine 15 inches	32
Chlorine 14 inches	30
Chlorine 14 inches	30
Hydrochloric acid 14 inches .	47
Chlorine again	30
Hydrochloric acid	56

In all cases the effect of the compound gas was found to exceed that of the elementary one; so that *the chemical change which renders chlorine more transparent to light renders it more opake to obscure heat.*

Great care is required in experiments on hydrochloric acid, and great care was bestowed on the above. Previously to the introduction of the gas the experimental tube was filled with perfectly dry air, so as to leave a perfectly dry residue on exhaustion. The gas was allowed to stream through the drying-tube until all traces of air were expelled; then a joint was suddenly broken, and the retort was connected with the experimental tube. The gas thus passed directly from the retort through the drying apparatus into the vacuum. It was difficult to avoid sending in with the gas a few particles of moisture; but these, if such existed, appeared to be dissipated by the dynamic

* Philosophical Transactions, 1861.

heating of the gas on entering the tube, and kept dissipated by the flux of heat passing through it. At all events the closest scrutiny could detect no trace of mist or turbidity within the tube; it was perfectly transparent throughout. The chlorine, on the contrary, was intensely coloured.

I made many experiments with chlorine which had been collected over water, but something (what I know not yet) appeared to be in all cases carried along with the gas from the water into the tube, which materially augmented its absorption.

The above experiments were made in the early part of this inquiry, and before I had become aware of all the peculiarities of my apparatus. Subsequent experiments reduced in some degree the absorption both of chlorine and hydrochloric acid. Very careful experiments made on the 29th of October gave the following absorption for these two gases, at a tension of 30 inches:—

Chlorine	39
Hydrochloric acid	53

The chlorine and hydrochloric acid were removed from the experimental tube in the following manner:—A cock and connecting piece were attached to one end of the experimental tube, and from them a length of india-rubber tubing led to the flue of the laboratory stove. A gas-holder of air was put in connexion with the other end of the experimental tube, a system of drying-tubes intervening between the latter and the holder. By a slight water-pressure a stream of dry air was carried gently through the tube to the flue, and in this way the gases, which if pumped out would have injured the pistons, were speedily removed. As the dry air replaced the gases, the needle gradually descended to zero, its arrival there being indicative of the complete displacement of the gas. The perfect dryness of the air thus made use of was beautifully proved. Had the air contained moisture, it would instantly on its mixture with hydrochloric acid have rendered the medium within the tube turbid. However slight this turbidity might be, and however invisible to the eye, the galvanometer would have revealed it. But there was no movement in an upward direction; the needle gradually sunk from the moment the air entered.

As regards the influence of chemical union in the absorption of radiant heat, a still more severe test than that furnished by the substances last examined is presented by bromine and hydrobromic acid; for the opacity of the former to light is far greater than that of chlorine, while the two compounds are equally transparent. The density of bromine, moreover, is 5·54, whereas that of hydrobromic acid is only 2·75. The difficulty of operating with the acid compound of bromine is at least equal to that attendant on hydrochloric acid, and several successive days were spent in endeavouring to arrive at safe conclusions in connexion with this subject. Bromine dried with phosphoric acid was introduced into a flask furnished with a screw cap, which enabled it to be attached to the experimental tube. By turning a stopcock, the pure vapour was allowed slowly to enter until the mercury column was depressed two inches. From more than twenty experiments made with this substance, I should infer that the absorption of the quantity

mentioned does not exceed

11,

while the absorption of hydrobromic acid of the same tension amounts to

30.

The hydrobromic acid was prepared by the action of glacial phosphoric acid (for a pure specimen of which I have to thank my friend Dr. FRANKLAND) on bromide of potassium. If the above figures represent the truth (and I have spared no pains to arrive at a right conclusion), we have here a most striking instance of *transparency to light and opacity to obscure heat being promoted by the self-same chemical act* *.

§ 4.

In the following Table is given the absorption of a number of gases at a common tension of one atmosphere.

TABLE I.

Name.	Absorption.
Air	1
Oxygen	1
Nitrogen	1
Hydrogen	1
Chlorine	39
Hydrochloric acid	62
Carbonic oxide	90
Carbonic acid	90
Nitrous oxide	355
Sulphuretted hydrogen	390
Marsh-gas	403
Sulphurous acid	710
Olefiant gas	970
Ammonia	1195

Air, oxygen, nitrogen, and hydrogen are all set down as equal to unity in the above Table. I do not mean thereby to affirm that there are no differences between these gases as regards their powers of absorption, but that the most powerful and delicate tests which I have hitherto applied have failed to establish a difference in a satisfactory manner. It is not improbable that the action of these gases may turn out to be less even than I have found it. For who can say that the best-constructed drying apparatus is really perfect? Besides, stopcocks must be greased, and hence may contribute an infinitesimal impurity to the air passing through them. I cannot even say that sulphuric acid, however pure, may not deliver a modicum of vapour to the current of air passing through it. At all

* A layer of liquid bromine, sufficiently opake to intercept the entire luminous rays of a gas-flame, is highly diathermanous to its obscure rays. An opake solution of iodine in bisulphide of carbon behaves similarly.—The details of these experiments shall be published in due time: they were publicly shown in my lectures many months ago.—June 18th, 1862.

events, if any further advance should be made in the purification of the gases, it will certainly only tend to augment the enormous differences exhibited in the above Table.

Ammonia, of the tension mentioned, stands highest in the above list as regards absorptive energy. I believe that a length of less than 3 feet of this gas, which to the vision is as transparent within the tube as the vacuum itself, is *perfectly black* to the rays emanating from the source here made use of. When the gas was in the tube, the interposition of a double metallic screen between the pile and source augmented the deflection very slightly. But I shall show further on that the ammonia in this experiment could not exhibit the full energy of its absorption, and that the length indicated is in all probability absolutely impervious to the heat issuing from our source.

It would be a mere affectation of accuracy to try to deal with smaller quantities of the first four substances mentioned in the Table than those with which I have here operated. Still, if such small quantities could be directly measured, the action of air, oxygen, hydrogen, and nitrogen, in comparison with that of the other substances at the same tension, would doubtless be greatly reduced. With the energetic gases the rays are most copiously struck down by the quantities which first enter the tube, the quantities which enter last producing in many cases an infinitesimal effect. Now I have shown in my last paper that for very small absorptions the effect is sensibly proportional to the quantity of gas present, and this would seem to justify us in assuming that for 1 inch of tension the absorption of air, oxygen, nitrogen, and hydrogen would be $\frac{1}{30}$th of the absorption at 30 inches. With all the other gases I have measured directly the absorption of a quantity possessing in each case a single inch of tension. Assuming the proportionality just referred to, and again calling the effect of air unity (the unit, however, being only $\frac{1}{30}$th of that in the last Table), the following are the relative absorptions:—

Table II.

Air	1
Oxygen	1
Nitrogen	1
Hydrogen	1
Chlorine	60
Bromine	160
Hydrobromic acid	1005
Carbonic oxide	750
Nitric oxide	1590
Nitrous oxide	1860
Sulphide of hydrogen	2100
Ammonia	7260
Olefiant gas	7950
Sulphurous acid	8800

Here we have the extraordinary result that for tensions of 1 inch of mercury *the*

absorption of ammonia is over seven thousand times, the absorption of olefiant gas seven thousand nine hundred and fifty times, while the action of sulphurous acid is eight thousand eight hundred times that of air.

It is impossible not to be struck by the position of chlorine and bromine in this Table. They are elements, and notwithstanding their colour and density, they take rank after the transparent elementary gases. The perfectly transparent olefiant gas absorbs more than one hundred and thirty times the amount absorbed by the untransparent chlorine, and nearly fifty times the quantity absorbed by the intensely brown vapour of bromine. I cannot think this fact insignificant. Hitherto chemists have spoken to us of elements, and we have helped ourselves to conceptions regarding them and their compounds in the only way possible to our mental constitution. But our conceptions remained purely subjective, nor were we acquainted with any physical trait which would in any degree justify these conceptions. Here, however, we seem to touch the ultimate particles of matter. Starting from the idea that a gas absorbs such vibrations as are isochronic with its own, in all cases the compound gas reveals itself to the mind's eye with its molecules on the whole swinging more slowly than the uncombined atoms of which it is composed. Their absorption of the longer undulations proves their general coincidence in period with those undulations. We load the atom by the act of chemical union, and thereby render its vibrations more sluggish, that is to say, more fit to synchronise with the slowly recurrent waves of obscure heat.

In the foregoing Table I have given the absorption of nitric oxide as 1590, which is less than that of nitrous oxide, though the molecule of the former contains a greater number of atoms than that of the latter. It will be noticed *that those gases which on combining suffer no condensation are less energetic absorbers than those which suffer a reduction of volume.* Whether this rule is universal I am as yet unable to say.

It is very difficult to operate with nitric oxide; the affinity of the gas for oxygen is so enormous that the slightest trace of this substance gives rise to the brown fumes of nitrous acid. On first sending this gas into the experimental tube, 1 inch of it gave an absorption of 2040; but the needle slowly went up afterwards, until it finally indicated an absorption of 5100. On looking across the tube at this time, the brown hue of nitrous acid was discernible.

In a second experiment I made the vacuum as perfect as possible; on allowing nitric oxide to enter, the absorption was 1860, but the needle soon afterwards declared an absorption of 3060, the brown fumes appearing as before.

On filling the experimental tube with nitrogen, then exhausting, and allowing nitric oxide to enter, the absorption of 1 inch of the gas was 1680. On filling the experimental tube previously with hydrogen the absorption was 1590, which is that given in the Table. On letting in a mixture of air and nitric oxide till the tube was filled, the action last mentioned was augmented nearly twentyfold. Nitrous acid is therefore an extremely energetic gas. The difference between it and bromine is enormous when the colours of both are equally dense.

A close inspection of MELLONI's Table* reveals, I think, the tendency of solid bodies also to become more transparent to heat as their composition becomes more simple. After rock-salt itself comes the element sulphur, and after it fluor-spar. But the case of lampblack will here occur to many, who regard it as the most powerful absorber and radiator yet discovered. No doubt the grouping of the atoms of an elementary substance may make it tantamount to a compound, and no doubt this is actually the case with lampblack; another eminent example of this kind is referred to in this paper in the section on ozone. LESLIE, however, found water to be a better radiator than lampblack, and WELLS found several substances which were more capable of being chilled by nocturnal radiation. On reflection, moreover, the following considerations arise. The lampblack of commerce and the soot of a lamp or candle, that is to say, the substances which have been hitherto employed in experiments on radiant heat, are copiously mixed with hydrocarbons, which are the most powerful absorbers and radiators in Nature. It might fairly be questioned whether the reputed experiments with lampblack really dealt with lampblack at all. But even the impure substance is to some extent transparent to radiant heat.

I have plates of black glass, rendered so by the solution of carbon in the glass while in a state of fusion, and which, though they are impervious to the rays of the most intense electric light, allow of a copious transmission of the rays of obscure heat. MELLONI's beautiful experiments on glass of this character are well known; indeed mine are but repetitions of his. Another of MELLONI's experiments which I have recently verified is the following. A plate of transparent rock-salt was placed over a smoky camphine lamp, and soot was deposited on its surface until it intercepted every ray of a brilliant jet of gas. The plate was placed between a source of heat possessing a temperature of 100° C., and a thermo-electric pile, a polished screen being placed between the salt and the source. As long as the screen remained, the needle of the galvanometer connected with the pile stood at zero; but the moment the screen was removed the needle promptly advanced, showing the instantaneous transmission across the layer of soot of a portion of the heat incident upon the salt. The actual numbers obtained in this experiment are these:—The deflection produced by the heat transmitted through the soot was 52°; which is equal to 90 units. The deflection produced when the layer of soot had been carefully removed, so as to leave both surfaces of the salt smooth and transparent, was 71°, which is equal to 300 units. The quantity transmitted through the soot is therefore to the total quantity as

$$90 : 300,$$

or as

$$30 : 100;$$

that is to say, the lampblack, which was perfectly opake to the light of a gas-jet, was transparent to fully 30 per cent. of the incident heat. On consulting MELLONI's Table, I was gratified to find that he made the transmission by a plate similarly prepared

* 'La Thermochrôse,' p. 164.

27 per cent.; while a layer so opake that it cut off the beams of the sun itself transmitted 23 per cent. of the rays emitted by a source heated to 100° C.

At page 93 of 'La Thermochrôse,' MELLONI examines the absorption of this substance for all sorts of rays, and by a series of ingenious experiments, and reasonings remarkable for their clearness and precision, he arrives at the conclusion that lampblack absorbs with the same intensity all descriptions of radiant heat*. At page 284, however, he cites and discusses with the same precision a series of experiments made with smoked rock-salt, in which he shows that the same layer of lampblack transmits 8 per cent. of the rays from a lamp of Locatelli, 10 per cent. of those of incandescent platinum, 18 per cent. of those from copper heated to 400° C., and fully 23 per cent. of those emitted by a source of 100° C. Now a transmission of 8 per cent. implies an absorption of 92; while transmissions of 10, 18, and 23 per cent. imply absorptions of 90, 82, and 77. But that the self-same layer of lampblack absorbs 77 per cent. of the rays from one source and 92 per cent. of the rays from another, is at variance with the statement that lampblack absorbs heat from all sources with the same intensity. Suppose the surface of a thermo-electric pile to be coated by a layer of lampblack of the same thickness as that which coated MELLONI'S plate of salt; 23 per cent. of the rays from a source of 100° C. would go right through such a layer and impinge upon the metal face of the pile; and if the latter were a good reflector, the heat incident upon it would be in great part retransmitted through the lampblack and lost to the instrument. For a source of 100° C., this loss would be many times greater than for a Locatelli lamp. Possibly, however, MELLONI meant simply to assert that for practical purposes the absorption by the face of his pile might be considered to be the same for all kinds of heat†.

§ 5.

I have now to record some new experiments on the action of *vapours* upon radiant heat. A number of glass flasks were prepared, of the shape and size of common test-tubes, each of which was furnished with a brass cap carefully cemented on to it. By means of this it could be attached to a stopcock, and thus connected with the experimental tube. The mode of operation was this. The liquid was introduced into the flask by means of a small glass funnel; the stopcock (S) was then attached to the flask and connected with a second air-pump, which was always kept at hand. The air above the liquid was removed, and the air dissolved in it was allowed to bubble away, until nothing remained but the pure liquid below and the pure vapour above it. The stopcock S was now shut off, and the flask united to the experimental tube. The exhaustion of the tube and stopcocks being complete, and the needle of the galvanometer at zero, the cock S was turned on and the mercury-gauge carefully observed at the same time.

* "Donc, le noir de fumée absorbe avec la même intensité toute sorte de rayonnements calorifiques" (p. 101).

† The sun, through the floating carbon of the London atmosphere, sometimes presents a most instructive appearance. Entirely shorn of his rays and of perfectly uniform brightness, his colour at times is as red as blood. This is doubtless mainly due to the comparative transparency of the floating carbon for the longer undulations.

No bubbling of the liquid was in any case permitted. The vapour entered silently and without the slightest commotion, and when the mercurial column was depressed to the extent required, the cock S was promptly turned off.

The energy with which the needle moves the moment a strong vapour enters is so extraordinary, that I was compelled to remove the stops which arrested the swing of the needle at 90°, lest the shock against them should derange the equilibrated magnetism of the astatic pair. The needle often swung far beyond a quadrant; and after it had come finally and permanently to rest, its position was observed in the following manner:—The dial of the galvanometer being well illuminated, a looking-glass was placed behind the instrument, at such an angle that when looked at horizontally the image of the dial was clearly seen. This image was observed by an excellent telescope, fixed at a distance of 11 feet from the galvanometer. Attached to the needle, and in continuation of it, was a bit of glass fibre of extreme fineness, which ranged over the graduated circle, and by means of it a very small fraction of a degree could be easily read off. I resorted to the expedient of observing from a distance, because I found that the approach of my person, perhaps through the diamagnetic action of my own body, had a sensible effect upon the needle of my instrument, which, I believe, surpasses in delicacy any hitherto constructed.

The *permanent* deflection of the needle was noted in all these experiments, and the value of the deflection, expressed in terms of one of the lower degrees of the galvanometer, was obtained from a table of calibration. To spare unnecessary labour, I omit the deflections in the following Table, and give the absorptions only produced by the vapours mentioned, at 0·1, 0·5, and 1·0 inch of tension.

TABLE III.

Name of substance.	Tensions. 0·1 inch.	0·5 inch.	1·0 inch.
Bisulphide of carbon	15	47	62
Iodide of methyl	35	147	242
Benzol	66	182	267
Chloroform	85	182	236
Methylic alcohol	109	390	590
Iodide of ethyl	158	290	390
Amylene	182	535	823
Sulphuric ether	300	710	870
Alcohol	325	622	
Formic ether	480	870	1075
Acetic ether	590	980	1195
Propionate of ethyl	596	970	
Boracic ether	620		

Let us compare some of the results of this Table of transparent vapours with the action of the highly coloured vapour of bromine. The absorption of bromine vapour at 1 inch

tension is about 6, and at 0·1 of an inch tension would probably not exceed 1; hence, at 0·1 of an inch tension, bisulphide of carbon has probably 15 times the absorbent power of bromine; but bisulphide of carbon is the feeblest of the compound vapours that I have yet discovered. The strongest of these, boracic ether, has, according to the above estimate, and at the tension stated, *more than* 600 *times the absorbing energy of the strongly coloured bromine.*

The whole of the numbers in the above Table are referred to atmospheric air as unity; 0·1 of an inch of bisulphide of carbon vapour, for example, absorbs 15 times as much as a whole atmosphere of air. Let us compare, for an instant, the action of boracic ether with that of air. We arrive at an approximate comparison in this way. The absorption of the tenth of an inch of boracic ether is something more than that of a whole inch of methylic alcohol; by diminishing the quantity of methylic alcohol to one-tenth, we reduce its absorption from 590 to 109. The absorption of one-tenth of an inch of boracic ether is 620°; suppose it to diminish in the proportion above found for methylic alcohol, we should have for 0·01 of an inch of boracic ether an absorption of 111; that is to say, for $\frac{1}{3000}$th of an atmosphere of boracic ether, we should have an absorption 111 times that of a whole atmosphere of oxygen, nitrogen, hydrogen, or atmospheric air.

With the transparent elementary gases it is impossible to measure directly the absorption of 0·1 of an inch; but assuming, as before, that up to an absorption of 1 the effect is proportional to the quantity of gas present, the absorption of each of the elementary gases, at a tension of 0·1 of an inch, would be about 0·0033; hence the absorption of boracic ether of 0·1 of an inch tension is to that of air at the same tension as

$$620 : 0{\cdot}0033,$$

which would give to the ether an energy 186,000 *times that of air.*

I have already spoken of the blackness of ammonia at 30 inches tension. Referring to Table I., its absorption is found to be 1195. In the last Table the vapour of acetic ether, possessing only one-thirtieth of the tension of the ammonia, produces apparently the same effect; its absorption is also 1195. Such facts give one entirely new ideas of the capabilities of matter, and our wonder will not be diminished by the results to be recorded further on.

With both gases and vapours we find that it does not follow that a gas which produces a larger effect than another at one tension should surpass that other at all tensions. Some gases start from a lower level than others, but finally attain an equal, or even a greater elevation. If their absorptions were represented by curves plotted from the same datum-line, these curves would in some cases approach, and in some cases cross each other. At a tension of 1 inch, for example, carbonic acid has more than double the absorptive power of carbonic oxide, whereas at a tension of 30 inches they are equal; indeed some of my experiments show carbonic oxide to have the advantage. On the 22nd of October, for example, I found the deflection produced by 2 inches of carbonic oxide to be 15°, while that of 2 inches of carbonic acid was 38°. The two gases

at a tension of 30 inches gave these results:—

Carbonic oxide	52
Carbonic acid	51·5

And again, on the 4th of November I obtained the following relative effects:—

	Tensions.	
	1·2 inches.	24 inches.
Carbonic oxide	12	57
Carbonic acid	37	54

The same remarks apply to vapours. Methylic alcohol, for example, starts at a lower level than the iodide of ethyl, but ascends more quickly, and finally reaches a much higher elevation. The same observation may be made of chloroform in comparison with benzol and the iodide of ethyl.

§ 6.

I have now to refer to a class of facts which surprised and perplexed me when I first observed them. As an illustration, I will first take the case of alcohol vapour. A quantity of this substance, sufficient to depress the mercury gauge 0·5 of an inch, produced an absorption which caused a deflection of 72° of the galvanometer needle.

While the needle pointed to this high figure, and previously to pumping out the vapour, I allowed dry air to stream into the tube, and happened while it entered to observe the effect upon the galvanometer. The needle, to my astonishment, sank speedily to zero, and went to 25° at the opposite side. The entry of the almost neutral air here not only abolished the absorption previously observed, but left a considerable balance in favour of the face of the pile turned towards the source. A repetition of the experiment brought the needle down to zero, and sent it to 38° on the opposite side. In like manner a very small quantity of the vapour of sulphuric ether produced a deflection of 30°; on allowing dry air to fill the tube the needle descended speedily to zero, and swung to 60° at the opposite side.

These results both perplexed and distressed me, for I imagined, on first observing them, that I had been throughout dealing with an effect totally different from absorption. I thought, at first, that my vapours had deposited themselves in opake films on my plates of rock-salt, and that the dry air on entering had cleared these films away, and allowed the heat from the source free transmission.

But a moment's reflection dissipated this supposition. The clearing away of such a film could at best but restore the state of things existing prior to its formation. It might be conceived of as bringing the needle again to 0°; but it could not possibly produce the negative deflection, which, in the case of ether vapour, amounted to the vast amplitude of 60°. Nevertheless I dismounted the tube, and subjected the plates of salt to a searching examination. I satisfied myself thus that no such deposition as that

above surmised took place. The salt remained perfectly transparent while in contact with the vapour.

Some of the experiments recorded in the Bakerian Lecture for this year (1860) had taught me that the dynamic heating of the air when it entered the exhausted tube was sufficient to produce a very sensible radiation on the part of any powerful vapour contained within the tube, but I was slow to believe that the enormous effect above described could be thus accounted for. My first care was to determine the difference of temperature between a thermometer placed within the tube at the end furthest from the source, and one placed without it. I then examined, by an extremely sensitive thermometer, the increase of temperature produced by the admission of dry air into the tube, and the decrease consequent on pumping out; and found the former to be a considerable fraction of the total heat transmitted from the source. Could it be that the heat thus imparted to the alcohol and ether vapours, and radiated by them against the adjacent face of the pile, was more than sufficient to make good the loss by absorption? The *experimentum crucis* at once suggested itself here. If the effects observed be due to the dynamic heating of the air, we ought to obtain them even when the sources of heat made use of in the experiments are entirely abolished; and we should thus arrive at the solution of the novel, and at first sight utterly paradoxical problem, *To determine the radiation and absorption of gases and vapours without any source of heat external to the gaseous body itself.*

For the sake of brevity, I will call the heating of gas by its admission into a vacuum, the *dynamic heating* of the gas; and the chilling accompanying its pumping out, *dynamic chilling*. It would also contribute to brevity if I were allowed to call the radiation and absorption of the gaseous body, consequent on such heating and chilling, *dynamic radiation* and *dynamic absorption*, though I fear the terms are not unobjectionable.

§ 7. *On Dynamic Radiation and Absorption.*

Both the source of heat and the compensating cube were dispensed with, and the thermo-electric pile was presented to the end of the cold experimental tube. By a little management, the slight inequality of radiation against both faces of the pile, arising from differences in the various parts of the laboratory, was obliterated, and the needle of the galvanometer thus brought to 0°.

The vapours were admitted in the manner already described, until a tension of 0·5 of an inch was obtained. The air was then allowed to enter through a drying apparatus by an orifice of a constant magnitude. Two stopcocks, in fact, were introduced between the drying-tube and the experimental tube; one of these was kept partially turned on, and formed the gauge for the admission of the air. When the tube was to be exhausted, the second stopcock was turned quite off. When the tube was to be filled, this stopcock was turned full on; but the *gauge-cock* was never touched during the entire series of experiments.

Before, however, the mode of experiment was thus strictly arranged, a few preliminary trials gave me the following results:—

Nitrous oxide on entering caused the needle to swing in a direction which indicated the heating of the gas; the limit of its excursion was 28°, after which it slowly sunk to 0°.

The pump was now worked; the propulsion of the first portions of the gas from the tube was so much work done by the residue. That residue became consequently chilled; into it the face of the pile adjacent poured its heat, and a swing of the needle on the negative side of 0° was the consequence. The limit of the excursion was 20°.

Olefiant gas, operated on in the same manner, produced on entering the tube a swing of 67°, showing radiation; and on pumping out, a swing of 41°, showing absorption. After the pumping out of the gas, and without introducing a fresh quantity, *dry air* was again admitted; the swing produced by the dynamic radiation of the residue of the gas (0·2 of an inch intension) was 59°. On pumping out *very quickly*, the dynamic absorption produced a deflection of nearly 40°.

A little of the vapour of sulphuric ether was admitted into the tube; on the admission of dry air afterwards the needle swung from 0° to 61°; on pumping out, the needle ran up to 40° on the opposite side.

These and other experiments, which I confess gratified me exceedingly, showed that, without resorting to any source of heat external to the gaseous body itself, its radiation and absorption might be determined with extreme accuracy, and the reciprocity of both phenomena rendered strikingly clear. In fact, at this very time I had been devising an elaborate apparatus for the purpose of examining the radiation of gases and vapours, with a view to comparing it with their absorption; but no such apparatus would have given me results equal in accuracy to those placed within reach by the discovery of dynamic radiation and absorption.

The following Table is the record of a series of experiments in connexion with this subject. The vapour in each case was admitted till the mercury column fell half an inch, and dry air was admitted afterwards.

TABLE IV.—Dynamic Radiation and Absorption of Vapours.

	Deflections.	
	Radiation.	Absorption.
Bisulphide of carbon	14°	8°
Iodide of methyl	19·5	8
Benzol	30	14
Iodide of ethyl	34	15·5
Methylic alcohol	36	
Chloride of amyl	41	23
Amylene	48	
Alcohol	50	27·5
Sulphuric ether	64	34
Formic ether	68·5	38
Acetic ether	70	43

The paradox already referred to is here solved, and the explanation given of the extraordinary effect observed in the case of the alcohol and ether vapours when dry air entered the experimental tube. Dynamic radiation, moreover, and dynamic absorption go hand in hand; and if we compare both with Table III., we shall find the order of the substances precisely the same, although one set of results are obtained with a source of heat external to the gaseous body, and the other with a source of heat and cold within the body itself. Had I sufficient time at my disposal, I could develope this subject with advantage. The results just recorded constitute my first regular series of experiments, and no doubt augmented experience will enable me to attain more perfect results.

I could not well obtain half an inch of my most energetically acting vapour, namely, boracic ether; but one-tenth of an inch admitted into the tube and dynamically heated and chilled, gave—

Radiation.	Absorption.
56°	28°

Seeing the astonishing energy with which some of these vapours absorb and radiate heat, it may be asked how far the quantity of vapour may be reduced before its action becomes insensible. At present I will not venture to answer this question fully; certainly we should be dealing at least with millionths of our smallest weights. But I will here lay before the Society an account of one experiment, the result of which can hardly fail to excite astonishment. The experimental tube being exhausted, one-tenth of an inch of boracic ether vapour was admitted into it; the barometer stood at 30 inches at the time, hence the tension of the vapour within the tube was $\frac{1}{300}$th of an atmosphere.

Dynamically heated by dry air, the radiation of this vapour produced a deflection of 56°.

The tube was then exhausted to 0·2 of an inch, and the quantity of vapour reduced thereby to $\frac{1}{10}$th part of its first amount; the needle was allowed to come to zero, and the residue of vapour was dynamically heated as before: its radiation produced a deflection of 42°.

The pump was again worked till a vacuum of 0·2 of an inch was obtained, this residue containing of course $\frac{1}{10}$th of the quantity of ether present in the last. On dynamically heating this residue, its radiation produced a deflection of 20°*.

Two additional exhaustions, succeeded by dynamic heating, gave the deflections 14° and 10° respectively.

Tabulating the results so as to place each deflection beside the vapour-tension which produces it, we have the following view of the experiment:—

* This is less than the truth; my assistant having executed three or four strokes of the pump inadvertently while the dry air was not shut off, removing thereby a considerable proportion of the vapour which ought to be present at this stage of the experiment.

TABLE V.—Dynamic Radiation of Boracic Ether.

Tension in parts of an atmosphere.	Deflection.
$\frac{1}{300}$th	56°
$\frac{1}{150}\times\frac{1}{300}=\frac{1}{45000}$th	42
$\frac{1}{150}\times\frac{1}{150}\times\frac{1}{300}=\frac{1}{6750000}$th	20
$\frac{1}{150}\times\frac{1}{150}\times\frac{1}{150}\times\frac{1}{300}=\frac{1}{1012500000}$th	14

The air itself, slightly warming the apparatus near the pile, produces a feeble radiation, amounting to 6° or 7°. I have purposely excluded the deflection 10°, in order to show that the effect was still diminishing when the experiment ended, the constant effect due to the air itself being not yet attained. I thus exclude two 0s from the denominator of my fraction which might fairly have appeared in it. The above result is, however, sufficiently extraordinary, showing as it does that the radiation of an amount of vapour possessing in our tube a tension of less than the thousand millionth of an atmosphere is perfectly measurable. It will also be borne in mind that the temperature imparted to this infinitesimal quantity of matter did not exceed 0·75 of a Centigrade degree.

These experiments, which I intend to develope on a future occasion, seem to give us new ideas as to the nature and capabilities of matter. A platinum wire raised to whiteness in a vacuum by an electric current, becomes comparatively cold in a second after the current has been interrupted; yet that wire, while ignited, was the repository of an immense amount of mechanical force. What has become of this? It has been conveyed away by a substance so attenuated that its very existence must for ever remain an hypothesis. But here is matter that we can weigh, measure, taste, and smell; that we can reduce to a tenuity, which, though expressible by numbers, defeats the imagination to conceive of it. Still we see it competent to arrest and originate quantities of force, which on comparison with its own mass are almost infinite, a small fraction of this force causing the double needle of the galvanometer to swing through considerable arcs. When we find common ponderable matter producing these effects, we have less difficulty in investing the luminiferous ether with those mechanical properties which have long excited the interest and wonder of all who have reflected upon the circumstances involved in the undulatory theory of light.

In the foregoing experiments dry air was used to warm the vapours, but similar differences ought to be exhibited by gases when heated by their own dynamic action. This is the case, as the following experiments show:—

TABLE VI.—Dynamic Radiation of Gases.

Name.	Radiation.
Air	7
Oxygen	7
Hydrogen	7
Carbonic oxide	19
Carbonic acid	21
Nitrous oxide	31
Olefiant gas	63

I also satisfied myself of the energetic radiation of the two following gases, which, however, were used in irregular quantities. They were admitted into the tube from a large bolthead, until a common tension was established between the gas in the tube and the gas in the bolthead.

	Radiation.	Absorption.
Ammonia 15 in. tension	58·5	33·5
Sulphurous acid 16 in. tension . .	45	24

Let us reflect for an instant on the condition of our tube with its $\frac{1}{2}$ inch of vapour at the moment when the latter has been heated by the entrance of the air. The gaseous column is heated throughout to the same temperature; the elastic condition of the luminiferous ether is the same for all the particles, and consequently their periods of vibration are all the same. Hence each molecule is in that precise condition which enables it to absorb most effectually the undulations emanating from its neighbours. The rays from the particles at the end of the tube most distant from the pile have to cross a space of nearly 3 feet before they reach the latter, this space being partially filled with molecules circumstanced as just described. Hence absorption to a comparatively greater extent must occur; and indeed we can imagine the tube so long that its frontal portion should furnish a vapour screen absolutely opake to the radiation of its hinder portion. Comparing ether vapour with olefiant gas, it is, I think, evident that the radiant points of the attenuated vapour which depresses the mercury column only 0·5 of an inch, are further apart than those of the gas which depresses the column 30 inches. Consequently there is a wider door open for the radiation of the distant ether particles towards the pile than for the distant particles of olefiant gas. The length of the whole column, in fact, might be available for the radiation of the vapour, and a part of it only available for the gas. Cut off this unavailable portion from the gas column, and we do not injure its efficacy; but cut off a similar length from the vapour column, and we may materially diminish its effect. Speaking generally, by reducing the column of ether and that of gas by the same amount, the diminution of radiation will be most sensibly felt where the radiant points are furthest asunder. Reasoning thus, it becomes evident that in a long tube the vapour may excel the gas in its amount of radiation, while with a short tube the gas may excel the vapour. Let us now test this reasoning by experiment.

The dynamic radiation of the following four substances has been tabulated thus:—

Sulphuric ether	64
Formic ether	68·5
Acetic ether	70
Olefiant gas	63

The action of olefiant gas is therefore smallest when the length of the radiating column is 2 feet 9 inches.

Experiments of the same character were made with a tube 3 inches long, or of the

former length, and the following results were obtained:—

Sulphuric ether	11
Formic ether	12
Acetic ether	15
Olefiant gas	39

The verification of the above theoretic reasoning is here complete. It is proved that *in a long tube the dynamic radiation of the vapour exceeds that of olefiant gas, while in a short tube the dynamic radiation of the gas far exceeds that of the vapour.*

§ 8.

The apparatus with which these experiments were made is capable of very diverse uses. Attached to a compression pump, with it the relation between the mechanical force expended in compressing a gas and the heat developed might be accurately determined. If oxygen, hydrogen, nitrogen, or air were the body compressed, the conversion of *vis viva* into heat might be declared by a modicum of vapour always kept in the tube, while a compound gas would tell its own tale.

Another interesting point might be, and indeed has been settled by the apparatus. Some years ago a discussion was carried on between Professors Challis and Stokes on Laplace's correction for the velocity of sound, Professor Challis contending that Laplace had no right to his correction, inasmuch as the heat developed by the local compression of a mass of air of indefinite extension would be instantly wasted by radiation. Experiments, he argued, conducted in confined vessels furnish no ground for drawing conclusions regarding what occurs in the atmosphere, where the heat developed has an indefinite space to lose itself. In our experimental tube, though it is mechanically closed, indefinite extension, as regards the radiation of heat, is secured in one direction, and the means also exist of measuring the flux of heat in this direction. What is true for one direction is of course true for all, so that the apparatus will inform us of what must occur in the open atmosphere. Now with the most powerful radiating gases which I have examined the radiation continues a very sensible time, while the heat acquired by air on entering the tube is often a source of inconvenience on account of the inability of the air to disperse its heat by radiation. The question is therefore experimentally decided in favour of Laplace and his supporter.

I would here dwell for a moment on this comparative absence of radiating power on the part of air, and of the elementary gases generally. The air is the sole source of the heat which has warmed the vapours in our experiments on dynamic radiation; it is related to them precisely as a hot polished metal plate is to the coat of varnish which makes it a radiator. The air and the metal, both elements or mixtures of elements, are incompetent to impart motion to the luminiferous ether without the intermediation of a second body. They possess the motion, but they are so related to the ether that they cannot communicate this motion to it *directly*, or only in an extremely feeble degree. The atoms of

air oscillate, but the ether does not swell. We have here a definite picture before the mind's eye, which, if the theory of an ether be true, is as certain as any conclusion of mathematics, and would hardly be rendered more certain if the physical vision were so sharpened as to be able to see the oscillating atom and the fluid in which it swings. I write thus strongly and definitely lest it should be imagined that I am dealing in vague conjectures in connexion with this subject. If I am vague, the mechanical theory of an ether must in reality bear the reproach. So far, however, from having a reproach to bear, the whole body of facts is in complete harmony with this theory.

Further, if, as all the facts declare, radiation and absorption are complementary acts, the one consisting in communication, the other in reception, and the one being strictly proportional to the other, no coincidence in period between the vibrations of a radiating body and those of oxygen, hydrogen, or air, could make any one of these substances a good absorber. The form of the atom, or some other attribute than its period of oscillation, must enter into the question of absorption. It is physically incapacitated from communicating motion, and hence in an equal degree incapacitated from accepting motion. The neutrality of elementary gases in the experiments on absorption already recorded does not arise from my accidentally choosing a source of heat whose periods do not synchronise with those of the gas; for however they might synchronise, the gas would still be a bad absorber. Even when the motion which their own absorbent power does not enable them to take up is mechanically imparted, or is communicated to them by contact, they are still incompetent to expend it upon the ether, which accepts all vibrations alike*.

§ 9.

Scents and effluvia generally have long excited the attention of observant men, and they have formed favourite illustrations regarding the divisibility of matter. Several chapters in the works of the celebrated Robert Boyle are devoted to this subject, and eminent men in all countries have speculated more or less upon the extraordinary tenuity of the matter which is competent to produce sensible effects upon our organs of smell. Such a subject would of course in itself form a wide inquiry, which it is quite out of my power to develope at present. I think, however, that the apparatus which we have thus far made use of enables us to deal with the question in a manner hitherto unattainable.

A number of dry aromatic plants† were obtained from Covent Garden, the leaves and flowers of which were stuffed into glass tubes 18 inches long and a quarter of an inch in diameter. By means of my second air-pump, a current of dry air was first passed

* I can hardly imagine the bands in the spectra of metallic compounds to be produced by the vibration of the compound atom. All my experiments show the vast influence of chemical union on the rate of oscillation; the metal itself and the compound of that metal could hardly, in my opinion, oscillate alike. Hence I infer that decomposition has occurred when the bright and constant spectral bands are seen.—June 20th.

† I mean "dry" in the common acceptation of the term. They were not green, but withered; doubtless, strictly speaking, they contained aqueous vapour.

over them for some minutes. They were then connected with the experimental tube, which had its sources of heat arranged as already described. The tube was first exhausted and the needle brought to 0°, and dry air was then passed over the scented herbs until the tube was filled. The consequent deflection was noted, and from it the absorbent action of the odorous substance was deduced.

Thyme thus treated exercised thirty-three times the absorption of the air in which it was diffused.

Peppermint exercised thirty-four times the action of the air.

Spearmint exercised thirty-eight times the same action.

Lavender produced thirty-two times the action of the air.

Wormwood forty-one times the action of the air.

The following perfumes were obtained from Mr. Atkinson of Bond Street, and examined in this manner. Small squares of dried bibulous paper, all of the same size, were rolled into cylinders about 2 inches in length; each of these was moistened by an aromatic oil, and introduced into a glass tube between the drying apparatus and the experimental tube. The latter being first exhausted, was afterwards filled by a current of dry air which had passed over the scented paper. Calling the action of the air which formed the vehicle of the perfumes 1, the following absorptions were observed in the respective cases:—

Table VII.

Name of perfume.	Absorption.
Pachouli	30
Sandal Wood	32
Geranium	33
Oil of Cloves	33·5
Otto of Roses	36·5
Bergamot	44
Neroli	47
Lavender	60
Lemon	65
Portugal	67
Thyme	68
Rosemary	74
Oil of Laurel	80
Cassia	109

It would be interesting to examine the absolute weights of the substances which produced these effects; but this I suppose is a task which chemistry is unable to accomplish. In comparison with the air which carried the odours into the tube, their weight must be almost infinitely small. Still we find that the least energetic in the list has thirty times the effect of the air, while the most energetic produces 109 times the same

effect. As regards the absorption of radiant heat, the perfume of a flower-bed may be more efficacious than the entire oxygen and nitrogen of the atmosphere above it.

After each scent had been introduced a stream of dry air was admitted at one end of the tube, while the pump was worked in connexion with the other. The perfume was thus cleared out until the needle returned to 0°. This was often a long operation, the odours clung with such tenacity to the apparatus. After the zero had been attained in the case of a strong perfume, a few minutes' rest of the pump sufficed to bring the scent from its hiding-places in the crevices and cocks of the apparatus, and almost to restore the original deflection. The quantity of those residues must be left to the imagination to conceive. If they were multiplied by billions they probably would not reach the density of the air.

Fearing that the more active perfumes might possibly prejudice the action of the more feeble ones which succeeded them, I made a series of experiments with the following essences, and obtained the results recorded:—

Camomile flowers	87
Spikenard	355
Aniseed	372

After this enormous effect I repeated the experiment with bergamot, and found its action to be exactly the same as that recorded in the Table.

I made a few experiments on musk, but obtained different results with it at different times. On the 16th of October I obtained some fresh musk from the perfumer's, placed it in a small glass tube, and carried dry air over it into the experimental tube. The first experiment gave me an absorption of

74,

the air which carried the perfume being unity. A second experiment, in which the air was admitted more quickly, the absorption was

72.

It would be idle to speculate upon the quantity of matter which produced this result. The stories regarding the unwasting character of this substance are well known; suffice it to say, that a quantity of its odour carried into the tube by a current of air of a minute's duration, produced an effect seventy-two times that of the air which carried it. Long-continued pumping failed to cleanse the tube and passages of the musk. It cannot be volatile, for an amount of ether vapour which produces a far greater action is speedily cleared away, while the cocks and connecting pieces of the air-pump had to be boiled in a solution of soda before they were fit for use after the experiments with this substance.

Two perfectly concurrent experiments with ordinary cinnamon, in which fragments of the substance were placed in a tube and had dry air passed over them, gave an absorption of

53.

Several kinds of tea, treated in the same manner, produced absorptions which varied between 20 and 28.

In the teas, cinnamon, musk, and the odorous plants already referred to, dry air had been passed over them for some time before they were examined. Still a small amount of aqueous vapour may have entered with the odours, and thus rendered the results to some extent of a mixed character.

§ 10. *Ozone.*

In my last memoir I alluded briefly to the action of ozone; but the experiments then made having been executed with a brass tube, I was very desirous of repeating them with a tube which could not be attacked by this extraordinary substance. Experiments with the glass tube, performed on the 16th, 17th, and 18th of last July, satisfied me that I had not over-estimated its power as an absorber of radiant heat.

In my first experiments I made use of large electrodes for the purpose of lessening the resistance to the passage of the current through the decomposing liquid. The oxygen thus obtained differed but little from ordinary oxygen.

This year I had three decomposing vessels constructed: in the first (No. 1) the platinum plates had about four square inches of surface, being rolled up to economise space; the plates of the second (No. 2) had two square inches of surface, while those of the third (No. 3) had only a square inch of surface each. Numerous experiments with these gave me the following constant results. Calling the absorption of ordinary oxygen 1,—

Electrolytic Oxygen.

From plates	Absorption.
No. 1	20
No. 2	34
No. 3	47

A series of experiments made on the following day gave these results:—

No. I.	21
No. II.	36
No. III.	47

I now cut away a portion of the plates of No. II. so as to make them smaller than those of No. III. The oxygen obtained with these plates gave an absorption of

65,

thus exceeding No. III. considerably. The plates of No. III. were now reduced so as to make them smallest of all; the oxygen which they delivered gave an absorption of

85.

I feared the development of heat with these smallest plates, and knowing heat to be very destructive of ozone, I surrounded the apparatus by a mixture of pounded ice and

salt. The absorption of the oxygen thus obtained with the smallest plates amounted to 136.

By the results already recorded we have been prepared for the effect of minute quantities of matter; otherwise we could not fail to be struck with astonishment on finding a quantity of this substance, which would elude all attempts on the part of the chemist to determine its amount, producing an effect so stupendous in comparison with common oxygen. I have, moreover, strong reason to believe that I understate considerably the effect of the ozone. The experiments exhibit in an extremely striking manner the great influence of the density of the current at the place where the oxygen is liberated on the amount of ozone developed.

§ 11.

All the results here recorded had been obtained, when, turning to DE LA RIVE's excellent treatise on Electricity, I found an allusion to the experiments of M. MEIDINGER on ozone. I had never previously heard any allusion made to this investigation, and was gratified to find in it the record of a very interesting piece of work.

M. MEIDINGER commences by showing the absence of agreement between theory and experiment in the decomposition of water, the difference showing itself very decidedly in a deficiency of oxygen *when the current was strong*. On heating his electrolyte, he found that this difference disappeared, the proper quantity of oxygen being liberated. He at once surmised that the defect of oxygen might be due to the formation of ozone; but in what way was still to be determined. If it were due to the great density of ozone in the tube which received the oxygen, the destruction of this substance by heat would restore the oxygen to its true volume. Strong heating, however, which destroyed the ozone, showed in repeated measurements no alteration of volume, hence M. MEIDINGER concluded that the defect which he had observed was not due to the ozone mixed with the oxygen itself. He finally concluded, and justified his conclusion by satisfactory experiments, that the loss of oxygen was caused by the formation of peroxide of hydrogen which was dissolved in the liquid, and thus withdrawn from the electrolytic gas. He was further led to experiment with electrodes of different sizes, and found the loss of oxygen much more considerable when a small electrode was used than with a large one; whence he inferred that the formation of ozone was facilitated by *augmenting the density of the current at the place where electrode and electrolyte meet*. Nothing could be more different from the method pursued by M. MEIDINGER than that by which I arrived at the same conclusion; and though I had no doubt of the accuracy of my experiments, it was pleasant to find them corroborated in such a remarkable and unexpected way. I may add, that since the perusal of M. MEIDINGER's paper I have repeated his experiments with my decomposition cells, and find that those which gave me the greatest absorption also show the greatest deficiency in the amount of oxygen liberated*.

* I have recently learned that M. DE LA RIVE was the first to observe the influence of the size of the electrodes on the development of ozone.

The quantities of ozone with which I have operated must be perfectly unmeasurable by ordinary means. The action of the substance is like that of olefiant gas, or boracic ether—bulk for bulk it might transcend either. No elementary gas that I have examined behaves at all like ozone. In its swing through the ether it must powerfully disturb the medium. If it be oxygen, it must be oxygen packed into groups of atoms, which encounter vast resistance in moving through the ether. I sought to decide the question whether it is oxygen or a compound of hydrogen in the following way. Heat destroys ozone. If it were oxygen only, heat would convert it into the common gas; if it were the hydrogen compound which some chemists consider it to be, heat would convert it into oxygen plus aqueous vapour. The gas alone admitted into my tube would give the neutral action of oxygen, but the gas plus the aqueous vapour I hoped might give a sensibly greater action. I caused the dry electrolytic gas to pass through a glass tube heated to redness direct into the experimental tube. I afterwards introduced a drying-tube between the place where the gas was heated and the experimental tube. Hitherto I have not been able to establish with certainty a difference between the dried and undried gas. Previously to heating, the electrolytic oxygen had been scrupulously dried; if the act of heating developed aqueous vapour, I can only say that the experimental means which I have employed are unable to detect it. For the present, therefore, I hold the belief that ozone is produced by the packing of the atoms of elementary oxygen into oscillating groups; and that heating dissolves the bond of union and allows the atoms to swing singly, thus disqualifying them for either intercepting or generating the motion, which as systems they were competent to intercept and generate.

§ 12.

Since these researches were commenced, an eminent experimenter has been led by his own inquiries in another field to enter upon the investigation of gaseous diathermancy. On the 7th of February of the present year (1861), Professor MAGNUS communicated to the Academy of Sciences in Berlin a memoir on the Transmission of Heat through Gases*. The published notices of my experiments, commencing in May 1859, had escaped his attention, and his work is therefore to be regarded as independent of mine. Considering the very different methods which we have pursued, the general agreement between us must be regarded as remarkable.

The starting-point of Professor MAGNUS's investigation was the interesting experiment of Mr. GROVE, in which a platinum wire raised to whiteness by an electric current is suddenly cooled by an atmosphere of hydrogen. This action, which we have hitherto been disposed to attribute to the mobility of hydrogen, and its consequent high convective power, Professor MAGNUS was led to regard as an effect of conduction; and the thought induced him to examine the conductibility of gases generally. The mode of experiment adopted led him, not I think to the establishment of gaseous conductivity, but to results substantially the same as those that I had previously obtained. In fact

* POGGENDORFF's 'Annalen,' reprinted in Philosophical Magazine, S. 4. vol. xxii. p. 85.

the very experiments devised to show conductivity showed in a very striking manner the existence of athermancy, or opacity to radiant heat, in the case of a considerable number of gases.

The experiments on radiation, where obscure heat was made use of, were thus conducted. Two glass vessels, one much larger than the other, had their bottoms fused together; the larger one being turned upside down, the smaller one stood upright on the top of it. The mouth of the larger vessel was ground down, so that it could be placed like an ordinary receiver on the plate of an air-pump and exhausted, while through proper openings different gases could be afterwards admitted into it.

To the plate of the air-pump on which the above vessel was placed, was attached a thermo-electric pile with wires leading from it, through the plate, to a galvanometer; the axis of the pile was vertical, one face of it being turned downwards towards the plate, and the opposite face turned upwards towards the common surface of the two vessels which had been fused together.

Water was placed in the uppermost vessel, and caused to boil by conducting hot steam through it. Its under surface became thus heated to a temperature of 100° C. But this under surface constituted the upper surface of the vessel underneath. This latter, therefore, possessed a temperature of 100° C.; and it formed the source of heat made use of in the experiments.

Here Professor MAGNUS had a radiating surface of glass—a good radiator—kept at a constant temperature by the hot water above it; at a distance from this surface and turned towards it was the thermo-electric pile, defended from the radiation of the surface, or exposed to it, at pleasure, by the action of a moveable screen. The entire space between the pile and the radiating surface could either be rendered a vacuum, offering no resistance to the passage of the calorific rays, or else be filled by a gas the diathermancy of which was to be examined.

The concurrence of the experiments made with this apparatus and those made with mine is, as I have stated, remarkable. Some differences, however, exist between my friend and myself, a few remarks on which will not be without their use to those who may afterwards enter upon this extensive field of inquiry.

Experimenting in the ordinary way with his thermo-electric pile—using one of its faces only—Professor MAGNUS finds that air and oxygen cut off each more than 11 per cent. of the heat emanating from his source, while hydrogen cuts off more than 14 per cent.* I, on the contrary, with the most delicate means I could apply, failed to establish the absorption of these gases by experiments made in the ordinary manner†. In fact it was their neutrality that drove me to devise the principle of compensation, briefly referred to at the commencement of this memoir. I was so particular in the experiments which led me to the above negative result, that if the absorption amounted to one-tenth of that found by Professor MAGNUS I do not think it could have escaped me. Nor do I think that if such an action existed MELLONI could have concluded that the absorption

* Page 30. † Philosophical Transactions, 1861.

of a column of air fifteen times the length of that employed by Professor MAGNUS was absolutely insensible.

In the account of the experiments already published, where my source of heat was also 100° C., I have set down the absorption of air, oxygen, and hydrogen at about 0·33 per cent.; which is for air and oxygen thirty times, and for hydrogen over forty times less than that found by Professor MAGNUS.

In fixing the above figure for the absorption of these gases, I protected myself by assigning what I knew to be the superior limit of the effect, but I was morally certain at the time, that as soon as I could combine sufficient power and delicacy I should make the effect less. This I have done in my present inquiry, and find the absorption of the above gases to be under 0·1 per cent., which in the case of oxygen is less than $\frac{1}{100}$th, and in the case of hydrogen less than $\frac{1}{170}$th of the effect obtained by Professor MAGNUS with a tube less than half the length of mine. Making every allowance for the difference between our two sources of heat, the discrepancy between us is still enormous. In fact my conclusion is that these gases are practical vacua to radiant heat, and that the mixture of oxygen and nitrogen which constitutes the body of our atmosphere is the same.

While, however, in the case of the elementary gases the discrepancy between Professor MAGNUS and myself consists in a defect on my part, or an excess on his, with the powerful gases I obtained a considerably stronger action than he does. Thus with olefiant gas his absorption amounts to less than 54 per cent., whereas in mine it amounts to more than 72. This last result is what might only be expected, inasmuch as the length of gas traversed by the radiant heat was in the one case a little under 15 inches, and in the other 33.

Professor MAGNUS has further published an account of experiments in which a powerful gas-flame surrounded by a glass cylinder furnished the source of heat; the latter being augmented by a parabolic mirror of polished metal, placed behind the lamp. In this case the gases were enclosed in a glass tube 1 metre long and 35 millims. in diameter, the two ends of which were stopped with plates of glass 4 millimetres thick.

Two series of experiments were executed with this tube, in one of which the interior surface was covered with black paper, while in the other the glass was uncovered within. The former method is that pursued by Dr. FRANZ, and the result obtained by Professor MAGNUS in the case of atmospheric air and oxygen closely agrees with that obtained for the same gases by Dr. FRANZ. Professor MAGNUS makes the absorption in the case of the blackened tube about 2½, and Dr. FRANZ about 3 per cent. for air and oxygen.

In the case of the unblackened tube, however, the absorption was found to be much more considerable. Here the absorption by air and oxygen amounted to 14·75 per cent., and with hydrogen it reached 16·23. This great difference between the unblackened and the blackened tube is ascribed by Professor MAGNUS to a change of quality which the heat undergoes by its reflexion from the interior glass surface.

One of my motives in introducing a glass tube into the present inquiry was, that I

might be enabled to investigate the interesting question raised by this surmise of Professor MAGNUS. I have failed, however, to obtain his result. My naked glass tube, which is nearly of the same length as his, gives me a result which is more than 140 times less than his in the case of air and oxygen, and more than 160 times less than what he has obtained with hydrogen. Our sources of heat are, it is true, different, but the disadvantage is on my side; for assuredly the rays from a gas-jet are, if anything, less affected by the transparent elementary gases than those from my source. Had I time, I would repeat the experiments with a flame; but this, I regret to say, is out of my power at present.

Another difference between Professor MAGNUS and myself has reference to the influence of aqueous vapour. With both the gas-flame and the boiling water as sources of heat, he finds the effect of dry air to be precisely the same as that of air which he has allowed to pass in minute bubbles through water, and thus saturated with aqueous vapour.

I was engaged in experiments on this substance when my other duties compelled me to close this inquiry for a time. I believe, however, I may safely say, that not only is the action of aqueous vapour on radiant heat measurable, but *this action may be made use of as a measure of atmospheric moisture, the tube used in my experiments being thus converted into a hygrometer of surpassing delicacy.* Unhappily, as in other cases touched upon in this memoir, I have been unable to give this subject the development I could wish; but the results which I am in a position to record are nevertheless interesting.

On a great number of occasions I compared the air sent in directly from the laboratory into the experimental tube with the same air after it had been passed through the drying apparatus. Calling the action of the dry air unity, or supposing it rather to oscillate about unity (for the temperature of my source varied a little from day to day), on the following days the annexed absorptions were observed with the undried air of the laboratory:—

Absorptions by undried air.

October 23rd	63
October 24th	62
October 29th	65
October 31st	56
November 1st	50
November 4th	58
November 8th	49
November 12th	62

Nearly $\frac{9}{10}$ths of the above effects are due to aqueous vapour; which, therefore, in some instances *exerted nearly sixty times the action of the air in which it was diffused.*

The experiments which I have made on aqueous vapour have been very numerous and varied. Differing as I did from so cautious and able an experimenter, I deemed it due to Professor MAGNUS and myself to spare no pains in securing myself against error. I

have experimented with air moistened in various ways, sometimes by allowing small bubbles of it to ascend through water, sometimes dividing it by sending it through the pores of common cane immersed in water. Between the drying apparatus and the experimental tube I have introduced tubes containing fragments of glass moistened with water, and allowed the air to pass over them; large effects were in all such cases obtained, the absorption being usually *more than eighty times that of dried air*. Fragments of unwetted glass, which had been merely exposed to the air of the laboratory, had dry air led over them into the experimental tube; the absorption was fifteen times that of dried air. A roll of bibulous paper, taken from one of the drawers of the laboratory, and to all appearance perfectly dry, was enclosed in a glass tube, and dry air carried between its leaves. The experiment was made five times in succession with the same paper, and the following absorptions were observed:—

	Absorption.
No. 1	72
No. 2	62
No. 3	62
No. 4	47
No. 5	47

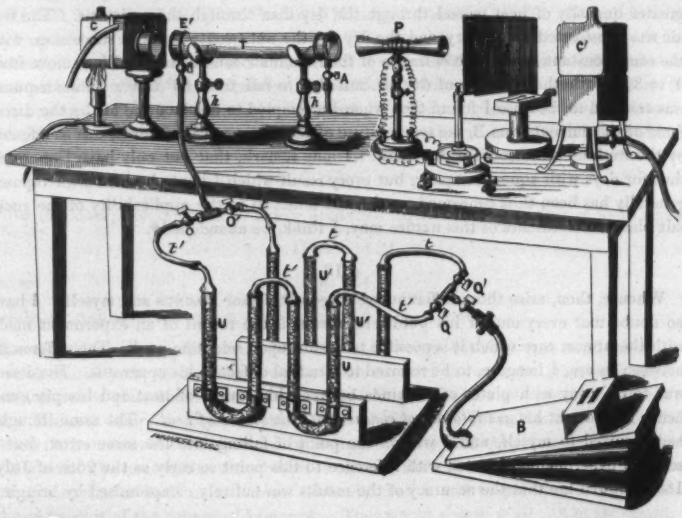
In fact, the action of aqueous vapour is exactly such as might be expected from the vapour of a liquid which Melloni found to be the most powerful absorber of radiant heat of all he had examined.

Every morning, on commencing my experiments, I had an interesting example of the power of glass to gather a film of aqueous vapour on its surface. Suppose the tube mounted, and the air of the laboratory removed as far as the air-pump was capable of removing it. On allowing dry air to enter for the first time, the needle would move from 0° to 50°. On pumping out it would return to 0°, and on letting in dry air a second time it would swing almost to 40°. Repeated exhaustions would cause this action to sink almost to nothing. These results were entirely due to the vapour collected during the night in an invisible film on the inner surface of the tube, and which was removed by the air on entering, and diffused through the tube. If the dry air entered at the end of the tube nearest to the source of heat, on the first and second admissions, and sometimes even on a third, the vapour carried from the warm end to the cold end of the tube was precipitated as a mist upon the latter, for a distance sometimes of nearly a foot. The mist always disappeared on pumping out. It is needless to remark that facts of this character, of which I could cite many, were not calculated to promote incautiousness or rashness on my part. I saw very clearly how easy it was to fall into the gravest errors, and I took due precautions to prevent myself from doing so.

Knowing that a solution of salt was almost as opake to radiant heat as water itself, I was careful to examine whether the effects which I had observed with aqueous vapour might not be due to the precipitation of the vapour on the surfaces of the plates of

salt used to stop my tube. The substance is well known to be very hygroscopic, and during the last three years the knowledge of this fact has rendered me careful to remove my polished plates every evening from the apparatus, and to keep them in perfectly dry air. Still, when it is remembered that the air on entering the tube is raised in temperature, and thus enabled to maintain a greater amount of vapour, and that the tube and plates of rock-salt form the channel for a flux of heat from the radiating source, the likelihood of precipitation occurring will seem but small. On examining the plates after the undried air of the laboratory was experimented with, no trace of precipitated moisture was observed upon their surfaces.

But to place the matter beyond all doubt, I abolished the plates of rock-salt altogether, and operated thus:—An india-rubber bag (B) was filled with air, and to its nozzle a T-piece, with the cocks Q Q′, was attached. The cock Q′ was connected with two tubes, U′ U′, each of which was filled with fragments of glass moistened with distilled water. The cock Q was connected with the tubes U U, each of which was filled with fragments of glass moistened by sulphuric acid. The other ends of these two series of tubes were connected with the cocks O O′, and from the T-piece between these cocks



a tube led to the end E′ of the open experimental tube T. The cock A at the other end of the experimental tube was placed in connexion with an air-pump. The pile P,

the screen S, and the compensating cube C′ were used as in the other experiments. E is the end of the front chamber, and C the source of heat. In some experiments I had the end E closed by a plate of rock-salt, in others it was allowed to remain open; a distance of about 12 inches intervening between the radiating surface and the open end E′ of the experimental tube.

Closing the cocks Q and O, and opening Q′ and O′, gentle pressure being applied to the bag B, a current of moist air was slowly discharged at the end E′ of the experimental tube. The pump in connexion with A was then worked, and thus by degrees when the air was sucked into the tube T. The deflection of the galvanometer was 30°, the moist moist air filled the tube as completely as the arrangement permitted*; this deflection being due to the predominance of the compensating cube over the radiating source C.

The cocks Q′ and O′ were now closed, and Q and O opened; proceeding as before, a current of *dry* air was discharged at E′, and this air was drawn into the tube T in the manner just described. The moist air was thus displaced by dry; and, while the displacement was going on, the galvanometer was observed through the distant telescope. The needle soon commenced to sink, and slowly went down to zero; proving that a greater quantity of heat passed through the dry than through the moist air. The wet air was substituted for the dry, and the dry for the wet twenty times in succession, with the same constant result; the entrance of the humid air caused the needle to move from 0° to 30°, while the entrance of dry air caused it to fall from 30° to 0°. The air-pump was resorted to, because I found that when I attempted to displace the air by the direct force of the current from B, the temperature of the pile, or of the source, was so affected by the fresh air as to confuse the result. I may remark, that not only have I operated thus for days with aqueous vapour, but every result which I have obtained with vapours generally has been thus confirmed, so that all doubt as to the applicability of the rock-salt plates to researches of this nature may, I think, be abandoned†.

§ 13.

Whence, then, arise those differences between Professor MAGNUS and myself? I have no doubt that every one of his published results is the record of an experiment made with the utmost care which it is possible to bestow upon scientific work. The differences between us are, I imagine, to be referred to a radical defect in his apparatus. His desire was to do away with plates of all kinds between his source of heat and his pile, and hence he brought his gas *into direct contact with his source of heat.* The same thought had occurred to myself, and I was on the point of falling into the same error; but a series of experiments executed with reference to this point so early as the 26th of July, 1859, showed me that the accuracy of the results was entirely compromised by bringing

* Still, of course, only partially.

† It is sheer want of time that prevents me from describing more particularly the numerous experiments executed with open tubes.

the gas to be examined into contact with the source. I obtained thus an action forty times what I knew it ought to be, and was confirmed in the view which caused me to interpose a vacuous chamber in front of the experimental tube. Let me here record a few experiments made on the 4th of last November in connexion with this subject.

I first satisfied myself that the drying apparatus was in perfect condition, the air of the laboratory producing, when sent through it, an absorption of 1. This same air was sent into the front chamber, that is, into direct contact with the source. The galvanometer needle moved as it does in the case of absorbent gases, and at the end of two minutes declared a loss of heat equivalent to an absorption of 50. The front chamber is 8 inches in length; the experimental tube is 33 inches long; hence a column of 8 inches, in contact with the radiating surface, produced at least fifty times the effect of a column more than four times as long when the air was separated from the radiating surface.

I made the foregoing experiment three times in succession, and after two minutes found the needle pointing to precisely the same degree; the lowering of the source was perfectly constant and regular, and in all cases showed a loss equivalent to an absorption of 50.

It will be remembered that Professor MAGNUS obtained a greater absorption with hydrogen than with either oxygen or air. This result is perfectly explained by reference to the quicker convection of this gas. I operated with hydrogen as I did with air, first satisfying myself that a column of it 33 inches long exercised an absorption less than unity. In fact it could not be measured. The same hydrogen introduced into the first chamber, and allowed to remain there for two minutes, caused a withdrawal of heat from the source equivalent to an absorption of

65.

Now the absorption of air in Professor MAGNUS's experiments is to that of hydrogen as

11·12 : 14·21,

or as 50 : 64,

while my results of convection are as 50 : 65.

The coincidence is so perfect that I am disposed to regard it as in part accidental.

Substantially the same remarks apply to the experiments with the glass tube stopped with plates of glass 4 millimetres thick. According to MELLONI, 61 per cent. of the rays of a Locatelli lamp are absorbed by a plate of glass only 2·6 millimetres thick. Professor MAGNUS surrounded his flame by a glass cylinder, and this, it may be urged, partially sifted the heat of the lamp before it reached the end of the tube. But in so doing the glass cylinder itself must become intensely heated, and to the heat of the cylinder the glass ends of the tube would be *opake*. They would absorb it all. Cold air admitted into such a tube is exactly similar to cold air let into my front chamber, it chills what is in part the source of heat, and maintains that chill by convection. The heat applied

may, in fact, be thus analysed. 1. We have a portion, almost wholly luminous, which went through the tube direct to the pile; 2, a portion *arrested* by the first glass plate; 3, a smaller portion *arrested* by the second glass plate; 4, we have the heat *radiated* by the first glass plate towards the second, and wholly absorbed by the latter; 5, we have the heat radiated by this latter against the pile. This analysis will probably enable us to understand how Professor MAGNUS obtained an absorption of only $2\frac{1}{2}$ per cent. with the blackened tube, and as much as 14·75 per cent. with the unblackened one. With the latter, the source, and the plate of glass nearest the source, send a copious flux down the tube to the plate at the opposite end; the oblique rays are in great part reflected by the interior surface, and thus reach the plate adjacent to the pile. With the blackened tube this oblique radiation is entirely cut off, the rays incident on the interior surface being absorbed. Thus the plate of glass adjacent to the pile must be much more intensely heated with the unblackened tube than with the blackened one. The difference in the amount of heat received by the pile-end plate in the respective cases is rendered very manifest by the experiments of Professor MAGNUS himself, for he finds that with the same source, twenty-six times the amount of heat transmitted by the coated tube is transmitted by the uncoated one. What, therefore, Professor MAGNUS ascribes to a change of quality by reflexion, would, if I am correct, be due to the higher heating in the case of the naked tube, and consequent *greater chilling by the cold air*, of the plate of glass close to the pile. To this must be added the effect produced by cooling the distant end of the tube itself, to which heat has been communicated from the first glass plate by the process of conduction, and the cooling of which comes most into play when the tube is uncovered.

The difference between Professor MAGNUS and myself as regards the action of aqueous vapour admits now of easy explanation. His effect being one of convection, and not of absorption, the quantity of vapour present in his experiments—probably not more than 1 per cent. of the volume of the gas, certainly not 2 per cent.—vanished as a convecting agent, in comparison with the air.

It is hardly necessary to repeat these reflections with reference to the experiments of Dr. FRANZ. The taking of the chilling of his plates for absorption, has caused him to find no difference of effect when he doubled the length of his tube. With a tube 450 millimetres long, he finds precisely the same absorption as with a tube of 900. He finds the action of carbonic acid to be the same as that of air, although at atmospheric tensions the action of the former is 90 times that of the latter*. He finds the vapour of bromine more destructive to radiant heat than nitrous acid gas, whereas the latter is beyond comparison the most destructive. The heat rendered latent by the evaporation of the

* The sensible equality of all the transparent gases and air was regarded as evident by Dr. FRANZ. "It might be seen," he writes, "from the outset that no decided difference would be observed between them" (p. 342). Similarly, Professor MAGNUS, speaking of aqueous vapour, writes, "Although it might be foreseen with certainty that the small amount of aqueous vapour in the air could have no influence on the radiation," &c. (p. 43).

bromine of course augmented the chill, and thus magnified the effect which in reality he was measuring. In reference to heating the glass plates by the flame, made use of in his experiments, I will cite a single passage from the memoir of Dr. FRANZ. It refers to the vapour of iodine produced by throwing the substance on a heated surface in a vessel closed with glass plates. "The mirror," he writes, "showed a deflection of only 178. But as the glass plates through which the heat radiated had not yet assumed a temperature high enough to reduce the iodine, which had been precipitated upon them in crystals, to a state of vapour . . . . . it was necessary to wait, and allow the radiation of the lamp to continue till all the iodine was driven from the bottle *." This shows how much the glass plates could be heated by the radiation of the lamp; this heat on a particular occasion being sufficient to dissipate the solid iodine which had coated the glass plates.

§ 14.

As a dam built across a river causes a local deepening of the stream, so our atmosphere, thrown as a dam across the terrestrial rays, produces a local heightening of the temperature at the earth's surface. This, of course, does not imply indefinite accumulation any more than the river dam does, the quantity lost by terrestrial radiation being, finally, equal to the quantity received from the sun. The chief intercepting substance is the aqueous vapour of the atmosphere †, the oxygen and nitrogen of which the great mass of the atmosphere is composed being sensibly transparent to the calorific rays. Were the atmosphere cleansed of its vapour, the temperature of space would be directly open to us; and could we under present circumstances reach an elevation where the amount of that vapour is insensible, we might determine the temperature of space by direct experiment. Colonel STRACHEY has written an admirable paper on the aqueous vapour of the atmosphere ‡, in which he shows that the amount of vapour diminishes much more speedily with the elevation than might be inferred from the law of DALTON.

It might be possible to reach a height where, by preserving one face of a thermo-electric pile at the temperature of the locality, the other, protected from all terrestrial radiation, turned to the zenith, would assume the temperature of space in that direction §, while the consequent galvanometric deflection would give us the means of determining the difference in temperature between the two faces of the pile. Knowing one, we should therefore be able to determine the other; knowing the temperature of the locality, we could infer from it the temperature of stellar space. Many eminent writers,

* "Es musste bei fortdauernder Strahlung der Lampe der Zeitpunct abgewartet werden."

† The mildness of an island climate must be in part due to this cause. The direct tendency of the vapour is to check sudden fluctuations of temperature. Where it is absent, as at the surface of the moon, such fluctuations must be enormous. The face turned towards the sun drinks in the solar rays without let or hindrance, while the radiation of the face turned from the sun pours unchecked into space.

‡ Proceedings of the Royal Society, vol. xi. p. 182.

§ A *well* of cold air would be formed within the reflector, the lowest stratum of the well sharing the temperature of the face of the pile.

it is true, have supposed the upper atmospheric regions to be colder than space, the depression of temperature being due to the radiation of the aërial particles, just as a grass-blade is lowered, by its radiation, below the air which surrounds it. This notion must, I think, be abandoned; for, as far as experiment goes, it leads us to conclude that air, and particularly air in the higher atmospheric regions, behaves as a vacuum both as regards radiation and absorption.

§ 15.

In his paper on the conduction of heat by gases, Professor Magnus examines the question of convection, and has adduced some striking experiments to show that the cooling of an incandescent wire in hydrogen is not due to the convection of the gas. He finds that when the wire is enclosed in a narrow tube, with only a thin film of the gas surrounding it, and where therefore currents, in the ordinary sense, are hard to be conceived of, the gas still exercises its cooling power. It had often occurred to me to make this experiment; and when I first heard of its successful performance by Professor Magnus I adopted his conclusion, that the cooling was due to conduction.

Reflection, however, caused me to change my opinion. Suppose the wire to be stretched along the axis of a wide cylinder containing hydrogen, we should have convection, in the ordinary sense, on heating the wire. Where does the heat thus dispersed ultimately go? It is manifestly given up to the sides of the cylinder. The transfer by convection is a transfer ultimately to the sides of the cylinder, and if we narrow our cylinder we simply hasten the transfer. The process of narrowing may continue till a tube like that used by Professor Magnus is the result; the convection between centre and sides will still continue, and produce the same cooling effect as before. Whether we assume conduction or convection, the tube surrounding the wire must be supposed to possess sufficient conducting power to carry the heat off, otherwise it would become incandescent itself by the accumulation of the heat.

The reasoning of Professor Magnus in connexion with this subject is of extreme ingenuity. He contends that there is no reason why stronger currents should establish themselves in hydrogen than in other gases. Currents are due to differences of density produced by the expansion of a portion of the gas by heat. Now hydrogen actually expands *less* than other gases, and hence the differential action on which the currents depend is less in this gas than in the others. Professor Magnus alludes to the friction of the particles against each other, but considers this ineffective.

This reasoning leads us to the threshold of a question which might form the subject of a long and profitable investigation. For a given difference of density, is not the mobility of hydrogen greater than that of the other gases? The experiments above recorded, where different gases were brought into direct contact with the source of heat, seem to answer this question in the affirmative. I have had no time to pursue the question regarding hydrogen; but I have made a few experiments, which show the influence of density on the mobility of a gas in a very striking manner.

Having first so purified atmospheric air as to render it sensibly neutral to radiant heat, I allowed 15 inches of it to enter the front chamber F, and there to come into contact with the source of heat. Convection of course immediately set in, and its amount was accurately measured by the quantity of heat withdrawn from the radiating surface; this quantity, expressed in the units adopted throughout this memoir, was

62.

The quantity of gas in the front chamber was now doubled, that is, it now had an atmosphere of tension; the withdrawal of heat then was expressed by the number

68.

In the last experiment we had double the number of atoms loading themselves with heat and carrying it away; if their motion had been as quick as that of the atoms when half an atmosphere was used, they would have withdrawn sensibly double the amount of heat; but the fact is that half an atmosphere carried off 64, while a whole atmosphere carried off 68; hence the absolute swiftness of the atoms in the case of the denser air must be very much less than in the case of the rarer. In fact, the amount of heat withdrawn will be proportional on the one hand to the number of carrying particles, and on the other to the velocity with which they move; hence if v and v' be these velocities, we have

$$\frac{62}{68}=\frac{v}{2v'}, \text{ or } \frac{v}{v'}=\frac{62}{34}.$$

Thus, while the atoms of the rarer gas travel 62 units in a second, those of the denser gas travel only 34.

This retardation can, I think, arise from nothing else than the resistance offered by the particles of the air to the motion of their fellows. It must be borne in mind that the smallness of the increment observed on doubling the amount of gas was not due to the partial exhaustion of the source by the first quantity of gas. The heat of the source was such that the withdrawal of 64 of our units could not sensibly affect the subsequent convection.

Here, then, we see what a powerful effect density, or the internal resistance which accompanies density, has on the mobility of a gas; and there is every reason to suppose that the mobility of hydrogen is due to the comparative absence, in its case, of internal resistance. However this may be, the foregoing experiment enables us to draw some important inferences.

Storms at great heights must be greatly facilitated by the mobility of the particles of the air. In fact storms are cases of convection on a large scale, and in our front chamber we had one in miniature. With the same difference of temperature on the summit of Mont Blanc, the motion of convection would be very nearly twice as great as at the sea-level.

In the summer of 1859 I was fortunate enough to induce my friend Professor FRANKLAND to accompany me to the summit of Mont Blanc, and to determine the comparative rates of combustion there and in the valley of Chamouni. Six candles were

purchased, burnt for an hour at Chamouni, and the loss of weight determined. The same candles were lighted for the same time on the summit of the mountain, and the consumption determined. Within the limits of error, the consumption above was equal to that below. The *light* below was immensely greater than that above, still the amount of stearine consumed in the two cases was sensibly the same. Professor FRANKLAND surmised this to be due to the greater mobility of the rarefied air, which allowed a freer interpenetration of the flame by the oxygen *, and the foregoing experiments show that the augmentation of mobility is just such as would account for the observed effect.

* The influence of interpenetration is well seen in the exposed gas-jets of London, particularly in the butchers' shops on a Saturday night. A gust of wind, which carries oxygen to the centre of a flame, suddenly deprives it of light. A simple and beautiful experiment consists of passing a lighted candle swiftly to and fro through the air; the white light reduces itself to a pale-blue band. BUNSEN's burner is an illustration in the same line.

VI. *On the Calculus of Symbols.*

By WILLIAM SPOTTISWOODE, *M.A., F.R.S.*

Made up of two Memoirs, one received November 4, Read November 21, 1861, the other received January 21, read January 30, 1862.

IN a paper published in the Philosophical Transactions for 1861, p. 79, Mr. W. H. L. RUSSELL has constructed systems of multiplication and division for functions of non-commutative symbols, subject to the same laws of combination as those in Professor BOOLE's memoir "On a General Method in Analysis," Philosophical Transactions, 1844. In this calculus there are four systems of multiplication and division, viz. internal and external, both (1) when the functions are arranged in powers of ϱ, (2) when in powers of π, or, as they may be termed, the ϱ-arrangement, and the π-arrangement. In the paper in question the author has confined himself, so far as division is concerned, to the case most useful in practice, in which the divisor is linear. And of this he has discussed in full only the ϱ-arrangement.

§ 1. *Internal division of the π-arrangement by a linear factor.*

Adopting the same notation as Mr. RUSSELL, I propose here to investigate, in the first place, the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an *internal* factor of

$$\varphi_n(\varrho)\pi^n+\varphi_{n-1}(\varrho)\pi^{n-1}+\ldots\varphi_0(\varrho),$$

and to determine the quotient. This is partially discussed in pp. 73–75.

Let $$\varrho\frac{d}{d\varrho}\psi=\psi',$$

then performing the actual divisions, for brevity writing ψ for $\psi(\varrho)$, and φ for $\varphi(\varrho)$,

$$\begin{array}{l} \psi_1\pi+\psi_0\Big)\varphi_1\pi+\varphi_0\Big(\dfrac{\varphi_1}{\psi_1} \\ \qquad\qquad \dfrac{\varphi_1\pi+\varphi_1\dfrac{\psi_0}{\psi_1}}{} \\ \qquad\qquad\quad \varphi_0-\varphi_1\dfrac{\psi_0}{\psi_1}. \end{array}$$

Hence the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of $\varphi_1(\varrho)\pi+\varphi_0(\varrho)$ will be

$$\varphi_0(\varrho)-\varphi_1(\varrho)\frac{\psi_0}{\psi_1}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Again,

$$\psi_1\pi+\psi_0\Big)\varphi_3\pi^3+\varphi_1\pi+\varphi_0\Big(\frac{\varphi_3}{\psi_1}\pi+\Big(\varphi_1-\varphi_3\frac{\psi_1'-\psi_0}{\psi_1}\Big)\frac{1}{\psi_1}$$

$$\varphi_3\pi^3+\varphi_3\frac{\psi_1'}{\psi_1}\pi+\varphi_3\frac{\psi_0}{\psi_1}\pi+\varphi_3\frac{\psi_0'}{\psi_1}$$

$$\Big(\varphi_1-\varphi_3\frac{\psi_1'+\psi_0}{\psi_1}\Big)\pi+\varphi_0-\varphi_3\frac{\psi_0'}{\psi_1}$$

$$\Big(\varphi_1-\varphi_3\frac{\psi_1'+\psi_0}{\psi_1}\Big)\pi+\Big(\varphi_1-\varphi_3\frac{\psi_1'+\psi_0}{\psi_1}\Big)\frac{\psi_0}{\psi_1}$$

$$\varphi_0-\varphi_1\frac{\psi_0}{\psi_1}+\varphi_3\Big\{\Big(\frac{\psi_0}{\psi_1}\Big)^2-\frac{\psi_0'\psi_1-\psi_0\psi_1'}{\psi_1^2}\Big\}.$$

Hence the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of $\varphi_3(\varrho)\pi^3+\varphi_1(\varrho)\pi+\varphi_0(\varrho)$ will be

$$\varphi_0(\varrho)-\varphi_1(\varrho)\frac{\psi_0(\varrho)}{\psi_1(\varrho)}+\varphi_3\Big\{\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)^2-\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)'\Big\}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Before proceeding further, we may remark that the remainder, after internally dividing $\varphi_4(\varrho)\pi^3+\varphi_3(\varrho)\pi^2+\varphi_1(\varrho)\pi+\varphi_0(\varrho)$ by $\psi_1(\varrho)\pi+\psi_0(\varrho)$, can differ from that last above found only in respect of the remainder arising from the division of $\varphi_4(\varrho)\pi^3$ by the factor in question; hence we have now only to divide the term $\varphi_4(\varrho)\pi^3$ by $\psi_1(\varrho)\pi+\psi_0(\varrho)$, and add the remainder so found to (2.), in order to have the condition required for the third degree. Proceeding to the division, writing $\chi=\frac{\psi_0}{\psi_1}$, and omitting for the present φ_3, which, since the division is internal, can be replaced as an external factor in the remainder, we have

$$\pi+\chi)\pi^3(\pi^3-\chi\pi+\chi^2-2\chi'$$

$$\pi^3+\chi\pi^2+2\chi'\pi+\chi''$$

$$-\chi\pi^2-2\chi'\pi-\chi''$$

$$-\chi\pi^2-\chi^2\pi-\chi\chi'$$

$$(\chi^2-2\chi')\pi+\chi\chi'-\chi''$$

$$(\chi^2-2\chi')\pi+\chi^3-2\chi\chi'$$

$$-\chi^3+3\chi\chi'-\chi''.$$

Hence the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of $\varphi_4(\varrho)\pi^3+\varphi_3(\varrho)\pi^2+\varphi_1(\varrho)\pi+\varphi_0(\varrho)$ will be

$$\varphi_0(\varrho)-\varphi_1(\varrho)\frac{\psi_0(\varrho)}{\psi_1(\varrho)}+\varphi_3(\varrho)\Big\{\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)^2-\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)'\Big\}-\varphi_2(\varrho)\Big\{\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)^3-3\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)'+\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)''\Big\}=0, \quad (3.)$$

the identity of which with Mr. Russell's condition, given in p. 75 of his paper, I have verified.

For the fourth degree,

$$\begin{array}{l}
\pi+\chi)\pi^4(\pi^3-\chi\pi^2+(\chi^2-3\chi')\pi-(\chi^3-5\chi\chi'+3\chi'') \\
\quad \pi^4+\chi\pi^3+3\chi'\pi^2+3\chi''\pi+\chi''' \\
\quad \overline{-\chi\pi^3-3\chi'\pi^2-3\chi''\pi-\chi'''} \\
\quad -\chi\pi^3-\chi^2\pi^2-2\chi\chi'\pi-\chi\chi'' \\
\quad \overline{(\chi^2-3\chi')\pi^2+(2\chi\chi'-3\chi'')\pi+(\chi\chi''-\chi''')} \\
\quad (\chi^2-3\chi')\pi^2+(\chi^2-3\chi\chi')\pi+(\chi^2\chi'-3\chi'^2) \\
\quad \overline{-(\chi^2-5\chi\chi'+3\chi'')\pi-\chi^2\chi'+3\chi'^2+\chi\chi''-\chi'''} \\
\quad \overline{-(\chi^3-5\chi\chi'+3\chi'')\pi-\chi^4+5\chi^2\chi'-3\chi\chi''} \\
\quad \overline{\chi^4-6\chi^2\chi'-4\chi\chi''+3\chi'^2-\chi'''}
\end{array}$$

and

$$\chi^4-6\chi^2\chi'-4\chi\chi''+3\chi'^2-\chi'''=-\chi(-\chi^3+3\chi\chi'-\chi'')+(-\chi^3+3\chi\chi'-\chi'')';$$

or if R_1, R_2, .. be the remainders of $\pi(\pi+\chi)^{-1}$, $\pi^2(\pi+\chi)^{-1}$, .., we have

$$R_2=-\chi R_1+R_1',$$

$$R_3=-\chi R_2+R_2',$$

$$R_4=-\chi R_3+R_3'.$$

And generally, if

$$\pi^n\ (\pi+\chi)^{-1}=\ Q_n+\ R_n(\pi+\chi)^{-1},$$

then

$$\pi^{n+1}(\pi+\chi)^{-1}=\pi Q_n+\pi R_n(\pi+\chi)^{-1},$$

the remainder of which must be contained in the last term. Performing the actual division, and remembering that $\pi R_n=R_n\pi+R_n'$,

$$\begin{array}{l}
\pi+\chi)R_n\pi+R_n'(R_n \\
\quad \overline{R_n\pi+R_n\chi} \\
\quad \overline{-\chi R_n+R_n'}.
\end{array}$$

Hence we have generally,

$$R_{n+1}=-\chi R_n+R_n',$$

and consequently, remembering that $R_0=1$, we have the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of $\varphi_n(\varrho)\pi^n+\varphi_{n-1}(\varrho)\pi^{n-1}+\ldots\varphi_0(\varrho)$,

$$\varphi_0(\varrho)R_0+\varphi_1(\varrho)R_1+\ldots\varphi_n(\varrho)R_n=0, \quad \ldots\ldots\ldots\ldots \quad (4.)$$

where

$$R_{i+1}=-\frac{\psi_0(\varrho)}{\psi_1(\varrho)}R_i+\varrho\frac{d}{d\varrho}R_i=\left(-\frac{\psi_0(\varrho)}{\psi_1(\varrho)}+\varrho\frac{d}{d\varrho}\right)R_i.$$

The law of the quotients is best seen by actual division. In case of $\varphi_2\pi^2+\varphi_1\pi+\varphi_0$, given above, the quotient may be written

$$\frac{1}{\psi_1}\times\begin{array}{|cc|c|}\hline \varphi_1 & \varphi_2 & \\ \hline 0 & 1 & \varpi \\ 1 & -\frac{1}{\psi_1}(\psi_1'+\psi_0) & 1 \\ \hline\end{array} \quad \ldots\ldots\ldots \quad (5.)$$

For the case of a cubic function of ϖ,

$$\psi_1\varpi+\psi_0)\varphi_3\varpi^3+\varphi_2\varpi^2+\varphi_1\varpi+\varphi_0\Big(\varphi_3\frac{1}{\psi_1}\varpi^3+\Big(\varphi_2-\varphi_3\frac{2\psi_1'+\psi_0}{\psi_1}\Big)\frac{1}{\psi_1}\varpi+\Big(\varphi_1-\varphi_2\frac{\psi_1'+\psi_0}{\psi_1}+\varphi_3\frac{2\psi_1'^2+3\psi_0\psi_1'+\psi_0^2-\psi_1\psi_1''-2\psi_1\psi_0'}{\psi_1^2}\Big)$$

$$\frac{\varphi_3\varpi^2+2\varphi_3\frac{\psi_1'}{\psi_1}\varpi^2+\varphi_3\frac{\psi_1''}{\psi_1}\varpi+\varphi_3\frac{\psi_0}{\psi_1}\varpi^2+2\varphi_3\frac{\psi_1'}{\psi_1}\varpi+\varphi_2\frac{\psi_0''}{\psi_1}}{\Big(\varphi_2-\varphi_3\frac{2\psi_1'+\psi_0}{\psi_1}\Big)\varpi^2+\Big(\varphi_1-\varphi_2\frac{\psi_1''+2\psi_0'}{\psi_1}\Big)\varpi+\Big(\varphi_0-\varphi_2\frac{\psi_0''}{\psi_1}\Big)}$$

$$\Big(\varphi_2-\varphi_3\frac{2\psi_1'+\psi_0}{\psi_1}\Big)\varpi^2+\Big(\varphi_2-\varphi_3\frac{2\psi_1'+\psi_0}{\psi_1}\Big)\frac{\psi_1'}{\psi_1}\varpi+\Big(\varphi_2-\varphi_3\frac{2\psi_1'+\psi_0}{\psi_1}\Big)\frac{\psi_0}{\psi_1}\varpi+\Big(\varphi_2-\varphi_3\frac{2\psi_1'+\psi_0}{\psi_1}\Big)\frac{\psi_0'}{\psi_1}$$

$$\Big(\varphi_1-\varphi_3\frac{\psi_1'+\psi_0}{\psi_1}+\varphi_3\frac{2\psi_1'^2+3\psi_0\psi_1'+\psi_0^2-\psi_1\psi_1''-2\psi_1\psi_0'}{\psi_1^2}\Big)\varpi+\Big(\varphi_0-\varphi_2\frac{\psi_0'}{\psi_1}+\varphi_3\frac{2\psi_1'\psi_0'+\psi_0\psi_0'-\psi_1\psi_0''}{\psi_1^2}\Big).$$

The quotient of which may be written

$$\frac{1}{\psi_1}\times\begin{array}{|ccc|c|}\hline \varphi_1 & \varphi_2 & \varphi_3 & \\ \hline 0 & 0 & 1 & \varpi^2 \\ 0 & 1 & -\frac{1}{\psi_1}(2\psi_1'+\psi_0) & \varpi \\ 1 & -\frac{1}{\psi_1}(\psi_1'+\psi_0) & \frac{1}{\psi_1^2}\begin{vmatrix} 2\psi_1'+\psi_0 & \psi_1 \\ \psi_1''+2\psi_0' & 2\psi_1'+\psi_0 \end{vmatrix} & 1 \\ \hline\end{array} \quad (6.)$$

Similarly, if the division be performed in the case of the quartic function, we shall find for the quotient of $(\varphi_4\varpi^4+\varphi_3\varpi^3+\varphi_2\varpi^2+\varphi_1\varpi+\varphi_0)(\psi_1\varpi+\psi_0)^{-1}$,

$$\frac{1}{\psi_1}\times\begin{array}{|cccc|c|}\hline \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \\ \hline 0 & 0 & 0 & 1 & \varpi^3 \\ 0 & 0 & 1 & -\frac{1}{\psi_1}(3\psi_1'+\psi_0) & \varpi^2 \\ 0 & 1 & -\frac{1}{\psi_1}(2\psi_1'+\psi_0) & \frac{1}{\psi_1^2}\begin{vmatrix} 3\psi_1'+\psi_0 & \psi_1 \\ 3\psi_1''+3\psi_0' & 2\psi_1'+\psi_0 \end{vmatrix} & \varpi \\ 1 & -\frac{1}{\psi_1}(\psi_1'+\psi_0) & \frac{1}{\psi_1^2}\begin{vmatrix} 2\psi_1'+\psi_0 & \psi_1 \\ \psi_1''+2\psi_0' & \psi_1'+\psi_0 \end{vmatrix} & -\frac{1}{\psi_1^3}\begin{vmatrix} 3\psi_1'+\psi_0 & \psi_1 & 0 \\ 3\psi_1''+3\psi_0' & 2\psi_1'+\psi_0 & \psi_1 \\ \psi_1'''+3\psi_0'' & \psi_1''+\psi_0' & \psi_1'+\psi_0 \end{vmatrix} & 1 \\ \hline\end{array} \quad (7.)$$

And likewise in general the quotient of $(\varphi_n\pi^n+\varphi_{n-1}\pi^{n-1}+..\varphi_0)(\psi_1\pi+\psi_0)^{-1}$ will be represented by a square table giving for the coefficient of φ_n

$$(-)^{n-1}\frac{1}{\psi_1{}^{n-1}}\times\begin{vmatrix} \frac{i-1}{1} & \psi_1' & + & \psi_0 & \psi_1 & .. & 0 \\ \frac{(i-1)(i-2)}{1.2} & \psi_1'' & + & \frac{i-1}{1}\psi_0' & \frac{i-2}{1}\psi_1' + \psi_0 & .. & 0 \\ . & . & & . & . & . & . \\ & \psi_1^{(i-1)} & + & \frac{i-1}{1}\psi_0^{(i-1)} & \psi_1^{(i-1)}+\frac{i-2}{1}\psi_0^{(i-1)} & .. & \psi_1'+\psi_0 \end{vmatrix} \quad (8.)$$

§ 2. *External division of the π-arrangement by a linear factor.*

I next investigate the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an *external* factor of

$$\varphi_n(\varrho)\pi^n+\varphi_{n-1}(\varrho)\pi^{n-1}+..\varphi_0(\varrho).$$

Performing the actual divisions, we have in the case of $n=1$,

$$\begin{array}{l} \psi_1\pi+\psi_0\Big)\varphi_1\pi+\varphi_0\Big(\dfrac{\varphi_1}{\psi_1} \\ \qquad \varphi_1\pi+\psi_1\Big(\dfrac{\varphi_1}{\psi_1}\Big)'+\psi_0\dfrac{\varphi_1}{\psi_1} \\ \hline \qquad \varphi_0-\psi_1\Big(\dfrac{\varphi_1}{\psi_1}\Big)'-\psi_0\dfrac{\varphi_1}{\psi_1}, \end{array}$$

or, as the remainder may be more conveniently written,

$$\varphi_0-\varphi_1'+\varphi_1\frac{\psi_1'-\psi_0}{\psi_1} \quad \ldots\ldots\ldots\ldots \quad (1.)$$

Again, in the case of $n=2$,

$$\begin{array}{l} \psi_1\pi+\psi_0\Big)\varphi_2\pi^2+\varphi_1\pi+\varphi_0\Big(\dfrac{\varphi_2}{\psi_1}\pi+\dfrac{1}{\psi_1}\Big\{\varphi_1-\varphi_2'+\varphi_2\dfrac{\psi_1'-\psi_0}{\psi_1}\Big\} \\ \qquad \varphi_2\pi^2+\psi_1\Big(\dfrac{\varphi_2}{\psi_1}\Big)'\pi+\psi_0\dfrac{\varphi_2}{\psi_1}\pi \\ \hline \qquad \Big\{\varphi_1-\psi_1\Big(\dfrac{\varphi_2}{\psi_1}\Big)'-\psi_0\dfrac{\varphi_2}{\psi_1}\Big\}\pi+\varphi_0; \end{array}$$

or, transforming the remainder as in the former case, and continuing the division,

$$\begin{array}{l} \Big\{\varphi_1-\varphi_2'+\varphi_2\dfrac{\psi_1'-\psi_0}{\psi_1}\Big\}\pi+\varphi_0 \\ \Big\{\varphi_1-\varphi_2'+\varphi_2\dfrac{\psi_1'-\psi_0}{\psi_1}\Big\}\pi+\psi_1\Big[\dfrac{1}{\psi_1}\Big\{\varphi_1-\varphi_2'+\varphi_2\dfrac{\psi_1'-\psi_0}{\psi_1}\Big\}\Big]+\dfrac{\psi_0}{\psi_1}\Big\{\varphi_1-\varphi_2'+\varphi_2\dfrac{\psi_1'-\psi_0}{\psi_1}\Big\} \\ \hline \qquad \varphi_0-\psi_1\Big[\dfrac{1}{\psi_1}\Big\{\varphi_1-\varphi_2'+\varphi_2\dfrac{\psi_1'-\psi_0}{\psi_1}\Big\}\Big]'-\dfrac{\psi_0}{\psi_1}\Big\{\varphi_1-\varphi_2'+\varphi_2\dfrac{\psi_1'-\psi_0}{\psi_1}\Big\}, \end{array}$$

which also may be transformed as follows:—

$$\varphi_0-\varphi_1'+\varphi_2''+(\varphi_1-2\varphi_2')\frac{\psi_1'-\psi_0}{\psi_1}+\varphi_2\left\{\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)^2-\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)'\right\}. \quad . \quad . \quad . \quad (2.)$$

A similar process of division will be found, in the case of $n=3$, to lead to the following remainder:—

$$\left.\begin{aligned}\varphi_0-\varphi_1'+\varphi_2''-\varphi_3'''+(\varphi_1-2\varphi_2'+3\varphi_3'')\frac{\psi_1'-\psi_0}{\psi_1}+(\varphi_2-3\varphi_3')\left\{\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)^2-\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)'\right\}\\ +\varphi_3\left\{\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)^3-3\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)'+\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)''\right\}.\end{aligned}\right\} \quad (3.)$$

If $\Psi_1, \Psi_2, \ldots$ represent the ψ-functions, coefficients of the φs in this expression, the law of their formation will be found to be as follows:—

$$\begin{aligned}\Psi_1 &= \frac{\psi_1'-\psi_0}{\psi_1},\\ \Psi_2 &= \Psi_1'-\Psi_1',\\ \Psi_3 &= \Psi_1\Psi_2-\Psi_2',\\ &\ldots\ldots\end{aligned}$$

And generally we may write

$$\Psi_{i+1}=\Psi_1\Psi_i-\Psi_i'.$$

And if $R_1, R_2, \ldots$ represent the remainders in the cases of $n=1$, $n=2$, .. respectively, we have

$$\begin{aligned}R_1&=\varphi_0-\varphi_1'+\varphi_1\Psi_1,\\ R_2&=\varphi_0-\varphi_1'+\varphi_2''+(\varphi_1-2\varphi_2')\Psi_1+\varphi_2\Psi_2,\\ R_3&=\varphi_0-\varphi_1'+\varphi_2''-\varphi_3'''+(\varphi_1-2\varphi_2'+3\varphi_3'')\Psi_1+(\varphi_2-3\varphi_3')\Psi_2+\varphi_3\Psi_3;\end{aligned}$$

whence

$$\begin{aligned}R_2&=R_1+\varphi_2''-2\varphi_2'\Psi_1+\varphi_2\Psi_2,\\ R_3&=R_2-\varphi_3'''+3\varphi_3''\Psi_1-3\varphi_3'\Psi_2+\varphi_3\Psi_3.\end{aligned}$$

With a view to forming the expression for R_n, let the symbol $\binom{1}{0}$ affixed to R_i signify that in the expression for R_n the suffixes of the φs have all been increased by unity. Then, by the principle of division,

$$\begin{aligned}R_{n+}&=\varphi_0-\psi_1\left[\frac{1}{\psi_1}R_n\binom{1}{0}\right]'-\frac{\psi_0}{\psi_1}R_n\binom{1}{0}\\ &=\varphi_0-R_n'\binom{1}{0}+\Psi_1R_n\binom{1}{0}\\ &=\varphi_0-\left(R_{n-1}-\varphi_n^{(n)}+\frac{n}{1}\varphi_n^{(n-1)}\Psi_1-\frac{n(n-1)}{1.2}\varphi_n^{(n-2)}\Psi_2+\ldots\right)'\binom{1}{0}\\ &\quad+\Psi_1\left(R_{n-1}-\varphi_n^{(n)}+\frac{n}{1}\varphi_n^{(n-1)}\Psi_1-\frac{n(n-1)}{1.2}\varphi_n^{(n-2)}\Psi_2+\ldots\right)\binom{1}{0}\\ &=\varphi_0-\left(R_{n-1}'-\Psi_1R_{n-1}\right)\binom{1}{0}+\varphi_{n+1}^{(n+1)}-\frac{n+1}{1}\varphi_{n+1}^{(n)}+\ldots;\end{aligned}$$

and in this expression

$$\phi_0-\left(R'_{n-1}-\Psi_1 R_{n-1}\right)\binom{1}{0}=R_n,$$

while the general term of the ϕ-series, *i. e.* the coefficient of $\phi_{n+1}^{(n-r)}$, will be

$$\begin{aligned}&(-)^{r-1}\left\{\frac{n(n-1)..(n-r+1)}{1.2..r}\Psi_r-\frac{n(n-1)..(n-r)}{1.2..(r+1)}\Psi_{r+1}-\frac{n(n-1)..(n-r+1)}{1.2..r}\Psi_1\Psi_r\right\}\\ =&(-)^{r-1}\frac{n(n-1)..(n-r+1)}{1.2..r}\left\{\Psi_r-\Psi_1\Psi_r-\frac{n-r}{r+1}\Psi_{r+1}\right\}\\ =&(-)^r\frac{(n+1)n(n-1)..(n-r+1)}{1.2..(r+1)}\Psi_{r+1},\end{aligned}$$

which proves the general case; so that generally

$$R_{n+1}=R_n\pm\phi_{n+1}^{(n+1)}\mp\frac{n+1}{1}\phi_{n+1}^{(n)}\Psi_1\pm\frac{(n+1)n}{1.2}\phi_{n+1}^{(n)}\Psi_2\mp..., \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

the upper or lower sign being taken according as $(n+1)$ is even or odd; where R_{n+1} is the remainder after external division $\phi_{n+1}(\varrho)\pi^{n+1}+\phi_n(\varrho)\pi^n+..\phi_0$ by $\psi_1(\varrho)\pi+\psi_0(\varrho)$.

For the quotients $Q_1, Q_2, ..$ we have immediately

$$\begin{aligned}Q_1&=\frac{1}{\psi_1}\phi_1,\\ Q_2&=\frac{1}{\psi_1}\left\{\phi_2\pi+R_1\binom{1}{0}\right\},\\ Q_3&=\frac{1}{\psi_1}\left\{\phi_3\pi^2+R_1\binom{2}{0}\pi+R_2\binom{1}{0}\right\}.\\ &.. \quad ...\\ Q_n&=\frac{1}{\psi_1}\left\{\phi_n\pi^{n-1}+R_1\binom{n-1}{0}\pi^{n-2}+R_2\binom{n-2}{0}\pi^{n-3}+..R_{n-1}\binom{1}{0}\right\}.\end{aligned}$$

This completes the solution of the problem of division by a linear factor, both internal and external.

§ 3. *To divide* $\Sigma_{n=0}^{n=N}\phi_n\pi^n$ *internally by* $\Sigma_{m=0}^{m=M}\psi_m\pi^m$.

The first term in the quotient will obviously be

$$\frac{\phi_N}{\psi_M}\pi^{N-M}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

and the product of this into the divisor may, by means of LEIBNITZ's theorem, be written thus:

$$\frac{\phi_N}{\psi_m}\Sigma_{p=0}^{p=N-M}[N-M, p]\,\Sigma_{m=0}^{m=M}\psi_m^{(N-M-p)}\pi^{m+p}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

where $\psi_m^{(N-M-p)}$ means the result of the operation π^{N-M-p} or ψ_m alone, and

$$[N-M, p]=\frac{(N-M)(N-M-1)..(N-M-p+1)}{1.2..p}.$$

Then the remainder after subtraction from the dividend may be written thus:

$$\Sigma_{m+p=0}^{m+p=N-1}\left\{\varphi_{m+p}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\varphi_N[N-M,p]\psi_m^{(N-M-p)}\right\}x^{m+p}, \quad . \quad . \quad . \quad (3.)$$

since the coefficient of x^N vanishes. With a view to the second term in the dividend, the first term of the remainder (3.) is

$$\left\{\varphi_{N-1}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\varphi_N[N-M,p]\psi_m^{(N-M-p)}\right\}x^{N-1}, \quad . \quad . \quad . \quad (4.)$$

in which the limits of p and m are subject to the further condition $p+m=N-1$. The terms under the sign of summation will be evaluated hereafter. Putting the expression

$$(4)=\Phi_1 x^{N-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

and, for the sake of symmetry,

$$\varphi_N=\Phi_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

the first and second terms in the quotient will be $\frac{\Phi_0}{\psi_M}x^{N-M}$, and $\frac{\Phi_1}{\psi_M}x^{N-M-1}$, respectively; and, in the same manner as (2.), the product of the second term of the quotient into the divisor may be written thus,

$$\frac{\Phi_1}{\psi_M}\Sigma_{p_1=0}^{p_1=N-M-1}\Sigma_{m=0}^{m=M}[N-M-1,p]\psi_m^{(N-M-p-1)}x^{m+p_1}, \quad . \quad . \quad . \quad . \quad (7.)$$

and the remainder thus:

$$\left.\begin{aligned}&\Sigma_{m+p=0}^{m+p=N-2}\left\{\varphi_{m+p}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\Phi_0[N-M,p]\psi_m^{(N-M-p)}\right\}x^{m+p}\\ -&\Sigma_{m+p_1=0}^{m+p_1=N-2}\left\{\frac{1}{\psi_M}\Sigma_{p_1=0}^{p_1=N-M-1}\Sigma_{m=0}^{m=M}\Phi_1[N-M-1,p_1]\psi_m^{(N-M-p_1-1)}\right\}x^{m+p_1}.\end{aligned}\right\} \quad . \quad (8.)$$

But since, when $p_1=N-M$, $[N-M-1,p_1]=0$, we may, without altering the value of (8.), change the superior limit of p_1 from $N-M-1$, to $N-M$; and by this means we may write the remainder (8.) in the following form:

$$\Sigma_{m+p=0}^{m+p=N-2}\left\{\varphi_{m+p}-\frac{1}{\psi_m}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}(\Phi_0[N-M,p]\psi_m^{(N-M-p)}+\Phi_1[N-M-1,p]\psi_m^{(N-M-p-1)})\right\}x^{m+p}. \quad (9.)$$

Similarly, calling the first term of (9.) $\Phi_2 x^{N-2}$, the third term in the quotient will be $\frac{\Phi_2}{\psi_M}x^{N-M-2}$, and the corresponding remainder

$$\left.\begin{aligned}&\Sigma_{m+p=0}^{m+p=N-3}\left\{\varphi_{m+p}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}(\Phi_0[N-M,p]\psi_m^{(N-M-p)}+\Phi_1[N-M-1,p]\psi_m^{(N-M-p-1)}\right.\\ &\qquad\left.+\Phi_2[N-M-2,p]\psi_m^{(N-M-p-2)})\right\}x^{m+p};\end{aligned}\right\} \quad (10.)$$

and so generally the $(r+1)$th term in the quotient will be $\frac{\Phi_r}{\psi_M}x^{N-M-r}$, where

$$\Phi_r=\varphi_{N-r}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\Sigma_{q=0}^{q=r-1}\Phi_q[N-M-q,p]\psi_m^{(M-N-p-q)}, \quad . \quad . \quad . \quad . \quad (11.)$$

and the $(r+1)$th remainder

$$\Sigma_{m+p=N-r-1}^{m+p=0}\left\{\varphi_{m+p}-\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\Sigma_{q=0}^{q=r}\Phi_q[N-M-q,p]\psi_m^{(N-M-p-q)}\right\}\pi^{m+p}. \quad . \quad . \quad (12.)$$

The final remainder is the $(N-M+1)$th; and the expression will be derived from (10.) by replacing r by $N-M$.

It remains to develope the terms under the sign of summation in the expressions for the Φs. In the first place $\Phi_0=\varphi_M$ simply. In the case of Φ_1, the limiting values of p and m are

$$p=0,\ 1,\ \ldots\ N-M,$$
$$m=0,\ 1,\ \ldots\ M,$$
$$p+m=N-1.$$

These give as the only admissible values

$$p=N-M\quad,\ N-M-1,$$
$$m=\quad M-1,\quad M\quad,$$

and consequently

$$\Phi_1=\varphi_{N-1}-\frac{\Phi_0}{\psi_M}\{\psi_{M-1}+(N-M)\psi_M'\}.$$

In the case of Φ_2, the only admissible values are

$$p=N-M\quad,\ N-M-1,\ N-M-2,$$
$$m=\quad M-2,\quad M-1,\quad M\quad,$$

giving

$$\Phi_2=\varphi_{N-2}-\frac{1}{\psi_M}\left\{\Phi_0\left(\psi_{M-2}+(N-M)\psi_{M-1}+\frac{(N-M)(N-M-1)}{1.2}\psi_M''\right)\right.$$
$$\left.+\Phi_1\left(\qquad\qquad\psi_{M-1}+(N-M-1)\qquad\psi_M'\right)\right\}.$$

Before proceeding further, it may be well to illustrate these formulæ by an example.

Taking the case of $N=4$, $M=3$, we may determine the quotient and last remainder of internal division of

$$\varphi_4(\varrho)\pi^4+\varphi_3(\varrho)\pi^3+\varphi_2(\varrho)\pi^2+\varphi_1(\varrho)\pi+\varphi_0(\varrho)$$

by

$$\psi_3(\varrho)\pi^3+\psi_2(\varrho)\pi^2+\psi_1(\varrho)\pi+\psi_0(\varrho),$$

and thence the conditions that the latter may be an internal factor of the former.

By the formulæ given above, we have

$$\Phi_0=\varphi_3,$$
$$\Phi_1=\varphi_2-\frac{\varphi_3}{\psi_3}\{\psi_1+2\psi_2'\},$$
$$\Phi_2=\varphi_1-\frac{1}{\psi_3}\left\{\varphi_3(\psi_0+2\psi_1'+\psi_2'')+\left(\varphi_2-\frac{\varphi_3}{\psi_3}(\psi_1+2\psi_2')\right)(\psi_1+\psi_2')\right\},$$

which will determine the quotient

$$\frac{1}{\psi_3}(\Phi_0\pi^2+\Phi_1\pi+\Phi_2).$$

The last remainder is

$$\Sigma_{m+p=1}^{m+p=1}\left\{\varphi_{m+p}-\frac{1}{\psi_2}\Sigma_{p=0}^{p=1}\Sigma_{m=1}^{m=1}\Sigma_{q=0}^{q=1}\Phi_q[N-M-q,p]\psi_m^{(N-M-p-q)}\right\}\pi^{m+p};$$

whence for

$$m+p=1,\ i.\,e.\ p=0,\ m=1,\ \text{or}\ p=1,\ m=0,$$

we have

	$q=0$	$q=1$	$q=2$
$p=0,$	$[2,0]=1,$	$[1,0]=1,$	$[0,0]=1,$
$p=1,$	$[2,1]=2,$	$[1,1]=1,$	$[0,1]=0.$

Hence for $m=1$ the above expression gives

$$\Phi_0\psi_1''+\Phi_1\psi_1'+\Phi_2\psi_1,$$

and for $m=0$ it gives

$$2\Phi_0\psi_0'+\Phi_1\psi_0.$$

Again, for $m+p=0$, *i. e.* $p=0$, $m=0$, we have

$$\Phi_0\psi_0''+\Phi_1\psi_0'+\Phi_2\psi_0.$$

Hence the total remainder is

$$\left\{\varphi_1-\frac{1}{\psi_2}(\Phi_0\psi_1''+\Phi_1\psi_1'+\Phi_2\psi_1+2\Phi_0\psi_0'+\Phi_1\psi_0)\right\}\pi+\left\{\varphi_0-\frac{1}{\psi_2}(\Phi_0\psi_0''+\Phi_1\psi_0'+\Phi_2\psi_0)\right\}.$$

It may be useful to compare these results with the actual division, in the above example.

$$\begin{array}{l}
\psi_2\pi^2+\psi_1\pi+\psi_0\Big)\varphi_4\pi^4+\varphi_3\pi^3+\varphi_2\pi^2+\varphi_1\pi+\varphi_0\Big(\frac{\Phi_0}{\psi_2}\pi^2+\frac{\Phi_1}{\psi_2}\pi+\frac{\Phi_2}{\psi_2}\\
\qquad\varphi_4\pi^4+2\varphi_4\frac{\psi_2'}{\psi_2}\pi^3+\varphi_4\frac{\psi_2''}{\psi_2}\pi^2\\
\qquad\quad+\varphi_4\frac{\psi_1}{\psi_2}\pi^3+2\varphi_4\frac{\psi_1'}{\psi_2}\pi^2+\varphi_4\frac{\psi_1''}{\psi_2}\pi\\
\qquad\qquad+\varphi_4\frac{\psi_0}{\psi_2}\pi^2+2\varphi_4\frac{\psi_0'}{\psi_2}\pi+\varphi_4\frac{\psi_0''}{\psi_2}\\
\hline
\Phi_0\pi^2+\Big(\varphi_3-\frac{\varphi_4}{\psi_2}(\psi_2''+2\psi_1'+\psi_0)\Big)\pi^2+\Big(\varphi_1-\frac{\varphi_4}{\psi_2}(\psi_1''+2\psi_0')\Big)\pi+\Big(\varphi_0-\frac{\varphi_4}{\psi_2}\psi_0''\Big)\\
\Phi_1\pi^2+\qquad\Phi_1\frac{\psi_2'}{\psi_2}\pi^2\qquad\qquad\qquad

Returning to the Φ functions, and writing for convenience the symbolical expression

$$\Sigma_{p=0}^{p=N-M-r}\Sigma_{m=0}^{m=M}[N-M-r, p]\psi_m^{(N-M-p-r)}=S_s[N-M-r, p]\psi_m^{(N-M-p-r)},$$
$$p+m=N-s,$$

where the suffix s indicates the number of units whereby the sum $p+m$ is less than N, we have

$$\Phi_0= \phi_N,$$

$$\psi_M\Phi_1= \psi_M\phi_{N-1}-S_1\Phi_0[N-M, p]\psi_m^{(N-M-p)}$$
$$= \begin{vmatrix} \phi_{N-1} & S_1[N-M, p]\psi_m^{(N-M-p)} \\ \phi_N & \psi_M \end{vmatrix}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13.)$$

$$\psi_M^2\Phi_2= \psi_M^2\phi_{N-2}-S_2(\Phi_0[N-M, p]\psi_m^{(N-M-p)} +\Phi_1\ [N-M-1, p]\psi_m^{(N-M-p-1)})$$
$$= \psi_M^2\phi_{N-2}-\Phi_0S_2\ [N-M, p]\psi_m^{(N-M-p)} -\Phi_1S_2[N-M-1, p]\psi_m^{(N-M-p-1)}$$
$$= \begin{vmatrix} \phi_{N-2} & S_2[N-M-1, p]\psi_m^{(N-M-p-1)} & S_2[N-M, p]\psi_m^{(N-M-p)} \\ \phi_{N-1} & S_1[N-M-1, p]\psi_m^{(N-M-p-1)} & S_1[N-M, p]\psi_m^{(N-M-p)} \\ \phi_N & 0 & S_0[N-M, p]\psi_m^{(N-M-p)}; \end{vmatrix} \quad \ldots \quad (14.)$$

and generally,

$$\psi_M\Phi_r= \begin{vmatrix} \phi_{N-r} & S_r\ [N-M-r+1, p]\psi_m^{(N-M-p-r+1)} \ldots S_r\ [N-M-1, p]\psi_m^{(N-M-p-1)} & S_r[N-M, p]\psi_m^{(N-M-p)} \\ \phi_{N-r+1} & S_{r-1}[N-M-r+1, p]\psi_m^{(N-M-p-r+1)} \ldots S_{r-1}[N-M-1, p]\psi_m^{(N-M-p-1)} & S_r[N-M, p]\psi_m^{(N-M-p)} \\ \cdot & \cdot \quad\quad\quad \cdot & \cdot \\ \cdot & \cdot \quad\quad\quad \cdot & \cdot \\ \phi_N & 0 \quad\quad\quad 0 & S_0[N-M, p]\psi_m^{(N-M-p)}. \end{vmatrix} \quad (15.)$$

These formulæ give, for the example discussed above,

$$\Phi_0= \phi_4$$

$$\psi_2\Phi_1= \begin{vmatrix} \phi_3 & \psi_1+2\psi_2' \\ \phi_4 & \psi_3 \end{vmatrix}$$

$$\psi_2^2\Phi_2= \begin{vmatrix} \phi_2 & \psi_1+\psi_2' & \psi_0+2\psi_1'+\psi_2'' \\ \phi_3 & \psi_3 & \psi_1+2\psi_2' \\ \phi_4 & 0 & \psi_3. \end{vmatrix}$$

And for the final remainder, the coefficient of ϖ,

$$\psi_2^2\left\{\phi_1-\frac{1}{\psi_2}\Phi_0(\psi_1'+2\psi_0')-\frac{1}{\psi_2}\Phi_1(\psi_1'+\psi_0)-\frac{1}{\psi_2}\Phi_2\psi_1\right\}= \begin{vmatrix} \phi_1 & \psi_1 & \psi_0+\psi_1' & 2\psi_0'+\psi_1'' \\ \phi_2 & \psi_2 & \psi_1+\psi_2' & \psi_0+2\psi_1'+\psi_2'' \\ \phi_3 & 0 & \psi_2 & \psi_1+2\psi_2' \\ \phi_4 & 0 & 0 & \psi_3. \end{vmatrix}$$

and the terms independent of $\varpi =$

$$\psi_3^2\left\{\varphi_0-\frac{1}{\psi_3}(\Phi_0\psi_3''+\Phi_1\psi_3'+\Phi_2\psi_3)\right\} = \begin{vmatrix} \varphi_0 & \psi_0 & \psi_0' & \psi_0'' \\ \varphi_3 & \psi_3 & \psi_1+\psi_3' & \psi_0+2\psi_1'+\psi_3'' \\ \varphi_2 & 0 & \psi_1 & \psi_1+2\psi_2' \\ \varphi_4 & 0 & 0 & \psi_2 \end{vmatrix}$$

both of which may be comprised under the single formula

$$\begin{vmatrix} \varpi & \varphi_0 & \psi_0 & \psi_0' & & \psi_0'' \\ 1 & \varphi_1 & \psi_1 & \psi_0+\psi_1' & & 2\psi_0'+\psi_1'' \\ 0 & \varphi_2 & \psi_2 & \psi_1+\psi_2' & & \psi_0+2\psi_1'+\psi_2'' \\ 0 & \varphi_3 & 0 & \psi_2 & & \psi_1+2\psi_2' \\ 0 & \varphi_4 & 0 & 0 & & \psi_2 \end{vmatrix}$$

§ 4. *To divide* $\Sigma_{n=0}^{n=N}\varphi_n\varpi^n$ *externally by* $\Sigma_{n=0}^{n=M}\psi_n\varpi^n$.

The first term in the quotient will be

$$\frac{\varphi_N}{\psi_M}\varpi^{N-M}, \quad \ldots \quad (1.)$$

the product of this into the divisor

$$\Sigma_{n=0}^{n=M}\psi_n\Sigma_{p=0}^{p=m}[m,p]\left(\frac{\varphi_N}{\psi_M}\right)^{(m-p)}\varpi^{N-M+p}, \quad \ldots \quad (2.)$$

and the remainder

$$\Sigma_{n=0}^{n=N}\varphi_n\varpi^n-\Sigma_{n=0}^{n=M}\psi_n\Sigma_{p=0}^{p=m}[m,p]\left(\frac{\varphi_N}{\psi_M}\right)^{(m-p)}\varpi^{N-M+p}; \quad \ldots \quad (3.)$$

the first term of which is

$$\left.\begin{aligned} &\left\{\varphi_{N-1}-\Sigma_{n=0}^{n=M}\psi_n[m,M-1]\left(\frac{\varphi_N}{\psi_M}\right)^{(m-M+1)}\right\}\varpi^{N-1} \\ =&\left\{\varphi_{N-1}-\psi_{M-1}\left(\frac{\varphi_N}{\psi_M}\right)-M\psi_M\left(\frac{\varphi_N}{\psi_M}\right)'\right\}\varpi^{N-1}. \end{aligned}\right\} \quad \ldots \quad (4.)$$

Let $\varphi_N=\Phi_0$, and let the coefficient of ϖ^{N-1} above written $=\Phi_1$; then the next term in the quotient will be $\frac{\Phi_1}{\psi_M}\varpi^{N-M-1}$, and the product of this into the divisor will take the same form as (2.), writing only Φ_1 for Φ_0, or φ_N. The remainder will be the same as (3.), with the addition of the term

$$-\Sigma_{n=0}^{n=M}\psi_n\Sigma_{p=0}^{p=m}[m,p]\left(\frac{\Phi_1}{\psi_M}\right)^{m-p}\varpi^{N-M+p-1}, \quad \ldots \quad (5.)$$

and the first term of the entire remainder will be

$$\left.\begin{aligned} &\left\{\varphi_{N-2}-\Sigma_{n=0}^{n=M}\psi_n\left[[m,M-2]\left(\frac{\Phi_0}{\psi_M}\right)^{(m-M+2)}+[m,M-1]\left(\frac{\Phi_1}{\psi_M}\right)^{(m-M+1)}\right]\right\}\varpi^{N-2} \\ =&\left\{\varphi_{N-2}-\psi_{M-2}\left(\frac{\Phi_0}{\psi_M}\right)-(M-1)\psi_{M-1}\left(\frac{\Phi_0}{\psi_M}\right)'-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\Phi_0}{\psi_M}\right)''\right. \\ &\qquad\qquad -\psi_{M-1}\left(\frac{\Phi_1}{\psi_M}\right)-M\psi_M\left(\frac{\Phi_1}{\psi_M}\right)'\Big\}\varpi^{N-2}; \end{aligned}\right\} \quad \ldots \quad (6.)$$

and if we make this expression $=\Phi_s x^{N-s}$, the next remainder will be

$$\Sigma_{n=0}^{n=N}\varphi_n x^n-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\left\{\left(\frac{\Phi_0}{\psi_m}\right)^{(m-p)}x^s+\left(\frac{\Phi_1}{\psi_m}\right)^{(m-p)}x+\left(\frac{\Phi_0}{\psi_m}\right)^{(m-p)}\right\}x^{N-M+p-s}, \quad (7.)$$

the first term of which is

$$\left.\begin{aligned}&\left\{\varphi_{N-s}-\Sigma_{m=0}^{m=M}\psi_m\left[[m,M-3]\left(\frac{\Phi_0}{\psi_M}\right)^{m-M+3}+[m,M-2]\left(\frac{\Phi_1}{\psi_M}\right)^{m-M+2}+[m,M-1]\left(\frac{\Phi_0}{\psi_M}\right)^{m-M+1}\right]\right\}x^{N-s}\\ &=\left\{\varphi_{N-s}-\psi_{M-s}\left(\frac{\Phi_0}{\psi_m}\right)-(M-2)\psi_{M-s}\left(\frac{\Phi_0}{\psi_m}\right)'-\frac{(M-1)(M-2)}{1.2}\psi_{M-1}\left(\frac{\Phi_0}{\psi_M}\right)''-\frac{M(M-)(M-2)}{1.2.3}\psi_M\left(\frac{\Phi_0}{\psi_M}\right)'''\right.\\ &\qquad-\psi_{M-s}\left(\frac{\Phi_1}{\psi_M}\right)-(M-1)\psi_{M-1}\left(\frac{\Phi_1}{\psi_M}\right)'-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\Phi_1}{\psi_M}\right)''\\ &\qquad\left.-\psi_{M-1}\left(\frac{\Phi_2}{\psi_M}\right)-M\psi_M\left(\frac{\Phi_2}{\psi_M}\right)'\right\}x^{N-s};\end{aligned}\right\}\quad(8.)$$

and generally the $(r+1)$th term in the quotient will be

$$\frac{\Phi_r}{\psi_M}x^{N-M-r},$$

and the $(r+1)$th remainder

$$\Sigma_{n=0}^{n=N}\varphi_n x^n-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\Sigma_{q=0}^{q=r}\left(\frac{\Phi_q}{\psi_M}\right)^{(m-p)}x^{N-M+p-q}. \quad (9.)$$

The formation of the Φs is as follows:—

$$\left.\begin{aligned}&\Phi_0=\varphi_N,\\ &\Phi_1=\varphi_{N-1}-\Sigma_{m=0}^{m=M}\psi_m\,[m,M-1]\left(\frac{\Phi_0}{\psi_M}\right)^{(m-M+1)},\\ &\Phi_2=\varphi_{N-2}-\Sigma_{m=0}^{m=M}\psi_m\left\{[m,M-1]\left(\frac{\Phi_1}{\psi_M}\right)^{(m-M+1)}+[m,M-2]\left(\frac{\Phi_0}{\psi_M}\right)^{(m-M+2)}\right\},\\ &\Phi_3=\varphi_{N-3}-\Sigma_{m=0}^{m=M}\psi_m\left\{[m,M-1]\left(\frac{\Phi_2}{\psi_M}\right)^{(m-M+1)}+[m,M-2]\left(\frac{\Phi_1}{\psi_M}\right)^{(m-M+2)}\right.\\ &\qquad\left.+[m,M-3]\left(\frac{\Phi_0}{\psi_M}\right)^{(m-M+3)}\right\},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\Phi_s=\varphi_{N-s}-\Sigma_{m=0}^{m=M}\psi_m\left\{[m,M-1]\left(\frac{\Phi_{s-1}}{\psi_M}\right)^{(m-M+1)}+[m,M-2]\left(\frac{\Phi_{s-2}}{\psi_M}\right)^{(m-M+2)}\right.\\ &\qquad\left.+-[m,1]\left(\frac{\Phi_{s-M+1}}{\psi_M}\right)^{(M-1)}+[m,0]\left(\frac{\Phi_{s-M}}{\psi_M}\right)^{(M)}\right\}.\end{aligned}\right\}\quad(10.)$$

The final remainder is given by the formula

$$\Sigma_{n=0}^{n=N}\varphi_n x^n-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\Sigma_{q=0}^{q=N-M}\left(\frac{\Phi_q}{\psi_M}\right)^{(m-p)}x^{N-M+p-q}; \quad (11.)$$

and the general term of this x^{N-s} is to be found as follows:

$$N-M+p-q=N-s,$$

i. e.

$$p-q=M-s.$$

Then we have for

$$\left.\begin{aligned}
&p=\mathrm{M}, \quad q=s,\\
&\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[m,\ \mathrm{M}\ ]\left(\frac{\Phi_s}{\psi_\mathrm{M}}\right)^{(m-\mathrm{M})}=\Phi_s;\\
&p=\mathrm{M}-1,\ q=s-1,\\
&\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[m,\ \mathrm{M}-1]\left(\frac{\Phi_{s-1}}{\psi_\mathrm{M}}\right)^{(m-\mathrm{M}+1)}=\mathrm{M}\psi_\mathrm{M}\left(\frac{\Phi_{s-1}}{\psi_\mathrm{M}}\right)'+\psi_{\mathrm{M}-1}\left(\frac{\Phi_{s-1}}{\psi_\mathrm{M}}\right);\\
&p=\mathrm{M}-2,\ q=s-2,\\
&\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[m,\ \mathrm{M}-2]\left(\frac{\Phi_{s-2}}{\psi_\mathrm{M}}\right)^{(m-\mathrm{M}+2)}=\frac{\mathrm{M}(\mathrm{M}-1)}{1.2}\psi_\mathrm{M}\left(\frac{\Phi_{s-2}}{\psi_\mathrm{M}}\right)''+(\mathrm{M}-1)\psi_{\mathrm{M}-1}\left(\frac{\Phi_{s-2}}{\psi_\mathrm{M}}\right)'+\psi_{\mathrm{M}-2}\left(\frac{\Phi_{s-2}}{\psi_\mathrm{M}}\right);\\
&p=1, \quad q=s-\mathrm{M}+1,\\
&\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[\ m,\ 1\ ]\left(\frac{\Phi_{s-\mathrm{M}+1}}{\psi_\mathrm{M}}\right)^{(m-1)}=\mathrm{M}\psi_\mathrm{M}\left(\frac{\Phi_{s-\mathrm{M}+1}}{\psi_\mathrm{M}}\right)^{(\mathrm{M}-1)}+(\mathrm{M}-1)\psi_{\mathrm{M}-1}\left(\frac{\Phi_{s-\mathrm{M}+1}}{\psi_\mathrm{M}}\right)^{(\mathrm{M}-2)}+\ldots\psi_1\left(\frac{\Phi_{s-\mathrm{M}-1}}{\psi_\mathrm{M}}\right);\\
&p=0, \quad q=s-\mathrm{M},\\
&\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[\ m,\ 0\ ]\left(\frac{\Phi_{s-\mathrm{M}}}{\psi_\mathrm{M}}\right)^{(m)}=\psi_\mathrm{M}\left(\frac{\Phi_{s-\mathrm{M}}}{\psi_\mathrm{M}}\right)^{(\mathrm{M})}+\psi_{\mathrm{M}-1}\left(\frac{\Phi_{s-\mathrm{M}}}{\psi_\mathrm{M}}\right)^{(\mathrm{M}-1)}+\ldots\psi_0\left(\frac{\Phi_{s-\mathrm{M}}}{\psi_\mathrm{M}}\right);
\end{aligned}\right\}\quad(12.)$$

the sum of all which will be found, on reference to the expressions for the formation of the Φs, to be equal to the first term of Φ_s, viz. $\varphi_{\mathrm{N}-s}$; and consequently the coefficient of $\varpi^{\mathrm{N}-s}$ vanishes for all values of s not exceeding the greatest value of q, viz. $\mathrm{N}-\mathrm{M}$. If, however, s is greater than $\mathrm{N}-\mathrm{M}$, by any number t, so that $s=\mathrm{N}-\mathrm{M}+t$, then the pairs of values

$$\begin{aligned}
&p=\mathrm{M}, && q=s,\\
&p=\mathrm{M}-1, && q=s-1,\\
&\quad\ldots && \quad\ldots\\
&p=\mathrm{M}-t+1, && q=s-t+1
\end{aligned}$$

are inadmissible, and the pairs

$$\begin{aligned}
&p=\mathrm{M}-t, && q=s-t,\\
&p=\mathrm{M}-t-1, && q=s-t-1,\\
&\quad\ldots && \quad\ldots\\
&p=0, && q=s-\mathrm{M}
\end{aligned}$$

alone remain; and consequently the coefficients of the powers of ϖ, for $s>\mathrm{N}-\mathrm{M}$, do not vanish, and the remainder consists of a series of terms, the index of the highest power of ϖ being

$$\mathrm{N}-\mathrm{M}+p-q=\mathrm{N}-s=\mathrm{N}-\mathrm{N}+\mathrm{M}-1=\mathrm{M}-1,$$

as it should be.

As an example, we may calculate by means of the formulæ given above, the final remainder in the external division of

$$\varphi_4(\varrho)\varpi^4+\varphi_3(\varrho)\varpi^3+\varphi_2(\varrho)\varpi^2+\varphi_1(\varrho)\varpi+\varphi_0(\varrho)$$

by

$$\psi_2(\varrho)\pi^2+\psi_1(\varrho)\pi+\psi_0(\varrho),$$

viz.

$$\Sigma_{n=0}^{n=4}\varphi_n\pi^n-\Sigma_{m=0}^{m=3}\Sigma_{p=0}^{p=m}\Sigma_{q=0}^{q=3}\psi_m[m,\,p]\left(\frac{\Phi_q}{\psi_m}\right)^{(m-p)}\pi^{m+p-q}.$$

The conditions

$$\begin{aligned}
&p=2,\ q=0 \text{ give } \Phi_4\pi^4\\
&p=2,\ q=1 \text{ —— } \Phi_1\pi^3\\
&p=2,\ q=2 \text{ —— } \Phi_2\pi^2\\
&p=1,\ q=0 \text{ —— } \left\{2\psi_2\left(\frac{\Phi_0}{\psi_2}\right)'+\psi_1\left(\frac{\Phi_0}{\psi_2}\right)\right\}\pi^3\\
&p=1,\ q=1 \text{ —— } \left\{2\psi_2\left(\frac{\Phi}{\psi_2}\right)'+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)\right\}\pi^2\\
&p=1,\ q=2 \text{ —— } \left\{2\psi_2\left(\frac{\Phi_2}{\psi_2}\right)'+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)\right\}\pi\\
&p=0,\ q=0 \text{ —— } \left\{\psi_2\left(\frac{\Phi_0}{\psi_2}\right)''+\psi_1\left(\frac{\Phi_0}{\psi_2}\right)'+\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\right\}\pi^2\\
&p=0,\ q=1 \text{ —— } \left\{\psi_2\left(\frac{\Phi_1}{\psi_2}\right)''+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'+\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\right\}\pi\\
&p=0,\ q=2 \text{ —— } \left\{\psi_2\left(\frac{\Phi_2}{\psi_2}\right)''+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'+\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\right\}.
\end{aligned}$$

Hence taking the sum of all the terms, the coefficients of π^4, π^3, π^2 vanish, and the final remainder is

$$\left\{\varphi_1-2\psi_2\left(\frac{\Phi_2}{\psi_2}\right)'-\psi_1\left(\frac{\Phi_2}{\psi_2}\right)-\psi_2\left(\frac{\Phi_1}{\psi_2}\right)''-\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'-\psi_0\left(\frac{\Phi_1}{\psi_2}\right)\right\}\pi+\varphi_0-\psi_2\left(\frac{\Phi_2}{\psi_2}\right)''-\psi_1\left(\frac{\Phi_2}{\psi_2}\right)'-\psi_0\left(\frac{\Phi_2}{\psi_2}\right).$$

These results may be compared with the actual division,

$$\begin{array}{l}
\psi_2\pi^2+\psi_1\pi+\psi_0\Big)\varphi_4\pi^4+\varphi_3\pi^3+\varphi_2\pi^2+\varphi_1\pi+\varphi_0\left(\dfrac{\Phi_0}{\psi_2}\pi^2+\dfrac{\Phi_1}{\psi_2}\pi+\dfrac{\Phi_2}{\psi_2}\right.\\
\qquad\qquad\varphi_4\pi^4+2\psi_2\left(\dfrac{\Phi_0}{\psi_2}\right)'\pi^3+\psi_2\left(\dfrac{\Phi_0}{\psi_2}\right)''\pi^2\\
\qquad\qquad\quad+\psi_1\left(\dfrac{\Phi_0}{\psi_2}\right)\pi^3+\psi_1\left(\dfrac{\Phi_0}{\psi_2}\right)'\pi^2\\
\qquad\qquad\qquad+\psi_0\left(\dfrac{\Phi_0}{\psi_2}\right)\pi^2\\
\hline
\Phi_1\pi^3+\left\{\varphi_2-\psi_2\left(\dfrac{\Phi_0}{\psi_2}\right)''-\psi_1\left(\dfrac{\Phi_0}{\psi_2}\right)'-\psi_0\left(\dfrac{\Phi_0}{\psi_2}\right)\right\}\pi^2+\varphi_1\pi+\varphi_0\\
\Phi_1\pi^3+\qquad\qquad 2\psi_2\left(\dfrac{\Phi_1}{\psi_2}\right)'\pi^2+\psi_2\left(\dfrac{\Phi_1}{\psi_2}\right)''\pi\\
\qquad\qquad+\psi_1\left(\dfrac{\Phi_1}{\psi_2}\right)\pi^2+\psi_1\left(\dfrac{\Phi_1}{\psi_2}\right)'\pi\\
\qquad\qquad\qquad+\psi_0\left(\dfrac{\Phi_1}{\psi_2}\right)\pi\\
\hline
\end{array}$$

$$\left.\begin{array}{rr} \Phi_3\varpi^3+\left\{\varphi_1-\psi_2\left(\frac{\Phi_1}{\Psi_2}\right)''-\psi_1\left(\frac{\Phi_1}{\Psi_2}\right)'-\psi_0\left(\frac{\Phi_1}{\Psi_2}\right)\right\}\varpi+\varphi_0 & \\ \Phi_3\varpi^3 & +2\psi_2\left(\frac{\Phi_2}{\Psi_2}\right)'\varpi+\psi_2\left(\frac{\Phi_2}{\Psi_2}\right) \\ & +\ \psi_1\left(\frac{\Phi_2}{\Psi_2}\right)\varpi+\psi_1\left(\frac{\Phi_2}{\Psi_2}\right) \\ & +\psi_0\left(\frac{\Phi_2}{\Psi_2}\right) \end{array}\right\}$$

$$\Phi_3\varpi+\left\{\varphi_0-\psi_2\left(\frac{\Phi_2}{\Psi_2}\right)''-\psi_1\left(\frac{\Phi_2}{\Psi_2}\right)'-\psi_0\left(\frac{\Phi_2}{\Psi_2}\right)\right\},$$

which agrees with the results found above.

§ 5. *To divide* $\Sigma_{n=0}^{n=N}\xi^n\varphi_n(\varpi)$ *internally by* $\Sigma_{m=0}^{m=M}\xi^m\psi_m(\varpi)$.

The first term of the quotient will be

$$\xi^{N-M}\frac{\varphi_N(\varpi-M)}{\psi_M(\varpi-M)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.)$$

and the product of this into the divisor,

$$\Sigma_{m=0}^{m=M}\xi^{N-M+m}\frac{\psi_m(\varpi)}{\psi_M(\varpi-M+m)}\varphi_N(\varpi-M+m). \quad \ldots \quad (2.)$$

The first term of the remainder will then be

$$\xi^{N-1}\left\{\varphi_{N-1}(\varpi)-\frac{\psi_{M-1}(\varpi)}{\psi_M(\varpi-1)}\varphi_N(\varpi-1)\right\}=\xi_{N-1}\frac{1}{\psi_M(\varpi-1)}\begin{vmatrix}\varphi_{N-1}(\varpi) & \psi_{M-1}(\varpi)\\ \varphi_N(\varpi-1) & \psi_M(\varpi-1),\end{vmatrix} \quad \ldots \quad (3.)$$

and consequently the second term in the quotient will be

$$\left.\begin{array}{l} \xi^{N-M-1}\left\{\varphi_{N-1}(\varpi-M)-\frac{\psi_{M-1}(\varpi-M)}{\psi_M(\varpi-M-1)}\varphi_N(\varpi-M-1)\right\}\frac{1}{\psi_M(\varpi-M)} \\ =\xi^{N-M-1}\frac{1}{\psi_M(\varpi-M)\psi_M(\varpi-M-1)}\begin{vmatrix}\varphi_{N-1}(\varpi-M) & \psi_{M-1}(\varpi-M)\\ \varphi_N(\varpi-M-1) & \psi_M(\varpi-M-1)\end{vmatrix} \end{array}\right\}. \quad \ldots \quad (4.)$$

The first term of the second remainder will then be

$$\left.\begin{array}{l} \xi^{N-2}\left\{\varphi_{N-2}(\varpi)-\frac{\psi_{M-1}(\varpi)}{\psi_M(\varpi-1)\psi_M(\varpi-2)}\begin{vmatrix}\varphi_{N-1}(\varpi-1) & \psi_{M-1}(\varpi-1)\\ \varphi_N(\varpi-2) & \psi_M\ (\varpi-2)\end{vmatrix}\right\} \\ =\xi^{N-2}\frac{1}{\psi_M(\varpi-1)\psi_M(\varpi-2)}\begin{vmatrix}\varphi_{N-2}(\varpi) & \psi_{M-1}(\varpi) & 0\\ \varphi_{N-1}(\varpi-1) & \psi_M(\varpi-1) & \psi_{M-1}(\varpi-1)\\ \varphi_N(\varpi-2) & 0 & \psi_M(\varpi-2)\end{vmatrix} \end{array}\right\}. \quad \ldots \quad (5.)$$

And it is not difficult to see that the first term of the rth remainder will be

$$\varrho^{N-r}\frac{1}{\psi_M(\pi-1)\psi_M(\pi-2)..\psi_M(\pi-r)}\begin{vmatrix}\varphi_{N-r}(\pi) & \psi_{M-1}(\pi) & .. & 0\\ \varphi_{N-r+1}(\pi-1) & \psi_M(\pi-1) & .. & 0\\ . & . & . & .\\ \varphi_N(\pi-r) & 0 & .. & \psi_M(\pi-r),\end{vmatrix} \quad .. \quad (6.)$$

in which determinant every column after the first consists of only two terms, viz. $\psi_{M-1}(\pi-s)$ and $\psi_M(\pi-s-1)$. Hence also the $(r+1)$th term of the quotient will be

$$\varrho^{N-M-r}\frac{1}{\psi_M(\pi-M)\psi_M(\pi-M-1)..\psi_M(\pi-M-r)}\begin{vmatrix}\varphi_{N-r}(\pi-M) & \psi_{M-1}(\pi) & .. & 0\\ \varphi_{N-r+1}(\pi-M-1) & \psi_M(\pi-1) & .. & 0\\ . & . & . & .\\ \varphi_N(\pi-r) & 0 & .. & \psi_M(\pi-r).\end{vmatrix} \quad (7.)$$

As to the other terms, than the first, of the various remainders. In the first remainder, the first term of which is given by (3.), the $(s+1)$th term will be found by making $n=N-s$, $m=M-s$, in the expression

$$\Sigma_{n=0}^{n=N}\varrho^n\varphi_n(\pi)-\Sigma_{m=0}^{m=M}\varrho^{N-M+m}\frac{\psi_m(\pi)}{\psi_M(\pi-M+m)}\varphi_N(\pi-M+m),$$

which gives

$$\varrho^{N-s}\left\{\varphi_{N-s}(\pi)-\frac{\psi_{M-s}(\pi)}{\psi_M(\pi-s)}\varphi_N(\pi-s)\right\}$$
$$=\varrho^{N-s}\frac{1}{\psi_M(\pi-s)}\begin{vmatrix}\varphi_{N-s}(\pi) & \psi_{M-s}(\pi)\\ \varphi_N(\pi-s) & \psi_M(\pi-s).\end{vmatrix}$$

Hence the entire first remainder may be expressed thus:

$$\Sigma_{s=0}^{s=M}\frac{\varrho^{N-s}}{\psi_M(\pi-s)}\begin{vmatrix}\varphi_{N-s}(\pi) & \psi_{M-s}(\pi)\\ \varphi_N(\pi-s) & \psi_M(\pi-s).\end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

Similarly, the general expression for the second remainder is

$$\Sigma_{n=0}^{n=N-1}\varrho^n\varphi_n(\pi)-\Sigma_{m=0}^{m=M}\varrho^{N-M+m-1}\frac{\psi_m(\pi)}{\psi_M(\pi-M+m)\psi_M(\pi-M+m-1)}\begin{vmatrix}\varphi_{N-1}(\pi-M+m) & \psi_{M-1}(\pi-M+m)\\ \varphi_N(\pi-M+m-1) & \psi_M(\pi-M+m-1),\end{vmatrix}$$

which may be transformed thus:

$$n=N-s, \quad m=M-s,$$

$$\Sigma_{s=0}^{s=M}\varrho^{N-s-1}\left\{\varphi_{N-s-1}(\pi)-\frac{\psi_{M-s}(\pi)}{\psi_M(\pi-s)\psi_M(\pi-s-1)}\begin{vmatrix}\varphi_{N-1}(\pi-s) & \psi_{M-1}(\pi-s)\\ \varphi_N(\pi-s-1) & \psi_M(\pi-s-1)\end{vmatrix}\right\}$$
$$=\Sigma_{s=0}^{s=M}\varrho^{N-s-1}\frac{1}{\psi_M(\pi-s)\psi_M(\pi-s-1)}\begin{vmatrix}\varphi_{N-s-1}(\pi) & \psi_{M-s}(\pi) & 0\\ \varphi_{N-1}(\pi-s) & \psi_M(\pi-s) & \psi_{M-1}(\pi-s)\\ \varphi_N(\pi-s-1) & 0 & \psi_M(\pi-s-1).\end{vmatrix} \quad (9.)$$

And generally the expression for the tth remainder may be written

$$\Sigma_{s=0}^{s=M}\varrho^{N-s-t+1}\frac{1}{\psi_M(\pi-s)\psi_M(\pi-s-1)\,..\,\psi_M(\pi-s-t+1)}\begin{vmatrix}\varphi_{N-s+t}(\pi) & \psi_{M-s}(\pi) & .. & 0\\ \varphi_{N-s+1}(\pi-s) & \psi_M(\pi-s) & .. & \psi_{M-t+1}(\pi-s)\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \varphi_N(\pi-s-t+1) & 0 & .. & \psi_M(\pi-s-t+1).\end{vmatrix} \quad (10.)$$

The last remainder is the $(N-M+1)$th. Then

$$t=N-M+1,$$

$$N-s-t+1=N-s-N+M-1+1=\ M-s$$

$$N-t+1=N\quad -N+M-1+1=\ M$$

$$M-t+1=M\quad -N+M-1+1=2M-N,$$

and the remainder in question

$$=\Sigma_{s=0}^{s=M}\varrho^{M-s}\frac{1}{\psi_M(\pi-s)\psi_M(\pi-s-1)\,..\,\psi_M(\pi-s-N+M)} \quad (11.)$$

$$\times\begin{vmatrix}\varphi_{M-s}(\pi) & \psi_{M-s}(\pi) & .. & 0\\ \varphi_M(\pi-s) & \psi_M(\pi-s) & .. & \psi_{2M-N}(\pi-s)\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \varphi_N(\pi-s-N+M) & 0 & .. & \psi_M(\pi-s-N+M),\end{vmatrix}$$

in which the coefficient of ϱ^M vanishes, as it should. The last term, viz. that independent of ϱ,

$$=\frac{1}{\psi_M(\pi-M)\psi_M(\pi-M-1)\,..\,\psi_M(\pi-N)} \quad (12.)$$

$$\times\begin{vmatrix}\varphi_0(\pi) & \psi_0(\pi) & .. & 0\\ \varphi_M(\pi-M) & \psi_M(\pi-M) & .. & \psi_{2M-N}(\pi-M)\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \varphi_N(\pi-N) & 0 & .. & \psi_M(\pi-N);\end{vmatrix}$$

and if $N=1$, the result agrees with that given by Mr. RUSSELL*.

§ 6. *To divide* $\Sigma_{n=0}^{n=N}\varrho^n\varphi_n(\pi)$ *externally by* $\Sigma_{m=0}^{m=M}\varrho^m\psi_m(\pi)$.

The first term of the quotient will be

$$\varrho^{N-M}\frac{\varphi_N(\pi)}{\psi_M(\pi+N-M)}. \quad (1.)$$

The first remainder

$$\left.\begin{aligned}&\Sigma_{n=0}^{n=N}\varrho^n\varphi_n(\pi)-\Sigma_{m=0}^{m=M}\varrho^{N-M+m}\frac{\psi_m(\pi+N-M)}{\psi_M(\pi+N-M)}\varphi_N(\pi)\\ &=\Sigma_{s=0}^{s=M}\varrho^{N-s}\left\{\varphi_{N-s}(\pi)-\frac{\psi_{M-s}(\pi+N-M)}{\psi_M(\pi+N-M)}\varphi_N(\pi)\right\},\end{aligned}\right\} \quad (2.)$$

* Philosophical Transactions, vol. cli. p. 72.

the first term of which is

$$\xi^{N-1}\left\{\varphi_{N-1}(\pi)-\frac{\psi_{M-1}(\pi+N-M)}{\psi_M(\pi+N-M)}\,\varphi_N(\pi)\right\},$$

whence the second term of the quotient will be

$$\left.\begin{aligned}&\xi^{N-M-1}\frac{1}{\psi_M(\pi+N-M-1)}\left\{\varphi_{N-1}(\pi)-\frac{\psi_{M-1}(\pi+N-M)}{\psi_M(\pi+N-M)}\,\varphi_N(\pi)\right\}\\&=\xi^{N-M-1}\frac{1}{\psi_M(\pi+N-M-1)\psi_M(\pi+N-M)}\begin{vmatrix}\varphi_{N-1}(\pi) & \psi^{M-1}\,\pi+N-M)\\ \varphi_N\ (\pi) & \psi_M\ (\pi+N-M).\end{vmatrix}\end{aligned}\right\}\quad\ldots\quad(3.)$$

Similarly, the second remainder will be

$$\left.\begin{aligned}&\Sigma_{n=0}^{n=N-1}\xi^n\varphi_n(\pi)-\Sigma_{m=0}^{m=M}\xi^{N-M+m-1}\frac{\psi_m(\pi+N-M-1)}{\psi_M(\pi+N-M-1)\psi_M(\pi+N-M)}\begin{vmatrix}\varphi_{N-1}(\pi) & \psi_{M-1}(\pi+N-M)\\ \varphi_N(\pi) & \psi_M\ (\pi+N-M)\end{vmatrix}\\&=\Sigma_{s=0}^{s=M}\xi^{N-s-1}\left\{\varphi_{N-s-1}(\pi)-\frac{\psi_{M-s}(\pi+N-M-1)}{\psi_M(\pi+N-M-1)\psi_M(\pi+N-M)}\begin{vmatrix}\varphi_{N-1}(\pi) & \psi_{M-1}(\pi+N-M)\\ \varphi_N(\pi) & \psi_M(\pi+N-M)\end{vmatrix}\right\}\\&=\Sigma_{t=0}^{t=m}\xi^{N-s-1}\frac{1}{\psi_M(\pi+N-M-1)\psi_M(\pi+N-M)}\begin{vmatrix}\varphi_{N-s-1}(\pi) & \psi_{M-s}(\pi+N-M-1) & 0\\ \varphi_{N-1}\ (\pi) & \psi_M\ (\pi+N-M-1)\psi_{M-1}(\pi+N-M)\\ \varphi_N\ (\pi) & 0 & \psi_M\ (\pi+N-M)\end{vmatrix}\end{aligned}\right\}(4.)$$

The tth remainder

$$\left.\begin{aligned}&=\Sigma_{s=0}^{s=M}\xi^{N-s-t+1}\frac{1}{\psi_M(\pi+N-M-t+1)\psi_M(\pi+N-M-t+2)\,..\,\psi_M(\pi+N-M)}\\&\times\begin{vmatrix}\varphi_{N-s-t+1}(\pi) & \psi_{M-s}(\pi+N-M-t+1)\,.. & 0\\ \varphi_{N-t+1}(\pi) & \psi_M(\pi+N-M-t+1) & ..\ \psi_{M-t+1}(\pi+N-M)\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \varphi_N(\pi) & 0 & .\ \psi_M(\pi+N-M);\end{vmatrix}\end{aligned}\right\}\quad\ldots\quad(5.)$$

and the last remainder, viz. the $(N-M+1)$th,

$$\left.\begin{aligned}&=\Sigma_{s=0}^{s=M}\xi^{M-s}\frac{1}{\psi_M(\pi)\psi_M(\pi+1)\,..\,\psi_M(\pi+N-M)}\\&\times\begin{vmatrix}\varphi_{M-s}(\pi) & \psi_{M-s}(\pi)\,.. & 0\\ \varphi_M(\pi) & \psi_M(\pi) & ..\ \psi_{2M-N}(\pi+N-M)\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \varphi_N(\pi) & 0 & ..\ \psi_M(\pi+N-M).\end{vmatrix}\end{aligned}\right\}\quad\ldots\quad(6.)$$

In the case considered by Mr. RUSSELL, viz. $M=1$, (6.) gives only the single term

$$\{\psi_1(\pi)\psi_1(\pi+1)\,..\,\psi_1(\pi+N-1)\}^{-1}\begin{vmatrix}\varphi_0(\pi) & \psi_0(\pi)\,.. & 0\\ \varphi_1(\pi) & \psi_1(\pi)\,.. & 0\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \varphi_{N-1}(\pi) & 0 & ..\ \psi_0(\pi+N-1)\\ \varphi_N(\pi) & 0 & ..\ \psi_1(\pi+N-1),\end{vmatrix}$$

and in this the coefficient of $\varphi_i(\pi)$ is $(-)^i \times$

$$\begin{vmatrix} \psi_0(\pi) & 0 & .. & 0 & 0 & .. & 0 & 0 \\ \psi_1(\pi) & \psi_0(\pi+1) & .. & 0 & 0 & .. & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & .. & \psi_0(\pi+i-1) & 0 & .. & 0 & 0 \\ 0 & 0 & .. & 0 & \psi_1(\pi+i) & .. & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & .. & 0 & 0 & .. & \psi_1(\pi+N-2) & \psi_0(\pi+N-1) \\ 0 & 0 & .. & 0 & 0 & .. & 0 & \psi_1(\pi+N-1) \end{vmatrix}$$

$$= \psi_0(\pi)\psi_0(\pi+1)\,..\,\psi_0(\pi+i-1)\psi_1(\pi+i)\,..\,\psi_1(\pi+N-2)\psi_1(\pi+N-1);$$

whence the whole expression

$$= \Sigma_{i=0}^{i=N}(-)^i\varphi_i(\pi)\frac{\psi_0(\pi)\psi_0(\pi+1)\,..\,\psi_0(\pi+i-1)}{\psi_1(\pi)\psi_1(\pi+1)\,..\,\psi_1(\pi+i-1)},$$

which agrees with the result given in the Philosophical Transactions, vol. cli. p. 73.

In the particular case of N=4, M=2, the final remainder in internal division is

$$\frac{1}{\psi_2(\pi-1)\psi_2(\pi-2)\psi_2(\pi-3)}\begin{vmatrix} \varphi_1(\pi) & \psi_1(\pi) & 0 & 0 \\ \varphi_2(\pi-1) & \psi_2(\pi-1) & \psi_1(\pi-1) & \psi_0(\pi-1) \\ \varphi_3(\pi-2) & 0 & \psi_2(\pi-2) & \psi_1(\pi-2) \\ \varphi_4(\pi-3) & 0 & 0 & \psi_2(\pi-3) \end{vmatrix}$$

$$+\frac{1}{\psi_2(\pi-2)\psi_2(\pi-3)\psi_2(\pi-4)}\begin{vmatrix} \varphi_0(\pi) & \psi_0(\pi) & 0 & 0 \\ \varphi_2(\pi-2) & \psi_2(\pi-2) & \psi_1(\pi-2) & \psi_0(\pi-2) \\ \varphi_3(\pi-3) & 0 & \psi_2(\pi-3) & \psi_1(\pi-3) \\ \varphi_4(\pi-4) & 0 & 0 & \psi_2(\pi-4); \end{vmatrix}$$

and in external division it is

$$\frac{1}{\psi_2(\pi)\psi_2(\pi+1)\psi_2(\pi+2)}\begin{vmatrix} \varphi_1(\pi) & \psi_1(\pi) & 0 & 0 \\ \varphi_2(\pi) & \psi_2(\pi) & \psi_1(\pi+1) & \psi_0(\pi+2) \\ \varphi_3(\pi) & 0 & \psi_2(\pi+1) & \psi_1(\pi+2) \\ \varphi_4(\pi) & 0 & 0 & \psi_2(\pi+2) \end{vmatrix}$$

$$+\frac{1}{\psi_2(\pi)\psi_2(\pi+1)\psi_2(\pi+2)}\begin{vmatrix} \varphi_0(\pi) & \psi_0(\pi) & 0 & 0 \\ \varphi_2(\pi) & \psi_2(\pi) & \psi_1(\pi+1) & \psi_0(\pi+2) \\ \varphi_3(\pi) & 0 & \psi_2(\pi+1) & \psi_1(\pi+1) \\ \varphi_4(\pi) & 0 & 0 & \psi_2(\pi). \end{vmatrix}$$

The expressions (10.) for the formation of the Φs admit of further development thus:

$$\Phi_0=\varphi_N$$

$$\Phi_1=\varphi_{N-1}-M\psi_M\left(\frac{\varphi_N}{\psi_M}\right)'-\psi_{M-1}\left(\frac{\varphi_N}{\psi_M}\right)$$

$$\begin{aligned}\Phi_2=\varphi_{N-2}&-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\varphi_N}{\psi_M}\right)''-(M-1)\psi_{M-1}\left(\frac{\varphi_N}{\psi_M}\right)'-\psi_{M-2}\left(\frac{\varphi_N}{\psi_M}\right)\\&+M^2\psi_M\left(\frac{\varphi_N}{\psi_M}\right)''+M\psi_{M-1}\left(\frac{\varphi_N}{\psi_M}\right)'+M\psi_M\left(\frac{\psi_{N-1}}{\psi_M}\right)'\left(\frac{\varphi_N}{\psi_M}\right)-M\psi_M\left(\frac{\varphi_{N-1}}{\psi_M}\right)'\\&\qquad+M\psi_{M-1}\left(\frac{\varphi_N}{\psi_M}\right)'+\psi_{M-1}\left(\frac{\psi_{N-1}}{\psi_M}\right)\left(\frac{\varphi_N}{\psi_M}\right)-\psi_{M-1}\left(\frac{\varphi_{N-1}}{\psi_M}\right)\\=\frac{(M+1)M}{1.2}&\psi_M\left(\frac{\varphi_N}{\psi_M}\right)''+(M+1)\psi_{M-1}\left(\frac{\varphi_N}{\psi_M}\right)'+\left\{-\psi_{M-2}+M\psi_M\left(\frac{\psi_{M-1}}{\psi_M}\right)'+\psi_{M-1}\left(\frac{\psi_{N-1}}{\psi_M}\right)\right\}\left(\frac{\varphi_N}{\psi_M}\right)\\&-M\psi_M\left(\frac{\varphi_{N-1}}{\psi_M}\right)'-\qquad\psi_{M-1}\left(\frac{\varphi_{N-1}}{\psi_M}\right)\\&+\qquad\varphi_{N-2};\end{aligned}$$

or writing, by analogy to the Φs,

$$\Psi_0=\psi_{M-1}$$

$$-\Psi_1=\psi_{M-2}-M\psi_M\left(\frac{\Psi_0}{\psi_M}\right)'-\psi_{M-1}\left(\frac{\Psi_0}{\psi_M}\right),$$

the expression for Φ_2 becomes

$$\begin{aligned}\Phi_2=&\frac{(M+1)M}{1.2}\psi_M\left(\frac{\varphi_N}{\psi_M}\right)''+(M+1)\Psi_0\left(\frac{\varphi_N}{\psi_M}\right)'+\Psi_1\left(\frac{\varphi_N}{\psi_M}\right)\\&-M\psi_M\left(\frac{\varphi_{N-1}}{\psi_M}\right)'-\qquad\Psi_0\left(\frac{\varphi_{N-1}}{\psi_M}\right)\\&+\qquad\psi_M\left(\frac{\varphi_{N-2}}{\psi_M}\right).\end{aligned}$$

And so likewise writing

$$\begin{aligned}\Psi_2=\psi_{M-2}&-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\Psi_0}{\psi_M}\right)''-(M-1)\psi_{M-1}\left(\frac{\Psi_0}{\psi_M}\right)'-\psi_{M-2}\left(\frac{\Psi_0}{\psi_M}\right)\\&-M\psi_M\left(\frac{\Psi_1}{\psi_M}\right)'-\qquad\psi_{M-1}\left(\frac{\Psi_1}{\psi_M}\right),\end{aligned}$$

it will be found that

$$\begin{aligned}\Phi_3=\varphi_{N-3}&-\frac{M(M-1)(M-2)}{1.2.3}\psi_M\left(\frac{\Phi_0}{\psi_M}\right)'''-\frac{(M-1)(M-2)}{1.2}\psi_{M-1}\left(\frac{\Phi_0}{\psi_M}\right)''-(M-2)\psi_{M-2}\left(\frac{\Phi_0}{\psi_M}\right)'-\psi_{M-3}\left(\frac{\Phi_0}{\psi_M}\right)\\&-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\Phi_1}{\psi_M}\right)''-(M-1)\psi_{M-1}\left(\frac{\Phi_1}{\psi_M}\right)'-\psi_{M-2}\left(\frac{\Phi_1}{\psi_M}\right)\\&-M\psi_M\left(\frac{\Phi_2}{\psi_M}\right)'-\psi_{M-1}\left(\frac{\Phi_2}{\psi_M}\right)\end{aligned}$$

$$= -\frac{(M+2)(M+1)M}{1.2.3}\psi_M\left(\frac{\varphi_N}{\psi_M}\right)''' - \frac{(M+2)(M+1)}{1.2}\Psi_0\left(\frac{\varphi_N}{\psi_M}\right)'' - (M+2)\Psi_1\left(\frac{\varphi_N}{\psi_M}\right)' - \Psi_2\left(\frac{\varphi_N}{\psi_M}\right)$$

$$+ \frac{(M+1)M}{1.2}\psi_M\left(\frac{\varphi_{N-1}}{\psi_M}\right)'' + (M+1)\Psi_0\left(\frac{\varphi_{N-1}}{\psi_M}\right)' + \Psi_1\left(\frac{\varphi_{N-1}}{\psi_M}\right)$$

$$- M\psi_M\left(\frac{\varphi_{N-2}}{\psi_M}\right)'' - \Psi_0\left(\frac{\varphi_{N-2}}{\psi_M}\right)$$

$$+ \psi_M\left(\frac{\varphi_{N-3}}{\psi_M}\right).$$

But the law of the expressions in the first form having been established above, it is unnecessary to pursue these latter formulæ further.

VII. *On the Theory of the Polyedra.* *By the Rev.* THOMAS P. KIRKMAN, *M.A., F.R.S., Honorary Member of the Literary and Philosophical Societies of Manchester and Liverpool.*

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THE following memoir contains a complete solution of the problem of the *classification and enumeration* of the P-edra Q-acra. The actual *construction* of the solids is a task impracticable from its magnitude; but it is here shown, that we can enumerate them with an accurate account of their symmetry, to any values of P and Q.

Section first discusses fully the *symmetry of polyedra*, and gives a sketch of the Tables which it is required to form for the solution of our problem, viz. Tables of the classified P-edra Q-acra and Q-edra P-acra, and of their faces, summits, and edges, symmetrical and unsymmetrical.

Section second proves that all these Tables are given, if *certain data* are first obtained. The rest of the memoir is occupied by the investigation and construction of these assigned *data*.

Section third contains the analysis of a *polar or monozone summit* of a P-edron Q-acron: the *deltotomous effaceables* of the summit are defined and restored: the *reticulation* which is laid bare by the removal of the rays of the summit is analysed and reduced.

Section fourth is devoted to the *construction* of polar and monozone *reticulations*, and to their *registration* with their *signatures* of form and symmetry, in groups which suffice for our problem.

Section fifth gives formulæ for the zoned and zoneless *coronation* of a polar or monozone *perfect reticulation*, whereby it becomes a polyedron; and Tables of the *perfect summits* thus obtained are sketched out.

Section sixth enumerates and registers the results of *deltotomous effacements* about *perfect summits*, that is, summits about which all deltotomous effaceables have been restored, or which have no effaced effaceables, whereby the summit analysed in section third is obtained and registered.

Section seventh analyses the *polar summits of a janal axis*; the *rhombotomous effaceables* of the opposite summits are defined, and, as well as their deltotomous ones, restored, about either pole; the *janal reticulation* laid bare by the removal of the summits is analysed and reduced.

Section eighth constructs, enumerates, and registers, with their signatures of symmetry, the *fundamental* and *primitive* janal reticulations.

Section ninth contains the construction, enumeration, and registration of *janal subnucleus reticulations*.

Section tenth gives the like concerning *janal nucleus reticulations*.

Section eleventh gives a similar account of *perfect janal mixed reticulations.*

Sections twelfth and thirteenth are devoted to the *janal coronation* of janal reticulations, and to the enumeration and registration of *perfect janal summits.*

Sections fourteenth and fifteenth enumerate and register the results of deltotomous and rhombotomous effacements about perfect janal summits.

Section sixteenth analyses a *polyarchipolar summit*; effaceables are restored about all like archipoles; the *polyarchaxine reticulation* laid bare by the removal of these polar summits is reduced, and afterwards constructed with enumeration and registration of results: the formulæ for *polyarchaxine coronation* are given, and the results of effacement about the principal axes are enumerated and registered.

Section seventeenth gives the analysis, construction, and enumeration of *contrajanal anaxine pairs of edges*, which have neither polar nor zoned symmetry, but of which one edge is the reflected image of the other, and diametrically opposed to it.

The rest of the memoir is devoted to the enumeration of *plane reticulations*, i. e. partitioned polygons, the knowledge of which is taken for granted in all that precedes.

Section eighteenth enumerates and registers the symmetrical and asymmetric *plane penesolids*, i. e. plane reticulations laid bare by the removal of an edge of a polyedron.

Section nineteenth enumerates and registers the *primary plane reticulations*, symmetric and asymmetric, with their signatures of symmetry.

Section twentieth constructs, enumerates, and registers with their signatures, the *zoned plane reticulations.*

Section twenty-first gives the like account of the *zoneless plane reticulations.*

It is unfortunate that my previous labours on the partitions of the R-gon, that is, on plane reticulations, are of little utility for this problem of the polyedra, by reason of their too great generality, and of their not giving the number of *marginal triangles* in each partition. Yet the fundamental theorem on the k-divisions of the R-gon* has been the key to the *greatest difficulty* in this theory, which is to find the number of the asymmetric plane reticulations which have a given *marginal signature.*

Of the two sections here presented to the public, the first (arts. I. . . . XXXV.) is in itself a complete treatise on the symmetry and classification of Polyedra; and the second, together with the preceding introduction, puts the reader clearly in possession of the main outline of the argument. Much remains of the entire work, which is, however, completely written, and in the possession of the Royal Society, as well as extensive applications of the method to the enumeration of polyedra. These will be intelligible only when the general methods and formulæ of this memoir have been investigated and exhibited.

Section 1.—*On the Symmetry of Polyedra.*

I. The symmetry of polyedra is—

1. Zoned symmetry;
2. Zoneless axial symmetry;

* Philosophical Transactions, 1857, p. 225.

3. Mixed symmetry, both zoned and zoneless axial;
4. Neuter symmetry, neither zoned nor zoneless axial;
5. No symmetry.

The greater part of the polyedra are entirely asymmetric.

In considering this symmetry, we have no regard to mere lengths or inclinations of edges. The symmetry is *descriptive*, not *metrical*. For example, if the cutting edge of a wedge be removed by any quadrilateral section, the solid acquires, for our purposes, all the symmetry of the cube; and this remains, however the figure may be distorted.

1. *Zoned Symmetry.*

II. The polyedra which have a zoned symmetry, and only such, are
a. Monozone polyedra;
b. *m*-zoned monaxine heteroids;
c. Zoned triaxines;
d. *m*-zoned monarchaxines, having one principal and *m* secondary axes;
e. Zoned polyarchaxines, having the axial systems of the regular polyedra.

The terms will be explained below.

Def. *A zone is any closed line drawn or drawable on a polyedron, which divides it into halves, either of which is the reflected image of the other, no regard being had to mere lengths of edges.*

The closed line may or may not be all in one plane, and the halves may or may not be *metrically* equal. We have a right to conceive, when it is convenient, that the solid is constructed with the greatest possible symmetry, in which case the halves will be exactly equal, and the zone will be a plane, having to them the geometrical relation which a mirror has to an object touching it and to its image.

This relation we shall assume as always existing, however the polyedron may be distorted; that is, we assume that any zoneless edge will meet, on any zonal plane, if it be produced, the edge which is its reflected image in respect of that zonal plane.

For example, any section of a cube which passes through two opposite faces, and contains either two edges or none of the solid, is a zone.

Every zone has a *zonal signature*, Z, which describes it by the *number* of its zoned *features*, i. e. its *zonal faces*, and its *zonal summits*, through which it passes, and its *zonal* and *epizonal edges*, but gives no account of the *number of edges* in the zoned faces or summits, nor of the *order* of the zoned features.

Any edge contained by the zone Z, *is a zonal edge of* Z. *Any edge cut by the zone, is an epizonal edge of* Z.

It will create no confusion if we denote both the zone and its signature by the same name Z.

We represent zonal and epizonal edges by the symbols 0 and 0 (zero faces), and we write the number of such edges as an index over the proper symbol. All such indices in a zonal signature are *coefficients*.

III. a. *Monozone polyedra.*—A monozone polyedron has no symmetry but that of a single zone. It has the zonal signature

$$Z=\{g, G, \mathbf{0}^a, 0^b\},$$

which records that Z has g zoned summits, G zoned faces, a zonal edges, and b epizonal edges.

No two AA′ *of the zoned features of a monozone polyedron have the same configuration, or are one the reflected image of the other.*

For if A were identical in configuration with A′, there would be a *symmetry* of *repetition*, not essential to the zone Z. And if A were the reflected image of A′, there would be a zone different from Z, passing between A and A′.

Every zoneless feature, B, *is twice read on a monozone polyedron, viz.* B *and its reflected image.*

The *undrawn* lines of Z are the *zonal traces* of the zoned faces and summits.

The *trace* of a zone is *agonal*, *diagonal*, or *monogonal*, according as it passes through one angle, two angles, or one angle only, of the face or summit.

Monogonal traces are seen only in *odd-angled* faces or summits.

A monozone 8-edron 12-acron is . . . . . whose zonal signature is

$$Z=\{2, \mathbf{2}, \mathbf{0}^1, 0^1\}.$$

IV. *Zoned axis.*—Any number of zonal planes may have a common line or axis, which is *a zoned axis.*

Def. *An axis is janal, whether it be zoned or zoneless*, if the configuration C, or if the reflected image of C, which is read at one extremity of the axis, can be read by an opposite eye at the other extremity, by turning the axis through any angle.

In particular, *a janal zoned axis* is said to be *objanal*, when the configuration C read at one *pole or extremity* is the configuration C′, read at the other pole (by an opposite eye in the axis), turned through two right angles, C′ being C inverted.

Also a *janal zoneless axis* is said to be *contrajanal*, if the configuration C read at one pole is the reflected image of the configuration C′ which can be read by an opposite eye at the other.

Def. *An axis, whether zoned or zoneless, is heteroid*, if the configuration C read at one pole cannot be read at the other, nor read inverted, nor reflected.

V. According to the character of the *polar features terminating an axis*, zoned or zoneless, the axis is *amphiedral*, *amphigonal*, *amphigrammic*, *gonogrammic*, *edrogrammic*, or *gonoedral*, terms which explain themselves.

An axis is m-zoned, when it is the intersection of m zones.

A zoned amphigrammic, edrogrammic, or gonogrammic axis is of necessity two-zoned; for a polar edge may be zonal in one zone, and epizonal in another, but cannot belong to a third.

VI. Theorem. *There are two different hemizonal sequences of configuration, and only two, read alternately upon the hemizones about a zoned axis.*

For, 1°, there cannot be fewer than two; because if two contiguous hemizones had exactly the same configuration, they would be each the reflected image of the other, and therefore not contiguous, but on opposite sides of a different zone.

And 2°, there cannot be more than two; for every hemizone A has on either hand, by definition of a zone, the same hemizone B, and B has on either hand the same hemizone A.

The propositions of the following article are easy deductions from the above theorem. We use the term perpendicular in a wide sense, in consideration of art. II.; that is, under the assumption of the greatest possible symmetry.

VII. When there is an even number of zones about a zoned axis, each is perpendicular to another, and has two identical hemizones.

If a $2m$-zoned axis is the only one of the solid, there are two different *entire zonal configurations* about the axis, viz. m zones Z alternating with the m zones Z′. The signatures of Z and Z′ may or may not be different; the configurations cannot be the same.

When there are $4m$ zones, each is perpendicular to one of the same configuration. When the number of zones is $4m+2$, each is perpendicular to one of a different configuration.

When the number of zones is odd, none is perpendicular to another.

When the axis is $2m$-zoned, there are, in each polar feature, m traces t alternating with m traces t'.

When the axis is $(2m+1)$-zoned, the polar face or summit has $2m+1$ identical traces.

About a $(2m+1)$-zoned axis there is *but one entire zonal configuration*, and consequently but one signature for all the zones. These zones have *not* each two identical hemizones.

Whether the r zones be odd or even, the $2r$ *semi-traces* present alternate configurations in the circuit of the pole.

Each trace has two like terminations, if r be even, and two different terminations, if r be odd.

VIII. Theorem. *If a zoned axis is the only axis of the polyedron, that axis is heteroid.*

For if not, the opposite poles will be either identical, or one the reflected image of the other.

If they are identical, there is an axis of *even repetition* perpendicular to the zoned axis, *i. e.* an axis about which in revolution of the solid, the same configuration is $2m$ times repeated to the eye; which is contrary to hypothesis.

If they are one the reflected image of the other, there is, at right angles to the zoned axis, a zone, whose intersections with those of the zoned axis are other axes of the solid; which is contrary to hypothesis. Wherefore the theorem is proved.

IX. b. *Zoned monaxine heteroids.*—A polyedron whose only axis is an m-zoned axis, is an m-*zoned monaxine heteroid polyedron.*

When m is odd, the zonal signature is

$$Z=\{(\sigma_p+g),\ (f_p+G)\ 0^a\ 0^b\},$$

where

$$\sigma_p+f_p=2$$

shows the number σ_p of polar summits, and f_p of polar faces, beside g *different* non-polar zonal summits, G different non-polar zonal faces, a different non-polar zonal edges, and b different non-polar epizonal edges.

The axis may be amphiedral, amphigonal, or gonoedral.

When m is even, the two zonal signatures are

$$Z=\{(\sigma_p+2g),\ (f_p+2G),\ 0_p^a\ 0_p^{a_1}\ 0^{2b}\},$$
$$Z_1=\{(\sigma_p+2g_1),\ (f_p+2G_1),\ 0_p^{a_1}\ 0_p^{a}\ 0^{2b_1}\},$$

where

$$\sigma_p+f_p+\alpha+\beta=2,$$
$$(\alpha=0=\beta \text{ if } m>2 \text{ (art. V.)})$$

describes the poles, which may give six different characters to the axis (V.), one for every solution of the equation $\sigma_p+f_p+\alpha+\beta=2$, where $\alpha=2$ differs not from $\beta=2$ in form.

We consider two zoned features, of which one is the reflected image of the other, to be the same in configuration, and enumerate the two as one in our Tables of zoned features.

The above signatures show g, G, a, b, and g_1 G_1 a_1 b_1 for the number of their *different* zoned non-polar features. For each zone has two identical hemizones, when m is even, but not when m is odd (VII.).

The polar edge which $(\alpha+\beta>0)$ is zonal or epizonal in Z, is epizonal or zonal in Z_1. There is nothing to prevent the two zones Z Z_1 from having the same *signature*; but they cannot have the same *configuration*. And we always consider two zonal signatures which differ only in p subscribed, and in the ways of writing the factors of the same number of features, as *numerically the same zonal signature*. This is important to be remembered, when we inspect our Tables in considering the zone Z.

Whether m *be odd or even, every different non-polar feature is read* 2m *times on the solid*, namely, once in each of 2m *interzonal regions*.

A 2-zoned monaxine heteroid 6-edron 8-acron is . . . . .

The zonal signatures are,

$$Z=\{(2.1)\ (2.\mathbf{1})\ 0_p^1\ 0_p^1\};$$
$$\sigma_p=0,\ g=1,\ G=\mathbf{1},\ \alpha=1=\beta,\ a=0=b;$$
$$Z'=\{(2.1)\ (2.\mathbf{2}),\ 0_p^1\ 0_p^1\};$$
$$\sigma_p=0,\ g_1=1,\ G_1=\mathbf{2},\ \beta=1=\alpha,\ a_1=0=b_1.$$

The axis is amphigrammic.

A 3-zoned monaxine heteroid 7-edron 9-acron is . . . . .

whose signature is

$$Z=\{(1),\ (\mathbf{2}_p+\mathbf{2}),\ 0^3\};$$
$$\sigma_p=0,\ g=1,\ f_p=\mathbf{2},\ G=\mathbf{2},\ a=0,\ b=3.$$

The polar triangle has three monogonal traces; the polar hexagon has three agonal traces.

X. *Principal zoned axis. Secondary axes.*—A *principal axis* has more zones or not fewer zones than any other. The zones which intersect in it are *principal zones.*

Theorem. *If there be but one principal zoned axis,* A, *it is janal; the solid has one, and but one, zone besides those of the axis, and this zone is perpendicular to the axis* A.

For Z'', a zone not containing the principal axis, cannot divide the solid into halves of which one is the reflected image of the other, unless the axis is janal and at right angles to Z''. As the axis can be perpendicular to but one zone, the truth of the theorem is evident.

This Z'' is called, when $m>2$, the *secondary zone of the polyedron*, and its intersections with the zones of the principal axis are the *secondary zoned axes of the polyedron.*

XI. Theorem. *A* 2m-*zoned principal axis has* 2m 2-*zoned secondary axes, all janal; the* 4m *secondary poles of these axes are in the secondary zone, of two alternate configurations, and the alternate secondary axes are different.*

The $2m$ axes are 2-zoned, being each the intersection of the secondary with a principal zone. The rest is evident, if we make a section of the solid in its secondary zone; for the 2m *traces* of this section are the 2m *axes* (VII.).

Theorem. *A* (2m+1)-*zoned principal axis has* 2m+1 *secondary axes. Their* 4m+2 *secondary poles are in the secondary zone, in the circuit of which they present two alternate configurations. The secondary axes are all* 2-*zoned, all alike, and all heteroid.*

XII. d. *Zoned monarchaxines.*—A polyedron having only one principal m-zoned axis is an m-*zoned monarchaxine janal polyedron*, which is also sufficiently described as an m-*zoned monarchaxine* (X.), where the number m does not include the secondary zone.

When $m=2$ no axis is principal, and there are three 2-zoned axes, any one of which is perpendicular to the other two. These three axes are of three different configurations; for if two semi-axes at right angles to each other were identical, they would reflect each other, and a zone would pass between them; whence it would follow that the third axis were no 2-zoned axis.

c. The solid in this case is called a *zoned triaxine polyedron.* It has three zones of different configurations.

The zonal signatures of a $2r$-zoned monarchaxine ($r>1$) are thus written, ZZ' being principal zones, and Z'' being secondary:

$$Z = \{2(\sigma_p+\varsigma_p+2g),\quad 2(f_p+\varphi_p+2G),\quad 0_p^{\alpha},\quad 0_p^{\beta},\quad 0^{\alpha},\ 0^{\beta}\},$$
$$Z' = \{2(\sigma_p+\varsigma'_p+2g'),\quad 2(f_p+\varphi'_p+2G'),\quad 0_p^{\alpha'},\quad 0_p^{\beta'},\quad 0^{\alpha'},\ 0^{\beta'}\},$$
$$Z'' = 2r\{(\varsigma_p+\varsigma'_p+2g''),\quad (\varphi_p+\varphi'_p+2G''),\quad 0_p^{\beta+\beta'},\quad 0_p^{\alpha+\alpha'},\quad 0^{2\alpha},\ 0^{2\beta}\},$$

where

$$\sigma_p+f_p=1$$

is the principal pole, summit or face, in the janal axis; and

$$\varsigma_p+\varphi_p+\alpha+\beta=1,\quad \varsigma'_p+\varphi'_p+\alpha'+\beta'=1,$$

ς_p denoting a polar summit, and φ_p a polar face, show the secondary poles, viz. r of either configuration (XI.). The numbers $gGab$, $g'G'a'b'$ record all the *different* zoned non-polar features of Z and Z', as $g''G''cd$ record those of Z''. The secondary polar edges are zonal in one zone and epizonal in another.

Every non-polar zoned feature of the r *zones* Z, *or of the* r *zones* Z', *occurs four times in each zone* Z *or* Z'. *Every non-polar zoned feature of* Z'' *occurs in it* 4r *times. Thus all such features are read* 4r *times.*

Every zoneless feature of the solid is read in it 8r *times*, in as many interzonal regions.

The principal axis is either amphiedral or amphigonal. The characters of the secondary axes, all janal, vary with the solutions of the equations enumerating their poles.

When $r=1$, the solid is a *zoned triaxine*, whose three signatures have one form, and may or may not be identical signatures. But the three configurations of the zones are always different.

The zonal signatures of a $(2r+1)$-zoned monarchaxine are

$$\begin{aligned} \mathrm{Z} &= \{(2\sigma_p+\varsigma_p+2g),\quad (2f_p+\varphi_p+2\mathrm{G}),\ \mathbf{0}_p^{\varsigma}\ 0_p^{\varphi}\ \mathbf{0}^{2\alpha}\ 0^{2\beta}\},\\ \mathrm{Z}'' &= (2r+1)\{\varsigma_p+2g'',\ (\varphi_p+2\mathrm{G}'',\quad \mathbf{0}_p^{\varphi}\ 0_p^{\varsigma}\ \mathbf{0}^{2\alpha}\ 0^{2\beta}\}, \end{aligned}$$

where

$$\sigma_p+f_p=1$$

describes the principal pole, and

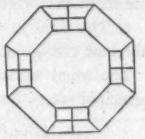
$$\varsigma_p+\varphi_p+\alpha+\beta=2$$

describes the secondary poles, of different configurations. Thus $\varsigma_p=2$ gives an amphigonal, $\varsigma_p=\alpha=1$ gives a gonogrammic axis, &c.

Every non-polar feature in the principal zone Z *is read twice in the solid in each of the* 2r+1 *zones. Every non-polar feature in the secondary* Z'' *is read* 4r+2 *times in* Z''.

Every zoneless feature of the solid is read 8r+4 *times*, once in each of 8r+4 interzonal regions.

A $(2r=)4$-zoned monarchaxine janal 22-edron 36-acron is here figured, of which the zonal signatures are



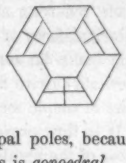
$$\begin{gathered} \mathrm{Z}=\{(\ .\ .\ ),\ 2(\mathbf{1}_p+\mathbf{1}_p),\ 0^{4.1}\},\\ \sigma_p=\varsigma_p=g=0,\ b=f_p=\varphi_p=1,\ \mathrm{G}=\alpha=\beta=a=0,\\ \mathrm{Z}'=\{(2.1_p+4.1),\ (2.\mathbf{1}_p),\ \mathbf{0}^{4.1}\},\\ \sigma_p=\varphi'_p=0,\ \varsigma'_p=g'=f'_p=1=a',\ \mathrm{G}'=\alpha'=\beta'=b'=0,\\ \mathrm{Z}''=4\{(1_p+2.1),\ (\mathbf{1}_p),\ \mathbf{0}^{2.1}\},\\ \varsigma_p=1=g''=\varphi_p=c. \end{gathered}$$

We see, on inspection of these signatures, that the principal axis is *amphiedral*, because Z has four polar faces, of which only two can be secondary poles. We see that Z'' has four polar summits in the two zones Z', and four polar faces in the two zones Z. Hence the secondary axes are alternately amphigonal and amphiedral.

The two zones Z have each four non-polar epizonal, and no zonal edges. The two zones Z' have each four non-polar zonal edges, and no epizonals.

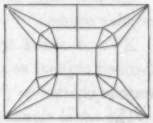
The secondary zone has eight non-polar zonal edges and no epizonals, and it has eight non-polar summits.

Hence we can subtract, on inspection of the signatures, all the zoned faces, summits, and edges from the entire number in the solid, and thus determine the number of *different zoneless features* in each of the eight interzonal spaces.

A 3-zoned monarchaxine 17-edron 27-acron is . . . . . 
of which the signatures are

$$Z = \{(1_p+2.1),\ (2.\mathbf{1}_p+\mathbf{1}_p),\ \mathbf{0}^{2.1}\ 0^{2.1}\},$$
$$Z'' = 3\{(1_p+2.1),\ (\mathbf{1}_p),\ \mathbf{0}^{2.1}\}.$$

We read in these signatures that the principal axis is amphiedral, because Z has four polar features of which two faces must be principal poles, because the principal axis is janal. Either zone shows that the secondary axis is *gonoedral*.

A zoned triaxine 30-edron 26-acron is . . . . . 
of which the signatures are

$$Z = \{(2.1_p+4.1)\ (2.\mathbf{1}_p)\ \mathbf{0}^{4.1}\},$$
$$Z' = \{(\ .\ .\ )\ (2.\mathbf{1}_p+4.\mathbf{1})\ 0_p^{2.1}\ 0^{4.1}\},$$
$$Z'' = \{(2.1_p+4.3)\ (\ .\ .\ )\ \mathbf{0}_p^{2.1}\ \mathbf{0}^{4.3}\}.$$

Z and Z′ have each two polar faces, wherefore their common axis is amphiedral. ZZ″ have each two polar summits, and have an amphigonal axis. Z′Z″ have an amphigrammic axis.

The non-polar zoned features are four summits, all alike in Z, and four zonal edges all alike; four epizonal edges alike, and four faces alike in Z′; twelve summits, as also twelve zonal edges, of three configurations in Z″.

There is nothing to prevent the three axes of a zoned triaxine having all one signature, and axes of *any* janal character. In every case the zonal signatures record accurately the configurations.

XIII. *System of principal poles. Zoned polyarchaxines.*—Let $A_1A_2A_3\ldots$ be a system of poles, zoned or zoneless, of one configuration, not all in one plane, in a polyedron P. If each pole be joined to those nearest it by lines drawn, if necessary, beneath the surface of P, it is evident that these lines will form a polyedron, Q, having edges all of one configuration, *i. e.* each being the intersection of an F-gon and an F′-gon. If F=F′, the polyedron Q is regular. If F<F′, let lines be drawn from the centres f of all the F-gons to all their angles. These lines produced will all pass through centres f' of other faces F. This is all inevitable, by reason of our hypothesis that the poles $A_1A_2A_3\ldots$ have the same configuration. Wherefore the system of lines ff' will form a regular polyedron, having as many edges as there are poles $A_1A_2A_3\ldots$.

Hence we have the

Theorem. *The number of like poles, zoned or zoneless, of any polyedron, which are not all in one plane, is equal to that either of the summits or of the edges of a regular polyedron.*

XIV. If there be a system of collateral polar summits $A_1A_2A_3\ldots$ of the same configuration in any polyedron P, it can be reduced by sections passing only through edges A_1A_2, A_2A_3, A_3A_4, ... to a regular polyedron having only the edges A_1A_2, A_2A_3, A_3A_4, ...

If a system of principal polar summits be not collateral, sections of the solid can be made, removing all the edges of those summits, and laying bare a system of as many principal faces of a solid having fewer edges. The reciprocal of this has a system of principal polar summits, which can be treated like the preceding one; and thus we shall inevitably arrive finally at a solid having collateral principal summits, which reduces by a set of simple sections, as above shown, to a regular polyedron.

It is thus proved that the only systems of principal summits, zoned or zoneless, are those of the regular polyedra.

The following description of zoned polyarchaxines is easily verified by considering the regular solids.

XV. *A zoned triarchaxine polyedron* has three 4-zoned janal principal axes, four 3-zoned objanal secondary axes, and six janal 2-zoned tertiary axes. The axes may have various characters (V.).

The zonal signatures of the zoned triarchaxine are

$$Z=\{(4s_p+4s_p''+8g)(4f_p+4f_p''+8G)0_p^{4s''}0_p^{4f''}0^{8a}0^{4b}\},$$
$$Z_\rho=\{(2s_p+4s_p'+2s_p''+4g_\rho)(2f_p+4f_p''+2f_p'''+4G_\rho)0_p^{2s''}0_p^{2f''}0^{4a}0^{4b_\rho}\}.$$

There are six zones Z_ρ, and three zones Z.

The principal poles are $6(s+f)=6.1.$

The secondary poles are $8(s'+f'')=8.1.$

The tertiary poles are $12(s_p''+f_p''+\alpha''+\beta'')=12.1.$

The secondary poles, as well as the tertiary, are of one name only.

The numbers g, G, a, b, g_ρ, G_ρ, a_ρ, b_ρ, g_ρ, &c., enumerate the non-polar zoned features which have all different configurations. The sum of these numbers $\overline{>}0$.

Every non-polar zoned feature is read 24 *times on the solid. Every zoneless feature is read* 48 *times on the solid, in as many interzonal regions.*

XVI. Def. *A janal zoned axis is homozonal*, when, the axis being horizontal, two opposite eyes can see in the poles at the same time, one the trace t vertical between $t't'$, and the other t' vertical between tt. When there are but two traces in the pole, one eye will see the trace t vertical and t' horizontal, while the opposite eye sees t' vertical and t horizontal; and this is the configuration seen by opposite eyes in the secondary axis of a zoned tetrarchaxine.

A zoned tetrarchaxine polyedron has four heteroid principal 3-zoned axes, and three secondary homozonal 2-zoned axes. It has six identical zones, whose signature is

$$Z=\{2(s_p+s_p+g),\ 2(f_p+f_p'+G)\ 0_p^{2s'}\ 0_p^{2s}\ 0^{2a}\}.$$

The principal poles are

$$4(s_p+f_p)=4.2.$$

The secondary poles are

$$6(s'_p+f''_p+\alpha')=6.1.$$

Each of the six zones has $g+G+a+b\geqq 0$ non-polar features all of different configuration.

Every zoned non-polar feature occurs 12 *times on the solid. Every zoneless feature is found* 24 *times, in as many interzonal regions.*

XVII. *A zoned hexarchaxine polyedron* has six objanal (IV.) principal 5-zoned axes, ten objanal secondary 3-zoned axes, and fifteen janal tertiary 2-zoned axes, which may have any terminations

There are fifteen identical zones, whose signature is

$$Z=\{2(s_p+s'_p+s''_p+2g),\quad 2(f_p+f'_p+f''_p+2G),\quad \mathbf{0}_p^{2\alpha''}0_p^{2a''}\mathbf{0}^{4b}0^{4\beta}\}.$$

The principal poles of the solid are

$$6(2s_p+2f'_p)=6.2,\quad (s_p+f_p=1);$$

the secondary poles are

$$10(2s'_p+2f''_p)=10.2,\quad (s'+f'=1);$$

the tertiary poles are

$$15(2s''_p+2f'''_p+2\alpha'')=15.2,\quad (s''+f''+\alpha''=1).$$

Every zoned non-polar feature is read four times in each zone, i. e. *sixty times on the solid. Every zoneless feature is read* 120 *times, in as many interzonal spaces.*

Observe that two zoned signatures are numerically identical, if they differ only in p subscribed, and in the mode of exhibiting the factors of a number. There is nothing to prevent any two of the signatures having the same number of summits, face, and edges, of art. (II. . . . XVII.) from being spoken of as the same signature Z; but the polarity and repetition of the features differ with the symmetry.

For example, the two solids

have, the former the zones

$$Z=\{(\,.\,.\,)\ (4.\mathbf{1}_p),\ .0_p^{4.1}\},$$
$$Z'=\{(4.1_p)\ (2.\mathbf{1}'_p),\ \mathbf{0}_p^{4.1}\},$$

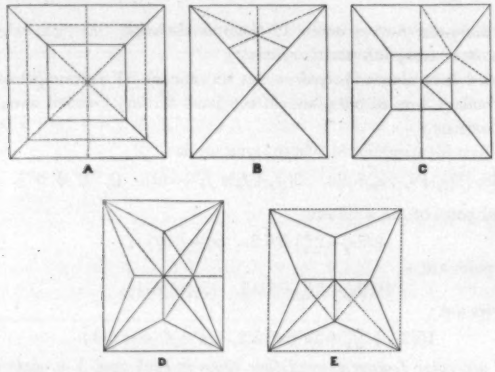
and the latter the zones

$$Z_1=\{(\,.\,.\,)\ (\mathbf{2}_p+2.1)\ 0^{2.2}\}$$
$$Z'=\{(2.2)\ (\mathbf{2}_p)\ \mathbf{0}^{2.1}\}.$$

We see that, neglecting p subscribed, Z and Z_1 are the same signature, as are Z' and Z'. The second solid is a 4-zoned monaxine heteroid, having only two polar zoned features. In the former every feature is polar.

The following will suffice for examples of zoned polyarchaxines. Here are five zoned

triarchaxines, in all of which, except the first, only half the solid is seen, the rest being conceived as below the page, which is a zonal plane. The dotted lines are undrawn zonal traces, which are, however, not all exhibited.



The zonal signatures are (art. XV.),—

A. $Z=\{(\ .\ .\ )(4.\mathbf{1}_p)0_p^{4.1}\}$, $\quad Z_l=\{(4.1'_p)(2.\mathbf{1}_p)\mathbf{0}_p^{2.1''}\}$;

B. $Z=\{(4.1_p)(\ .\ .\ )\mathbf{0}_p^{4.1}\}$, $\quad Z_l=\{(2.1_p)(4.\mathbf{1}'_p)0_p^{2.1''}\}$.

These two are reciprocals, both regular polyedra.

C. $Z=\{(4.1_p+4.1''_p)(.\ .)\mathbf{0}^{2.1}\}$, $\quad Z_l=\{(2.1_p+4.1'_p+2.1''_p)(4.\mathbf{1}_l)\mathbf{0}^{4.1_l}\}$;

D. $Z=\{(4.1_p)(.\ .)\mathbf{0}_p^{4.1''}\}$, $\quad Z_l=\{(2.1_p+4.1'_p)(4.\mathbf{1})0_p^{2.1''}\mathbf{0}^{4.1_l}\}$;

E. $Z=\{(4.1_p)(4.\mathbf{1}''_p)\}$, $\quad Z_l=\{(2.1_p+4.1'_p)(2.\mathbf{1}''_p)\mathbf{0}^{4.1_l}\}$.

The solid C has only amphigonal axes, and has eight non-polar edges in Z, all alike, and four non-polar faces, and four non-polar edges, in Z_l, either all alike. D has principal and secondary amphigonal, and tertiary amphigrammic, axes. The 12-edron E has principal and secondary amphigonal and tertiary amphiedral axes. Four of the six principal polar summits are in the plane of the page. The principal poles of E are tessaraces; the eight secondary poles are triaces; the twelve tertiary poles are quadrilaterals.

None of the solids drawn have zoneless faces, but they may have any number, if we load each of the forty-eight interzonal regions with the same polyedron R, zoned or zoneless. If R is zoneless, we shall impose R twenty-four times and its reflected image twenty-four times, so that the traces may be all preserved. The polyedron R may be polar or not, and symmetrical or not.

What precedes is sufficient to show that our signatures accurately record the zonal configuration. What more is required for placing the polyedron upon record, we shall see when we treat of registration.

2. *Zoneless Symmetry.*

XVIII. The polyedra which have a zoneless symmetry are—

a. r-ple monaxine heteroid polyedra;
b. r-ple monaxine contrajanal polyedra;
c. r-ple zoneless monarchaxine polyedra;
d. zoneless triaxine polyedra;
e. zoneless polyarchaxine polyedra, which have the axial system of the regular polyedra.

An *axis* zoned or zoneless is said to be of m-*ple repetition*, if the same feature presents itself m times in the same posture in a revolution of the solid about that axis.

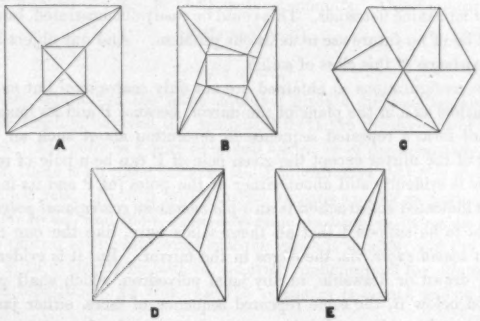
Every r-*zoned axis is an axis of* r-*ple repetition*; for though it has $2r$ like interzonal spaces about it, reflecting each other, no zoneless feature presents itself more than r times in the same posture in a revolution about the zoned axis.

Every amphigrammic, edrogrammic, or gonogrammic axis, zoned or zoneless, is of necessity an axis of 2-ple repetition.

By an r-*ple axis*, when *zoned* is not added, we understand a zoneless axis of r-ple repetition.

Heteroid r-*ple monaxine polyedron.*—A polyedron which has no zone, and but one zoneless r-ple heteroid axis, is an r-*ple monaxine heteroid polyedron.* The axis may be of any character. In this solid *every non-polar feature is read* r *times, and no more*, namely, once in each repeated sequence which presents itself in revolution about the axis.

Such solids are the following:—



A has a 2-ple gonoedral axis; B has a 4-ple amphiedral axis; C has a triple gonoedral, D has a 2-ple amphigrammic, and E a 2-ple edrogrammic axis.

XIX. *Monaxine contrajanal polyedron.*—If we place any r-ple heteroid monaxine P,

($r>1$), by a $2rm$-gonal face upon a mirror, and then turn P through an angle of m summits, while the image of P remains unmoved, that image, together with P so turned, forms an r-*ple monaxine contrajanal polyedron*. If, for example, $2rm=4r$, there will be in the mirror a repeated sequence of four edges ABCD, and the configurations read by opposite eyes in the axis supposed parallel to the page and perpendicular to the lines will be ($m=2$),

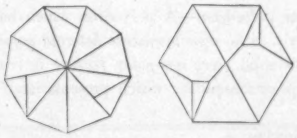
A B C D A B C D A B...
C D A B C D A B C D...

One eye sees A beyond C, and to the right of that B beyond D; the opposite eye sees A beyond C, and to the left of that B beyond D. The configurations are *contrajanal* (IV.).

We may take the last-drawn solid for the polyedron P, and conceive it laid by its square polar face in a mirror, and then turned through $m=1$ summit, while the image remains unmoved. The solid with the image will form a 2-ple monaxine contrajanal.

But it will be found impossible to produce this contrajanal configuration unless by employing a $2rm$-gonal polar face, and by turning the solid through m summits, $r>1$ being the index of repetition.

The monaxine heteroids,



which have each a repeated sequence ABC, will give by this process attempted, no zoneless figures but monaxine heteroids. This could be easily demonstrated, but such demonstration would be of no future use to us in our problem. And our object here is simply to prove the existence of this class of solids.

Further, the configurations so obtained are not only *contrajanal* but *monaxine*. For there is no zoneless axis in the plane of the mirror, because P and its image in our construction do not form a repeated sequence in revolution about such an axis; and no point of P out of the mirror except the given pole of P can be a pole of repetition.

And as there is evidently still about either of the poles (of P and its image) an r-ple repetition, the indicated construction is an r-*ple monaxine contrajanal polyedron*.

But it is not to be supposed that all these solids have, like the one constructed, a closed circle of *zonoid edges*, viz. the edges in the mirror. But it is evident that such a circle is either drawn or drawable, in any janal polyedron, which shall present in the faces above and below it, the same repeated sequence of faces, either janal or contrajanal, to the two poles. Our object here is not to discuss the form, but to establish the existence, of these polyedra.

XX. *One principal zoneless axis. Zoneless* r-*ple monarchaxines*.—A principal axis of repetition, zoned or zoneless, has a higher repetition than another.

Let the r-ple heteroid monaxine P be placed by its polar mr-gonal face A in any way on A′ the same face of P′ identical with P, so that the summits of A and A′ may coincide.

As the two opposite poles $\alpha\alpha'$ of the solid PP′ are identical, there is an axis β of double repetition, zoned or zoneless, at right angles to the axis $\alpha\alpha'$, and, as this axis $\alpha\alpha'$ is r-ple, there must be r axes β, having r identical poles p and r identical poles p_ρ, one of each kind in each repeated sequence about $\alpha\alpha'$.

Let $r>2$, and let $\alpha\alpha'$ be the only principal zoneless axis of PP′. Then β is not zoned; for if it were, it would have at least two zones, neither of which would contain the zoneless poles $\alpha\alpha'$. There would then be at least four poles α, one between each pair of hemizones about β, and $\alpha\alpha'$ would not be the only principal axis. Wherefore β is zoneless, and there are r zoneless axes β.

Let pp' be two like poles of axes β most nearly contiguous. They form with the principal poles $\alpha\alpha'$ a repeated sequence; wherefore there is a zoneless pole p_ρ of double repetition between p and p', of a different configuration from p, and there must be r zoneless poles p alternate with r zoneless poles p_ρ around the axis $\alpha\alpha'$.

The solid PP′ ($r>2$) is an *r-ple monarchaxine polyedron. It has* r *secondary axes, of double repetition, which are, according as* r *is even or odd, all janal and alternately different, or all heteroid and alike; and, in either case, the* 2r *poles present alternate configurations.*

In our construction there is a sequence of *zonoid edges* (in the faces A) in general effaceable; but we shall find that there are r-ple monarchaxines which have no such edges effaced, or effaceable, by effacements which shall preserve all summits.

Zonoid signature.—Every r-ple monarchaxine has a *zonoid signature*, which gives an exact enumeration of the *secondary poles*, whether they be faces, poles, or edges, but no account of the number of edges in polar faces or summits. This signature has the form

$$\zeta=r(\sigma_p+f_p+0^\alpha_p),$$

where

$$(\sigma+f+\alpha)=2,$$

and where every solution of the latter equation gives a different system of secondary poles. The distinction between the symbols **0** 0 (II.) in the account of zoneless polar edges vanishes. *The poles in* ζ *may be of two names, or of one name only.*

When $r=2$, there is no principal axis, the demonstration that β is zoneless fails, and β *may be* a zoned axis. In such case there are as many axes $\alpha\alpha'$ as there are zones about β perpendicular to $\alpha\alpha'$, and there cannot be more, because $\alpha\alpha'$ being an axis of repetition, must be central in the interzonal space in which it appears; and the limiting zones of that space must evidently be both of the same configuration, for otherwise $\alpha\alpha'$ would be no axis of repetition. In this case the symmetry is mixed, and will presently be discussed.

When $r=2$ and β is a zoneless axis, it follows that β is janal, otherwise $\alpha\alpha'$ would be no axis of repetition; and as the poles of $\alpha\alpha'$ and β form a repetition, there must be a third janal axis γ of even repetition at right angles to $\alpha\alpha'$ and to β.

One of these two axes $\beta\gamma$ may be an m-ple zoneless principal axis, of which the other two axes are secondaries.

Zoneless triaxines.—When $m=2$, the three zoneless axes $\alpha\alpha'$, β, and γ are all double janal axes, and the solid is a *zoneless triaxine polyedron.*

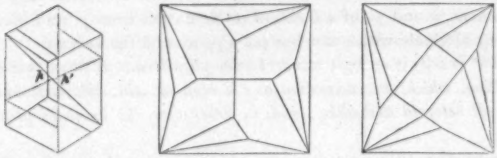
The zonoid signature of a zoneless triaxine, when it is recorded, has the form

$$\zeta=2\{\sigma_p+f_p+0_p^*\}, \quad (\sigma_p+f_p+\alpha=3),$$

showing six poles of three, of two names, or of one only. But we shall see that the registration of these signatures in zoneless triaxines is of no use for our purpose.

Every non-polar feature of an r*-ple monarchaxine or* (r=2) *triaxine is read* 2r *times upon the solid, namely,* r *times about either extremity of the* r*-ple janal axis.*

The following are such solids, in the two last of which only half the solid is seen, the other half, identical with that seen, being supposed below the paper.



The solid A, in which the dotted lines are below the page, is a zoneless triaxine whose zonoid signature is ($r=2$),

$$\zeta=2\{1_p+\mathbf{1}_p+0_p^1\},$$

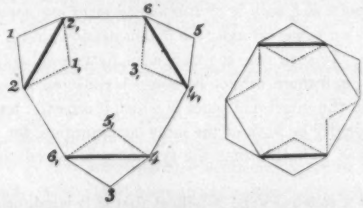
the three axes being amphigonal (pp'), amphiedral, and amphigrammic.

The solids B and C are 4-ple monarchaxines, half seen, whose zonoid signatures are

$$\zeta=4\{1_p+0_p^1\} \text{ for B and for C.}$$

The former has an amphigonal, the latter an amphiedral principal axis.

A simple mode of constructing zoned monarchaxines is to draw such *reticulations* as these, where the dark edges are effaceables:



The first is a 3-zoned monarchaxine, the second a zoneless triaxine, reticulation, having in the page an amphigonal and an amphigrammic axis. If we crown the former in one

polar face with the hexace 123456, and in the lower polar face with the hexace $1_p2_p3_p4_p5_p6_p$, the solid completed will be a triple monarchaxine. If we crown the opposite polar faces of the latter with tessaraces upon the marginal triangles, we form a zoneless triaxine polyedron.

XXI. *System of principal zoneless axes. Zoneless polyarchaxines.*—The following propositions have been sufficiently established by the proof in arts. XIII., XIV., that every system of principal axes is that of a regular polyedron.

We give an account in the zonoid signature of a polyarchaxine, of its *secondary and tertiary poles.*

A zoneless triarchaxine has three principal 4-ple janal axes, four secondary triple janal axes, and six double janal tertiary axes. Its zonoid signature is

$$\zeta=\{8(\sigma_p+f_p')+12(\sigma_p''+f_p''')+0_p^{\text{iv}}\},$$

where

$$\sigma_p'+f_p'=1$$

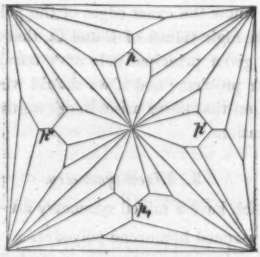
describes the secondary, and

$$\sigma_p''+f_p'''+\alpha''=1$$

the tertiary poles.

The poles of the same rank are of one name only. Every non-polar feature is read twenty-four times in the solid, viz. four times about each pole of every principal axis.

Such a solid is here half-drawn, the portion unseen being identical with that seen, and below the page.



This triarchaxine 72-edron 62-acron has amphigonal principal and ($pp_1p'p''$) secondary axes, and amphigrammic tertiary axes. If we efface the twelve polar edges of the tertiary axes, we obtain a 60-edron 62-acron having amphiedral tertiary axes.

A *zoneless tetrarchaxine polyedron* has four triple heteroid principal axes, and three double janal secondary axes. Its zonoid signature is

$$\zeta=\{+6(\sigma_p'+f_p'')+0_p^{\text{iv}}\},$$

where

$$\sigma_p'+f_p''+\alpha')=1$$

describes the secondary poles.

The principal poles may be of one name, or of two names.

Every non-polar feature is read twelve times on the solid, viz. thrice about the pole p of every principal axis.

A *zoneless hexarchaxine polyedron* has six quintuple janal principal axes, ten triple janal secondary axes, and fifteen double janal tertiary axes. Its zonoid signature is

$$\zeta=\{20(\sigma_p'+f_p'),\quad 30(\sigma_p''+f_p'')+0_p^{2\alpha''}\},$$

where

$$\sigma_p'+f_p'=1$$

gives the secondary, and

$$\sigma_p''+f_p''+\alpha''=1,$$

the tertiary poles.

Every non-polar feature is read sixty times on the solid, viz. five times about each extremity of every principal axis.

Zoneless polyarchaxines are easily constructed on zoned ones, by drawing lines in every principal, or secondary, or tertiary face, by which the zones are destroyed, and the repetition preserved; or by crowning like polar faces with polyedra which have a zoneless repetition of equal rank.

We shall see, by our processes of construction, that we always know the *name* of the pole opposite to any zoned or zoneless pole that we may be handling; for we always know the character (V.) of the axis; but the number of edges in that opposite pole is not thereby given.

We know of course the edges of every pole that we construct, but when the axis is heteroid, we do not always know the exact edges of the pole opposed.

Hence it may happen, in high values of P and Q, that we do not always know the exact feature opposite to a given principal pole of a tetrarchaxine. But this is not of the least consequence in our problem; and if we should wish to know what is the exact feature so opposite, a question that never arises in our argument, we can easily determine the point by other considerations.

3. *Mixed symmetry.*

XXII. The polyedra which have a mixed symmetry are—

a. r-zoned homozone polyedra.

b. r-ple monozone monaxine polyedra.

Homozone axes.—Let M M′ be two identical r-zoned polyedra having a $2rm$-gonal polar face F. There are m edges of F between two contiguous traces (VII.), wherefore the traces of F are either all agonal, or all diagonal. Let the identical polar faces FF′ of M and M′ be so united that the trace t of F shall cover the trace t of F′ of different configuration from t (VII.). Let pp' be two contiguous terminations of traces in the united faces FF′; and let PP′ be the poles of the r-zoned janal axis of the solid MM′. The sequence $PpP'p'$ is a repetition; there is therefore a *pole of even repetition* between p and p', which is a zoneless pole, because no zone by hypothesis intervenes between p and p';

and there will be $2r$ of these zoneless poles; one central between every two hemizones about the r-zoned axis. And as the same sequence $PpP'p'$ is read about them all, the $2r$-zoneless poles have all one configuration, or configurations of which one is the reflected image of another.

Let PP′ be the only principal axis of the solid; then, as the pole P can recur only twice in a revolution about the zoneless poles, they are *all of double repetition*. Wherefore there are r-janal and similar axes of zoneless double repetition in a plane perpendicular to the r-zoned axis PP′.

1. Let r be odd; then every zone of the axis PP′ is perpendicular to one of the r zoneless axes, or symmetry would be impossible; and it is consequently *a repeating zone*, of which every non-polar feature is an *objanal monozone feature*, i. e. a feature f diametrically opposite to another which is to an opposite eye the inverted image of f (art. IV.).

Further, when r is odd, the poles $\alpha\alpha'$ of any one of the r zoneless axes are *contrajanal poles*: for, if not, the axis $\alpha\alpha'$ will be strictly janal, such that two opposite eyes in the axis will read exactly the same configurations from left to right: therefore the axis PP′ perpendicular to $\alpha\alpha'$ will be an axis of even repetition, since the same pole α recurs exactly in half a revolution about PP′; which is absurd, because, r being odd, PP′ is an axis of odd repetition (XVIII.).

Therefore $\alpha\alpha'$ is, when r is odd, a contrajanal axis.

2. Let r be even; then because the axis PP′ is of even repetition, the configurations read by two opposite eyes in the axis $\alpha\alpha'$ are strictly identical, and $\alpha\alpha'$ is a zoneless 2-ple strictly janal axis.

When r is even, either zone of M conspires to form the zone of the solid (MM′), as is evident from the position of the trace t upon the trace t'. For this reason the solid is called *homozone*, the two zones of M being confounded together. And this name is conveniently used to designate the solid (MM′), whether r be odd or even.

The axis PP′ is an r-*zoned homozone axis*.

When $r>2$, the axis PP′ is a principal axis, and the solid is a *homozone monarchaxine polyedron*. When $r=2$, there is no principal axis (XX.), as the zoned axis has, like the two zoneless ones, but a 2-ple repetition. The 2-zoned homozone is a *triaxine homozone polyedron*.

The zonal and zonoid signatures of the r-zoned homozone polyedron, for r odd or even, are

$$Z=\{(2\sigma_r+2g)(2f_r+2G)\ 0_r^r\ 0_r^n\ 0^{2n}\ 0^{2s}\},$$

where

$$\sigma_r+f_r+\alpha=1, \text{ and } \alpha=0, \text{ if } r>2,$$

$$\zeta=2r\{\varepsilon_r+\varphi_r+0_r^q\},$$

where

$$\varepsilon_r+\varphi_r+\theta=1$$

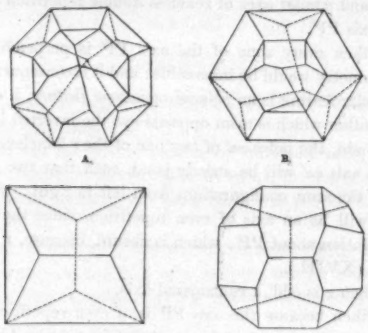
describes the zoneless pole.

Every non-polar zoned feature is read on the solid $2r$ *times*, namely, once in each hemizone about the zoned axis.

Every non-polar zoneless feature is read 4r *times*, viz. twice about each of the 2r zoneless poles.

In our construction the edges of the united faces FF' form a circuit of zonoid edges, generally effaceable, so that the zoneless axes may become amphiedral. But we shall learn that there are homozones which have no such lines effaced or effaceable.

Homozone polyedra are the following:—



In the first the 4-zoned axis has polar tessaraces, and the four 2-ple zoneless axes are amphigrammic. The signatures of A are—

$$Z=\{(2.1_r+2.1)(2.\mathbf{3})\ \mathbf{0}^{2.1}\ 0^{2.3}\}(f_r=\alpha=0),$$
$$\zeta=\ 8\{0^1_r\},\quad (\epsilon_r=\phi_r=\mathbf{0}).$$

The signatures of the 3-zoned homozone B are—

$$Z=\{(2.1_r+2.1)\ (2.\mathbf{3})\ (\mathbf{0}^{2.1}\ 0^{2.3}\},$$
$$\zeta=\ 6\{0^1_r\}.$$

It has polar triaces, and three amphigrammic 2-ple zoneless axes.

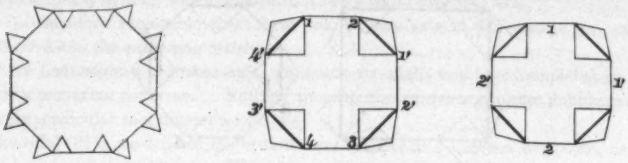
C is a 2-zoned homozone, *i. e.* a triaxine homozone, whose zoned axis is amphigrammic, the two zoneless ones being amphigonal. The signatures are—

$$Z=\{(2.1)(2.\mathbf{2})\ \mathbf{0}^1_r\ 0^1_r\ 0^{2.1}\}(\sigma_r=f_r=g=\alpha=0),$$
$$\zeta=\{4.1_r\}(\phi_r=\theta=0).$$

The fourth, D, has an amphigrammic zoned axis, and two amphiedral zoneless axes. Its signatures are—

$$Z=\{(2.1)(2.\mathbf{2})\ \mathbf{0}^1_r\ 0^1_r\ 0^{2.1}\},$$
$$\zeta=\{4.\mathbf{1}_r\},\ (\epsilon_r=\theta=0).$$

Homozones may be easily constructed by drawing janal reticulations like the following:



The interior marginal triangles are supposed to present in the inferior face of the reticulation precisely the configuration formed by the exterior ones in the upper polar face.

The first is a 4-zoned homozone reticulation, which becomes a 4-zoned homozone polyedron, if crowned in both the upper and lower faces by octaces whose rays pass to the eight marginal triangles.

The second is a 4-zoned monarchaxine reticulation, and becomes a 2-zoned homozone polyedron, if crowned in the upper face by an octace upon the four upper marginal triangles and upon the four summits 1 2 3 4, and if crowned below by an octace upon the four lower triangles and upon the summits 1′ 2′ 3′ 4′. The same reticulation in the third figure becomes a 2-zoned homozone polyedron, if crowned above by a hexace upon the four upper triangles and on the points 1 2, and by a hexace below on the lower marginal triangles and on the points 1′2′, whereby the points 1 2 1′ 2′ become four zoned triaces.

XXIII. *Monaxine monozone polyedra.*—Let F be a polar face of any $\overline{3+r}$-ple zoneless axis α of a polyedron P, and let P be placed on a mirror by the face F. The solid (PP′) formed by P and its image P′ is a $\overline{3+r}$-ple monozone monaxine polyedron.

The zoneless axis ($\alpha\alpha'$) of the solid (PP′) is *contrajanal* (IV.), and in the plane of the mirror there is a $\overline{3+r}$ times *repeating zone*, where r may be odd or even.

The zonal signature of the solid is, putting $r_1=3+r$,

$$Z=r_1\{g,\ G,\ \mathbf{0^e},\ 0^4\},$$

which has gr_1 zoned summits of g configurations, Gr_1 zoned faces of G configurations, &c.

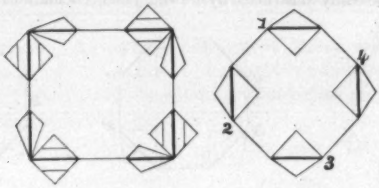
Every zoneless feature except the two zoneless poles is read $2r_1$ *times on the* r_1-ple *monozone monaxine.*

The designation of monaxine contrajanal belongs to these solids in strictness, as well as to those of art. XIX. But there can never be any confusion in our terms, if the zoneless polyedra of XIX. be called by that name, which is to be understood as zoneless, if the term monozone be wanting.

A 2-ple monaxine monozone is . . . . . whose zonal signature

$$Z=2\{1,\ \mathbf{2},\ 0^1\}\ (a=0).$$

The zoneless axis is amphigrammic, 2-ple and contrajanal. Monaxine monozones are readily constructed on simple reticulations like these:



The former is a 4-ple monaxine monozone reticulation, which becomes a 4-ple monaxine monozone polyedron, if crowned in both the opposite polar faces by octaces upon the eight marginal triangles.

The latter is a 4-zoned monarchaxine reticulation, which becomes a 4-ple monaxine monozone polyedron, if crowned both above and below by octaces through the same four points 1 2 3 4 and through the marginal triangles. By such coronation the principal zones of the reticulation are destroyed, but the 4-ple repetition is preserved, and the secondary zone of the reticulation in the plane of the page is preserved also.

XXIV. Theorem. *There cannot be more than one principal axis in a mixed symmetry.*

For we have proved (XIII., XIV.) that the only systems of principal axes are those of the regular polyedra.

In a zoned polyarchaxine there can be no zoneless poles; for if there were one, there would be one at least in every interzonal region, *i. e.* there would be 24, 48, or 120; and such a number of similar poles has been proved impossible in XIII.

For a like reason there can be no zoned pole about any of the zoneless poles of a polyarchaxine, and consequently no zone; for if there were one zone there would be many, and therefore zoned axes and poles. Hence the theorem is proved.

From this theorem it follows that there cannot be in the solid PP′ of the preceding article any other axis than $\alpha\alpha'$. For if there were another, either (1) $\alpha\alpha'$ would be a principal axis, or (2) there would be one principal axis of more than $(3+r)$-ple repetition, or (3) there would be no principal axis.

1. If $\alpha\alpha'$ be a principal axis, every secondary axis β will be at right angles to $\alpha\alpha'$, otherwise there would be more than one axis $(\alpha\alpha')$ in the sequence repeated about β, which is impossible by the preceding theorem; and β will be a $2m$-ple axis, because the pole α occurs in half a revolution about β; but there is no axis of even repetition at right angles to $\alpha\alpha'$ by the reasoning of XXII. because $\alpha\alpha'$ is contrajanal. Therefore $\alpha\alpha'$ is no principal axis.

2. If there be a principal axis A different from $\alpha\alpha'$, since it is not at right angles to the zone FF′, there will be, by the definition of a zone (II.), more than one such axis A, which is impossible by the preceding theorem.

3. If there be no principal axis, there will be at least about the pole of $\alpha\alpha'$, $3+r$

identical poles of $(3+r)$-ple repetition, and these will form by the reasoning of XIII. a polyarchipolar system, which is absurd, if there be no principal axis.

It is therefore impossible that there can be any axis in PP′ except $\alpha\alpha'$, unless $r_1=2=3+r$ in the preceding article.

If we had taken α a double axis, we might or might not have completed in that article a monaxine monozone. Nothing prevents the existence of other double axes α, each perpendicular to a different zone (F).

The solid (PP′) constructed might thus have been a $(2m+1)$-zoned homozone, having $2m+1$ contrajanal double axes (XXII.).

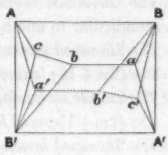
4. *Neuter Symmetry.*

XXV. The polyedra which have a neuter symmetry are of one species only, viz.,

Contrajanal anaxine polyedra.—Let F be any $2m$-gonal zoneless non-polar face of a polyedron P, and let P be placed on a mirror by the face F, and turned through two right angles, while the image P′ remains unmoved. The solid PP′ constructed by P so turned, and by the image P′, *may be a contrajanal anaxine polyedron.* It will be such, if it has neither pole nor zone.

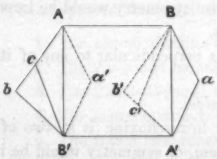
It is evident that to any edge ab, on one side of the mirror, there corresponds another diametrically opposite edge a_1b_1, on the other side, in the image, such that the configuration seen along ab to the right is that read along a_1b_1 to the left. The edges of the solid (PP′) in all cases form *janal anaxine pairs* (ab, a_1b_1) of edges diametrically opposite, unless P be such that by our construction we have completed a zoned or polar symmetry, in which case certain of the pairs will be zoned or polar.

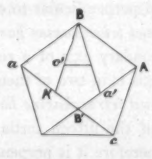
Every feature f *is twice read on a contrajanal anaxine polyedron, namely,* f *and its reflected image.*



Here is a contrajanal anaxine 10-edron 10-acron, where the dotted lines are supposed below the page. The configuration read along ab to the left is that read by an opposite eye along $a'b'$ to the right, and the same is true of any other pair bc, $b'c'$. This solid is formed by crowning with edges ab, $a'b'$ the *contrajanal penesolid* below on the left.

The same solid is constructed by drawing the *janal anaxine pair*,





AB′, A′B, in the quadrilaterals A′c′Ba, Aa′B′c, of the 2-ple monaxine monozone polyedron on the right, whereby both the zone and the axis are destroyed.

Observe that the term *janal* along with *anaxine* always means *contrajanal*.

XXVI. Def. *A janal anaxine pair on any polyedron are any two edges* ab, a_1b_1 *diametrically opposite, non-polar, and zoneless, such that the configuration read along* ab *to the right is exactly that read by an opposite eye along* a_1b_1 *to the left.*

A *janal anaxine* A-gon (*or* A-ace) *is any zoneless non-polar face* (*or summit*) *whose* A *edges form with those of an opposite* A-gon (*or* A-ace) A *janal anaxine pairs.*

For example, A′ and A are janal anaxine triaces in the last-drawn polyedron.

Thus we see that there are janal anaxine edges in solids which are not janal anaxine polyedra.

XXVII. It is important that we should here determine what kinds of polyedra have janal anaxine pairs.

1. Let P be a polyedron of zoned or mixed symmetry which has janal anaxine pairs. The janal anaxine pair ab, a_1b_1 will either meet on a zone Z, or ab will meet $a'b'$ and a_1b_1 will meet $a'_1b'_1$ on a zone Z (II.). The pair ab, a_1b_1 being diametrically opposite may be supposed parallels. Then the zoneless ab meets its reflected image $a'b'$, and a_1b_1 meets $a'_1b'_1$ on Z, and the angle $(ab, a'b')$ has exactly the configuration of $(a'_1b'_1, a_1b_1)$, and is diametrically opposite thereto.

Hence Z must be *a repeating zone*, to which an axis of even repetition is perpendicular, for the same configuration recurs about that axis in half a revolution.

It is then requisite and sufficient, in order that a zoned polyedron have janal anaxine pairs, *that it have a zone perpendicular to an axis, zoned or zoneless, of even repetition.*

XXVIII. The $(2m+1)$-zoned monarchaxine (XII.) has none of its $2m+1$ secondary axes perpendicular to a zone; for each of these axes is in one of the $2m+1$ zones.

The $(2m+1)$*-zoned monarchaxine has no janal anaxine edge.*

The $2m$-zoned monarchaxine polyedron has every secondary axis in one zone and perpendicular to another (XI., VII.).

The 2m*-zoned monarchaxine has janal anaxine pairs.*

The $(2m+1)$*-ple monaxine monozone has them not,* the axis being of odd repetition.

The 2m*-ple monaxine monozone has such edges.*

The $(2m+1)$*-zoned homozone has them* (XXII.).

The 2m*-zoned homozone has them not.*

The zoned triarchaxine polyedron has each of its six tertiary axes in two of its nine zones, and perpendicular to one of them, otherwise symmetry would be impossible.

The zoned triarchaxine has janal anaxine edges.

No secondary axis of a zoned tetrarchaxine is perpendicular to any of its six zones, for this axis is in two of them.

The zoned tetrarchaxine has no janal anaxine edges.

Each of the fifteen tertiary axes of a zoned hexarchaxine is in two of the fifteen zones, wherefore it is perpendicular to one of them, or symmetry would be impossible.

The zoned hexarchaxine has janal anaxine edges.

XXIX. 2. Let P be a polyedron of zoneless symmetry which has janal anaxine edges.

The edge ab makes with the zoneless pole α of P a configuration which is the reflected image of that made by the opposite edge a_1b_1 with the opposite pole α of a janal axis. The only zoneless polyedron whose opposite janal poles can have such a configuration is the r-ple monaxine contrajanal polyedron (XIX.).

And we see, by considering the repeated sequence of faces above and below any closed line drawn or drawable on the polyedron equidistant from both poles, as, for example, (XIX.),

A B C D A B C D
C D A B C D A B, for r even (=2),

and

A B C D A B C D A B C D
C D A B C D A B C D A B, for r odd (=3),

that the edge $\frac{A}{C}$ diametrically opposite to $\frac{A}{C}$ in the former does not, but that the edge $\frac{C}{A}$ diametrically opposite to $\frac{A}{C}$ in the latter, does present, to two eyes either in the plane of the sequence or at the poles, a contrajanal configuration as compared with the first edge $\frac{A}{C}$.

The same thing is easily proved by taking any sequence of $2m$ faces, and comparing the cases of r odd and r even.

The same thing is also proved thus. If ab, a_1b_1 are a janal anaxine pair about a zoneless axis of r-ple repetition, the line bisecting the two edges is a diameter (XXV.), and the plane containing it and the axis is a diametral plane. There are about the axis $r-1$ other similar janal anaxine diameters symmetrically disposed; and such symmetry is evidently impossible unless $r-1$ be even, that is, unless r be odd.

The r*-ple monaxine contrajanal polyedron has janal anaxine pairs if* r *be odd, but not if* r *be even.*

In all other zoneless polyedra, the opposite poles of a janal axis have configurations which are exact repetitions of each other to opposite eyes, wherefore the configuration read by one eye along an edge ab, which configuration includes the pole, cannot be the reflected image of that read along a_1b_1 by an opposite eye.

No polyedron of zoneless symmetry, except the monaxine contrajanal, has janal anaxine edges.

XXX. We have proved that janal anaxine edges are found in the polyedra following:—

1. The zoned triaxines (XI.);
2. The $2m$-zoned monarchaxines (XII.);
3. The $(2m+3)$-zoned homozones (XXII.);
4. The zoned triarchaxines (XV.);
5. The zoned hexarchaxines (XVII.);
6. The $2m$-ple monaxine monozones (XXIII.);
7. The $2m+1$-ple monaxine contrajanals (XIX.).

Every zoneless and non-polar edge on these solids is a janal anaxine edge.

The signatures, zonal and zonoid, give the number of polar, zonal, and epizonal edges of the polyedron, whereby that of the zoneless and non-polar is accurately known; and as the number of repetitions of every feature is known by what precedes, *the exact number of different janal anaxine pairs upon all these solids is given by our signatures.*

Registration of P-*edra* Q-*acra and* Q-*edra* P-*acra.*

XXXI. It has been shown that all possible symmetrical P-edra Q-acra are comprised in the preceding classes. They are thus registered.

TABLES A.

1. *Zoned Symmetry.*

1. Monozone P-edra Q-acra (III.),

$$(\mathbf{PQ})\{Z\}=A,$$

where A is the number of monozones which have the zonal signature Z.

2. *Zoned monaxine heteroids* (VIII., IX.),

$$(\mathbf{PQ})Y^{r}_{kst}\ \{ZZ'\}=B,$$

$$(\mathbf{PQ})Y^{r+3}_{kst}\{\ \ Z\}=C,$$

where B is the number of these solids having the zonal signatures ZZ′ and a heteroid $2r$-zoned axis, of which Y expresses the character and nothing more (V.). In the same way, we record that there are C $(2r+3)$-zoned heteroids, having a given character of axis, and a given zonal signature Z.

We are content to know about these solids what is here registered, without asking what are the exact polar features; for we have a separate table of polar summits and faces, in which every polar A-ace and A-gon is recorded, with its zones, and with the character of its axis. If the question should arise, which, however, never does arise in our problem, what is the exact feature opposite to a given heteroid pole, it can easily be determined by reference to our processes of construction.

3. *Zoned triaxines* (XII.),

$$(\mathbf{PQ})(Y^{1}_{ja}Y^{2}_{ja}Y^{3}_{ja})\{ZZ'Z''\}=D,$$

where the zonal signature and characters of the janal axes of the D polyedra are recorded.

4. *Zoned monarchaxines* (XII.),

$$(\mathbf{PQ})X^{p}_{ja}Y^{2}_{ja}Y^{3}_{ja}\{ZZ'Z''\}=E,$$

$$(\mathbf{PQ})X^{p+3}_{ja}Y^{2}_{kst}\{\ ZZ''\ \}=F,$$

where X denotes an axis whose polar features are exactly registered, the principal poles being always known on these solids, and recorded with their zonal traces.

5. *Zoned triarchaxines* (XV.),

$$^{3}(\mathbf{PQ})X^{1}_{ja}Y^{3}_{obj}Y^{3}_{ja}\{ZZ_{ll}\}=G.$$

The principal janal poles are always registered with their traces. The secondary and tertiary poles may not be exactly given.

6. *Zoned tetrarchaxines* (XVI.),

$$^{4}(\mathbf{PQ})Y^{1}_{hol}Y^{2}_{jn}\{Z\}=H.$$

The number of edges in the two heteroid principal poles, and in the janal secondary poles, may not always be known in this Table.

7. *Zoned hexarchaxines* (XVII.),

$$^{6}(\mathbf{PQ})X^{1}_{obj}Y^{2}_{obj}Y^{3}_{jn}\{Z\}=I.$$

2. *Zoneless Symmetry.*

8. *Zoneless r-ple monaxine heteroids* (XVIII.),

$$(\mathbf{PQ})Y^{r}_{hol}=J\ (r>1),$$

where the absence of zonal and zonoid signature, and the heteroid axis, are characteristic of the class.

9. *Zoneless r-ple monaxine contrajanals* (XIX.),

$$(\mathbf{PQ})X^{r}_{conjn}=K\ (r>1),$$

where the polar feature is exactly registered.

10. *Zoneless triaxines* (XX.),

$$(\mathbf{PQ})y^{2}_{jn}y^{n}_{jn}y^{n}_{jn}=L.$$

Here are symbols y of double janal zoneless axes, of which not even the characters (V.) are registered. We shall see that it suffices for our problem to know the number L of *all* zoneless triaxine P-edra Q-acra. All the poles of them will be found in the following Table, which enumerates with exact description all janal poles. All that we care to know more of these L solids is how many amphigrammic axes they contain, and this we shall readily determine in the proper place (XLI.).

11. *Zoneless 2r-ple, &c. monarchaxines* (XX.),

$$(\mathbf{PQ})X^{2r}_{jn}\ Y^{1}_{jn}\ Y^{2}_{jn}\ \{\zeta\}=M,\ (r>1),$$
$$(\mathbf{PQ})X^{2r+1}_{jn}Y^{1}_{hol}\quad\{\zeta\}=N,$$
$$(\mathbf{PQ})X^{2}_{jn}\ \ Y^{1}_{hol}\quad\{\zeta\}=N',$$
$$(\mathbf{PQ})Y^{2}_{jn}\ \ Y^{2}_{(amphi.)}\{\zeta\}=N''.$$

In all these ζ denotes the zonoid signature common, as well as the other characters specified, to all the solids registered in the number M, N, &c.

The only zoneless monarchaxines, in which there can be any doubt about the number of edges in their principal poles, are those which have triple janal axes perpendicular to amphiedral, amphigonal, or amphigrammic secondary axes (XXXVIII.).

12. *Zoneless triarchaxines* (XXI.),

$$^{3}(\mathbf{PQ})X^{1}_{jn}Y^{2}_{jn}Y^{2}_{jn}\{\zeta\}=P,$$

in which the principal poles are exactly specified, and where the zonoid signature gives the characters of the secondary and tertiary poles.

13. *Zoneless tetrarchaxines* (XXI.),

$$^{4}(PQ)Y^{1}_{1st}Y^{2}_{ja}\{\zeta\}=Q.$$

We may not know always the edges in the pole opposed to a given tetrarchipole.

14. *Zoneless hexarchaxines* (XXI.),

$$^{6}(PQ)X^{1}_{ja}Y^{1}_{ja}Y^{2}_{ja}\{\zeta\}=R.$$

There are but few values of P and Q for which polyarchaxine polyedra are possible; and the ambiguities above denoted with respect to their principal poles can only exist for high values of P and Q, such as will not be calculated for a millennium or two.

3. *Mixed Symmetry.*

15. *(r+3)-ple monaxine monozones* (XXIII.),

$$(PQ)_{mo.mo}X^{r+2}_{cojo}\{Z\}=S,$$
$$(PQ)_{mo.mo}Y^{2}_{cojo}\{Z\}=S',$$

where the exact polar features are always known, except for 2-ple axes. This is, however, sufficient for our purpose, as we shall see (XXXIX.).

16. *Homozones* 2r*-zoned and* (2r+1)*-zoned* (XXII.).

$$(PQ)_{hom}X^{2r}_{ja}\ Y^{2}_{ja}\ \{\ Z\zeta\}=T,\ (r\geqq 1),$$
$$(PQ)_{hom}X^{2r+1}_{obja}Y^{2}_{cojo}\{\ Z\zeta\}=T'\ (r>1),$$
$$(PQ)_{hom}Y^{2}_{obja}\ Y^{2}_{cojo}\{^{3,6}Z\zeta\}=U,$$
$$(PQ)_{hom}Y^{1}_{ja}\ \ Y^{2}_{ja}\ \{\ ^{4}Z\zeta\}=W.$$

In all these entries the first axis is zoned, the second zoneless. In the third, U, $^{3,6}Z$ denotes a zonal signature of the triarchaxine form, or hexarchaxine, or possibly of both forms. When such a zone occurs in homozones along with a zonoid signature ζ, showing poles of a name not excluded from Z, we may not always know the exact edges of the zoned pole (vide XXXVII., XXXIX.).

When this ambiguity does not exist, the numbers U and W (where ^{4}Z denotes a signature of the tetrarchaxine form) will not be found in our Table; and they can only present themselves for high values of P and Q, which will not be calculated for the next thousand years. But it is enough for us that the numbers U W are exactly known, as this suffices for our problem.

4. *Neuter Symmetry.*

17. *Janal anaxine polyedra* (XXV.),

$$(PQ)_{ja.an}=\Xi,$$

in which there is neither zone nor pole, Ξ being the entire number of the solids.

5. *Asymmetric Polyedra.*

18. $$(\mathbf{PQ})_{\infty}=\Theta,$$

giving the entire number Θ of the solids.

This completes the Tables A of P-edra Q-acra, in which it is understood that every zonal and zonoid signature, and every zoneless repetition (r) will be found entered.

Another such Table is made of the Q-edra P-acra, which will have the form

$$(\mathbf{QP})\quad\{Z\}=A,$$
$$(\mathbf{QP})Y^{2r}\{Z\}=B, \&c.,$$

where the signatures will differ from the preceding only in the exchange of faces for summits, and of zonal for epizonal edges.

Registration of janal poles of P-*edra* Q-*acra and* Q-*edra* P-*acra.*

TABLES B.

XXXII. A janal zoned polar A-gon, which is always the termination of a janal zoned axis, is described by its zonal traces in one of the following manners:—

$$A_{ja}^{m.di},\quad A_{ja}^{m.ag},\quad A_{ja}^{2h.agdi},\quad A_{obj}^{n.di},\quad A_{obj}^{n.ag},\quad A_{ja}^{n.mo},\quad A_{obj}^{n.mo},$$

where m is an odd or even number of traces diagonal or agonal, as the case may be, h is any number of diagonal alternate with as many agonal traces, and n is any odd number of traces, agonal, diagonal or monogonal, as the case may be, in the pole of a janal or objanal axis.

A janal polar zoneless A-gon is described thus by its repetition,

$$A_{ja}^{r},\quad A_{cobja}^{r}$$

where r may be odd or even, the axis being janal or contrajanal. The summits of the A-gon form a sequence of configuration r times repeated in the circuit of the A-gon.

Def. *A janal zoned axis is heterozone, if the polyedron has zones of more than one configuration.*

Thus all janal zoned axes are conveniently divided into *heterozone* and *homozone*. A homozone axis or pole has but one zonal signature, as the solid has zones only of one configuration (XXII.). A heterozone axis or pole has always two or three zonal signatures. The $2r$-zoned heterozone pole has two different zones about its axis; the $(2r+1)$-zoned heterozone has only one zonal configuration about the principal axis; but it has also a secondary zone of different configuration (VII.). Hence the term heterozone applies usefully whether the number of principal zones be odd or even.

Heterozone janal polar faces.

$$A_{ja}^{2r.ag}\ sF\{ZZ'Z''\}=a,$$
$$A_{ja}^{2r.di}\ sF\{ZZ'Z''\}=b,\quad \{r\geqq 1\},$$
$$A_{ja}^{2r.agdi}sF\{ZZ'Z''\}=c,$$

where Z'' always denotes the zone perpendicular to the axis of the registered janal pole.

The number s is that of the summits of the solid not in the A-gon, *i. e.* inferior to the A-gon; and F is the number of faces different from it, that is, inferior to it.

We have here

$$P=F+1, \quad Q=A+s.$$

We read that there are a A-gonal janal poles having r agonal traces of the zone Z, and r agonal traces of Z'; and a zone Z'' perpendicular to the axis of the A-gons. We read also c A-gonal janal poles having r agonal traces of Z and r diagonal traces of Z', with the secondary zone Z''.

When $2r=4$, the numbers a, b, c, when ZZ'Z are the zoned signatures, will comprise the archipolar A-gons of triarchaxines having the zones ZZ'.

When $2r=2$, the numbers a, b, c enumerate the polar A-gons secondary in zoned monarchaxines, those of zoned triaxines, the tertiary polar A-gons of zoned triarchaxines and hexarchaxines, for the signatures may be ZZ'Z, or ZZZ, the case of the hexarchaxines. In this latter case only the term heterozone is improperly applied to any polar face above registered.

All $(2r+3)$-zoned heterozone A-gons are registered thus:

$$A_{ja}^{(2r+3)ag}\,sF\{ZZ''\}=d,$$
$$A_{ja}^{(2r+3)di}\,sF\{ZZ''\}=e,$$
$$A_{ja}^{(2r+3)mo}\,sF\{ZZ''\}=f.$$

Principal janal polar faces of zoned Polyarchaxines.

$$^{3}A_{ja}^{4y}\,sF\{ZZ''\}=\theta_1,$$
$$^{6}A_{ja}^{5y}\,sF\{\ Z\ \}=\theta_2,$$

where $4y$ means $4ag$ or $4di$, and $5y$ means $5ag$, $5di$ or $5mo$, as the case may be.

Radical janal polar faces of zoned Polyarchaxines.

$$^{3}A_{rad.ja}^{r}\,sF\{ZZ''\}=\theta_1,$$
$$^{6}A_{rad.obj}^{r}\,sF\{\ Z\ \}=\theta_2.$$

It is necessary, as we shall hereafter learn (§ 16.), to have a separate Table of all the *radical* archipoles of the polyarchaxines, which are, however, supposed to be included in the numbers θ_1 and θ_2 above written.

A radical polyarchipole janal or heteroid, whether face or summit, is a principal pole of a polyarchaxine collateral with other principal poles of the solid.

The triarchipoles will be found also under the numbers a, b, c $(r=2)$; for nothing prevents them being constructed and handled as monarchaxine poles.

The hexarchipoles may be constructed and handled as homozone poles, and they will consequently be found also in the Table following. This will, however, create no confusion in our results. There are no *janal* tetrarchipoles ^{4}A.

Homozone janal polar faces (XXII.).

$$A_{ja}^{2rg}\ sF\{Z\zeta\}=g,$$
$$A_{\omega g}^{(2r+2)g}sF\{Z\zeta\}=h,$$

where y stands for the *traces ag, di, mo, agdi*, as the case may be. Z is the zonal, and ζ the zonoid signature.

When $r=1$, the g homozones recorded are *homozone triaxines* (XXII.).

The number h will contain every secondary polar A-gon of all triarchaxines, and every primary and secondary polar A-gon of all hexarchaxines; for all these faces are in repeating zones to which an axis of 2-ple (zoned) repetition is perpendicular.

The number $g(r=1)$ will contain every secondary polar A-gon of a tetrarchaxine, for this solid has homozone secondary axes (XVI.).

Zoneless janal polar faces.

XXXIII.

$$A_{ja}^{r+2}sF\{\ \zeta\ \}=i,$$
$$A_{co.ja}^{r+2}sF\{Z'_{mo.mo}\}=j,$$
$$A_{co.ja}^{r+2}sF_{mo.co}\ \ =k.$$

The first are monarchaxine janals (XX.), with zonoid signature; the second are monaxine monozones, with the signature of the zone perpendicular to the zoneless axis (XXIII.); the third are monaxine contrajanals of $(r+2)$-ple repetition (XIX.).

Zoneless janal polyarchipolar faces.

$$^{3}A_{ja}sF\{\zeta\}=\varphi_1,$$
$$^{6}A_{ja}sF\{\zeta\}=\varphi_2.$$

Radical zoneless janal polyarchipoles (§ 16.).

$$^{3}A_{rad.ja}sF\{\zeta\}=\varphi'_1,$$
$$^{6}A_{rad.ja}sF\{\zeta\}=\varphi'_2.$$

The register of janal polar faces is completed by the *2-ple janal and contrajanal polar faces*, the contrajanal axis being perpendicular to a zone. The janal are entered thus,

$$A_{ja}^{2}sF=l,$$

without zonoid signature; for since we can construct no janal symmetry on such a pair of opposite A-gons (as we shall see), but that of a zoneless triaxine, we require no account of the axes perpendicular to α that terminated by the A-gon.

This axis α being janal, must be perpendicular to two other $2m$-ple janal axes, β and γ (XX.); and the polyedron may be either a $2r$-ple monarchaxine, of which α is a secondary axis, or it may be a zoneless triaxine, of which α, β, γ (XX.) are three axes; or it may be, when β or γ is zoned, a $2r$-zoned homozone (XXII.) $(r\geqq 1)$.

Whatever the polyedron may be, any constructions on the opposite 2-ple A-gons will

degrade the symmetry about any higher axis β, zoned or zoneless, to which α may be secondary, and the janal constructions will be all zoneless triaxines. For the entire symmetry about a $2r$-ple axis β cannot be preserved, if $r>1$, by janal constructions on the poles of one only of the r axes α perpendicular to β; that is, β, by constructions on α only, is degraded to a 2-ple repetition.

When β is a $2r$-zoned homozone axis, γ and α are identical in configuration; and any janal construction on the poles of α destroys the zones, and β becomes a zoneless 2-ple axis.

When the 2-ple axis is contrajanal, and perpendicular to a zone, it is entered thus,

$$A^{s}_{conj\alpha}{}^{s}F\{Z''\}=m,$$

with the zone to which it is perpendicular. The solid on which A is found may be either a $(2r+3)$-ple homozone (XXII.), or a 2-ple monaxine monozone. In either case the A-gon is the reciprocal of a 2-ple A-ace constructed by our processes as a monaxine monozone, having a 2-ple contrajanal axis perpendicular to a given zone, and the entry is made as above. The only possible polar janal constructions on the two contrajanal polar faces are 2-ple monaxine monozones.

Janal amphigrammic poles.

The register of janal poles is completed by the janal polar edges of amphigrammic axes, zoned and zoneless.

The polar edge is registered as the intersection of two A-gons; and we write

$$s'=Q-2A+2$$

for the number of summits of the P-edron Q-acron not in the A-gons, and

$$F'=P-2$$

for that of the faces distinct from them.

$$(AA)^{2nqrdi}_{j\alpha}{}^{s}F'\{ZZ'Z''\}=n,$$

$$(AA)^{2nqrdi}_{j\alpha}{}^{s}F'\{\ Z\zeta\ \}=p,$$

$$(AA)^{n}_{j\alpha}\ {}^{s}F'\qquad\quad=q.$$

We read that there are n different polar edges of janal A-gons epizonal in Z and zonal in Z', Z'' being perpendicular to the amphigrammic axis.

We read that there are p polar edges of A-gons, both zonal and epizonal in Z, and perpendicular to zoneless axes of given zonoid signature.

And there are q amphigrammic zoneless axes, whose polar edges are intersections of A-gons. We have no occasion here to register any zonoid signature of other axes.

The n edges may be tertiary poles of zoned triarchaxines or hexarchaxines, or secondaries in zoned monarchaxines, or in zoned triaxines. In the first of these four cases we shall read $Z=Z'$, in the second, $Z=Z'=Z''$.

The p edges are either zoned poles of homozone triaxines, or possibly secondaries of zoned tetrarchaxines (XL.), if Z is a tetrarchaxine zonal signature.

The q zoneless polar edges may be tertiaries of zoneless hexarchaxines or triarchaxines, or secondaries of zoneless tetrarchaxines or monarchaxines, or poles of zoneless triaxines or homozone triaxines, or secondaries in m-zoned homozones, or axes of 2-ple monaxine monozones, or of 2-ple monaxine contrajanals. In any case we require no account of their symmetry, janal or contrajanal, as these polar edges can never be, like polar faces, the subjects of constructions. It suffices for our purpose that we know exactly the number of all janal zoned and zoneless amphigrammic axes.

Our object here is to state clearly what our Tables are supposed to contain. The mode of obtaining the Tables will be discussed in the sequel.

Registration of polar and non-polar faces of P-*edra* Q-*acra and* Q-*acra* P-*edra.*

TABLES C.

XXXIV. In these Tables will be found all the janal poles of the preceding Tables, as well as all heteroid polar faces. No symmetry is registered here but that of the face, therefore there is no account of poles secondary to the one considered. But the character of the axis (V.) is always suffixed, by one of the abbreviations,

am.go, am.ed, am.gr, go.gr, go.ed, ed.go.

We shall write x as the symbol of such abbreviation.

Zoned polar faces.

$$\begin{aligned}
&A_x^{2r\,di} \quad s\mathrm{F}\{ZZ'\} = a,\\
&A_x^{2r\,og} \quad s\mathrm{F}\{ZZ'\} = b,\\
&A_x^{2r\,ogdi} \quad s\mathrm{F}\{ZZ'\} = c,\\
&A_x^{(2r+1)\,di} \quad s\mathrm{F}\{\ Z\ \} = d,\\
&A_x^{(2r+1)\,og} \quad s\mathrm{F}\{\ Z\ \} = e,\\
&A_x^{(2r+1)\,mo} \quad s\mathrm{F}\{\ Z\ \} = f.
\end{aligned}$$

Here s and F are the summits and faces inferior to the A-gon.

We read that there are c A-gons which have r agonal traces of Z and r diagonal traces of Z″, of which A-gons some will be janal and others heteroid poles.

Zoned Tetrarchipoles.

$$\begin{aligned}
&{}^4A_x^{2y} \quad s\mathrm{F}\{Z\} = \theta_2,\\
&{}^4A_{rad\,x}^{2y} \quad s\mathrm{F}\{Z\} = \theta_2,
\end{aligned}$$

which, being heteroid poles, could not appear in the preceding Tables of polyarchipoles. The y denotes the traces.

Zoneless polar faces.

$$A_x^r sF = g \quad (r > 1),$$

showing that g A-gons have an r-ple zoneless repetition and the character x of axis, of which some will be janal and others heteroid poles.

Zoneless Tetrarchipoles.

$${}^{4}A^{2}_{\pi}\ sF\{Z\}=\phi_{3},$$
$${}^{4}A^{2}_{rad\pi}sF\{Z\}=\phi'_{3}.$$

Zoned non-polar faces.

$$A^{d'}\,sF\{Z\}=d',$$
$$A^{e'}sF\{Z\}=e',$$
$$A^{f'}sF\{Z\}=f'.$$

Here there is no axis to be characterized.

Objanal monozone faces.

$$A^{d'}_{\alpha\beta\gamma}sF\{Z\}=j,$$
$$A^{e'}_{\alpha\beta\gamma}sF\{Z\}=k,$$
$$A^{f'}_{\alpha\beta\gamma}sF\{Z\}=l.$$

These are certain of the above *zoned non-polar faces*, which have an objanal symmetry by reason of their being faces in a *repeating zone*, to which an axis (α) of even repetition, zoned or zoneless, is perpendicular. This axis (α) appears not in this Table, but appears in general in the Tables which we shall learn to construct of *perfect objanal monozone summits*, which are the reciprocals of the faces here registered. Whatever α may be, the only janal symmetrical constructions possible on these faces are objanal monozone summits or reticulations, or simply janal anaxine pairs of edges, of whose construction we shall treat hereafter (§ 17.). The enumeration of our results is not dependent on our knowledge of the zoned or zoneless axis (α).

Zoneless non-polar faces.

Janal anaxine faces (XXVI.).

$$A_{ja.ax}sF=h.$$

Asymmetric faces.

$$A_{as}sF=i.$$

This number i includes the h A-gons of the preceding entry of janal anaxine faces.

No face is enumerated under the numbers ghi, which is the reflected image of another.

Similar Tables, B and C, are supposed completed for the Q-edra P-acra. And they are all obtained from those of the reciprocal summits which we shall learn to construct, *and which are conceived as included in these Tables.*

The janal polar summits (Table B) and the polar summits (Table C) can be constructed for both Q-edra P-acra and P-edra Q-acra, by writing summits for faces and zonal for epizonal edges in all the signatures above given.

Registration of edges of P-*edra* Q-*acra and* Q-*acra* P-*edra.*

TABLES D.

Polar edges.

XXXV. $(AA)_x{}^{s'}F'\{ZZ_i\}=a'$,

$(AA)_x^{s'}F'_{az} \quad =b'$,

where x is *am.gr*, *go.gr*, or *ed.gr* (V.). Here *az.* means *azone* or zoneless.

$P=F'+2$,

$Q=s'+2.A-2$.

The amphigrammic polar edges may be either janal or heteroid. When they are janal, they will of course be found both in this Table and in the Table B of janal poles (XXXII., XXXIII.).

Non-polar zoned edges.

$(AB)_{np}{}^{s'}F'\{Z\}=c'$ $(B\overline{\geq}A)$,

$(AA)_{np}{}^{s'}F'\{Z\}=d'$,

which record c' edges, each the intersection of a B-gon and A-gon, and each epizonal in the zone Z, and d' edges of A-gons zonal in Z.

The number d', and also the number c', when B=A, will not include polar edges of A-gons above entered in the zone Z, a signature which may belong equally to polar and to monozone polyedra (IX.).

Janal anaxine edges.

$(AB)_{ja.ax}{}^{s'}F'=e'$ $(B\overline{\geq}A)$,

showing that there are e' different janal anaxine edges of intersection of an A-gon and a B-gon, on all the P-edra Q-acra.

Asymmetric edges.

$(AB)_{as}{}^{s'}F'=f'$ $(B\overline{\geq}A)$,

which records the entire number f', including e' last registered, of asymmetric intersections of an A-gon with a B-gon on all the P-edra Q-acra.

One at least of the faces about each of the f' edges here registered is zoneless and non-polar, otherwise the edge would not be zoneless and non-polar. One of the faces may be polar or zonal non-polar. Thus we see that asymmetric or janal anaxine edges are found in faces which are not *asymmetric nor janal anaxine faces* (XXVI.).

Our Tables D of P-acra Q-edra would give us a Table D of the above edges in terms of the summits of the edges; but such a Table is of no use to us.

When the Tables A, B, C, D (XXXI. to XXXV.) are completed for P-edra Q-acra and Q-edra P-acra, so that no two features are recorded of which one is the reflected image of another, our problem of enumeration and classification of P-edra Q-acra is perfectly solved; and we shall see that we have the power of continuing such Tables to higher values of P and Q.

It is necessary to determine what *data* are required, and how they are to be employed, for the completion of these Tables; and next to find the means of obtaining these *data*. This will all be discussed, and the requisite general formulæ will be recorded, in the following sections.

SECTION 2.—*Problem of Classification and Enumeration of the* P-*edra* Q-*acra*.

XXXVI. (a) Let us suppose given for all polyedra of fewer than P+Q−2 edges, as far as we require them, the Tables A, B, C, D (XXXI. . . XXXV.).

(b) Let us suppose given for all the P-edra Q-acra, and for all the Q-edra P-acra, P and Q being definite numbers, all the polyarchaxine polyedra, with their poles and signatures, as entered in XXXI., XXXII., XXXIV.

(c) We suppose known also for P-edra Q-acra and for Q-edra P-acra all *janal poles*, with their signatures, as entered in Tables B (XXXII., XXXIII.).

(d) Let us conceive that all polar faces and summits of the same solids are known, as entered in Tables C (XXXIV.), and all polar edges, as entered in Tables D (XXXV.).

(e) Likewise all objanal monozone faces of the same solids, as entered in Tables C (XXXIV.).

(f) And also all monozone faces which have a diagonal trace of a single zone, *i. e.* the number d′ of Tables C (XXXIV.), for all signatures A and Z.

(g) And, finally, let us suppose that all the edges are given of Table D (XXXV.), for P-edra Q-acra, and for Q-edra P-acra.

We suppose given in the data (a), (b), (c), (d), (e), (f), (g), the numbers GHIPQR of Tables A (XXXI.), and all the Tables B, C, D, except only the numbers e′, f′, h, i of Tables C.

All the remaining numbers entered in Table A, and the numbers e′, f′, h, i *of Table C, can be determined by the data* (a), (b), (c), (d), (e), (f), (g).

When this has been proved, and when we have shown that we can obtain the data (a)... (g), our problem will be solved.

XXXVII. We have first to show that the data (a), (b)... (g) suffice for the determination of the sought numbers in Tables A and C. These numbers we shall consider in the order following:—EFTT′UKMNN′N″S′SDWLBCJAΞΘ in Table A (XXXI.), and e′, f′, h, i in Tables C (XXXIV.).

One difficulty of our problem lies in the danger incurred of enumerating the same solid more than once. Thus, all janal and objanal principal secondary or tertiary poles of zoned or zoneless polyarchaxines will be constructed by our processes simply as janal poles of a single axis: and we cannot be certain, without consideration, when we construct a 5-ple, 4-ple, 3-ple, or 2-ple axis, zoned or zoneless, that we do not thereby complete a polyarchaxine.

For example, if we load all the principal faces but two opposite ones of a regular 12-edron with 5-gonal pyramids (pentaces, $\delta\acute{\alpha}\kappa\eta$), we have in the two void faces an amphiedral axis of a 5-zoned homozone polyedron. If in our processes we were to charge this

on its two pentagon poles with pentaces, we should certainly thereby construct an amphigonal 5-zoned homozone axis. But it would be an error to enumerate the construction among the homozones, for we have evidently here only completed a hexarchaxine.

In like manner, in constructing a 2-zoned janal axis, we cannot be sure that we have not been completing, by secondary or tertiary poles, a monarchaxine or a polyarchaxine.

E. F.

E. 2r-*zoned monarchaxines* (XXXI.).—When $r>2$, every $2r$-zoned heterozone polar face of Table B (XXXII.) gives a distinct $2r$-zoned monarchaxine of the number E; for no polyarchaxine has a $(6+2m)$-ple pole. Wherefore E is given for $r>2$, by (a, b, c) Table B (XXXII.), and the exact poles of the $2r$-zoned axis are known and can be registered for either polar faces or for their reciprocal summits.

Let $r=2$, $2r=4$; and let $J_{,}$ be the entire number of 4-zoned A-gonal poles having given traces (XXXII.) and the zonal signatures $\{ZZ'Z''\}$. Some of these $J_{,}$ poles may be principal poles of triarchaxines. When Z'' is not identical with one of the zones ZZ', this cannot happen, because no triarchaxine has three zonal signatures.

In the case then of $ZZ'Z''$ all different, every one of the $J_{,}$ poles gives a different 4-zoned monarchaxine, and they can be registered thus in Table A,

$$(\mathbf{P}Q)A^{y}_{j^a}Y^{z}_{j^a}Y^{z}_{j^a}\{ZZ'Z''\}=J_{,}=E,$$

an entry which gives all that is indicated under the number E; for the principal pole is seen, and the signatures $ZZ'Z''$ give exactly the characters of the other poles and axes. Here y denotes the traces of the A-gonal pole.

Let next $J_{,,}$ be the number of 4-zoned A-gonal polar faces in Table B which have $Z''=Z'$ and definite traces and signatures; and let j be the number of triarchaxines in 5, Table A, which have a principal A-gonal face with the same traces and signatures $\{ZZ'\}$ (a) XXXVI. We have the number E in this case thus:—

$$J_{,,}-j=(\mathbf{P}Q)A^{y}_{j^a}Y^{z}_{j^a}Y^{z}_{j^a}\{ZZ'Z'\}=E.$$

F. $(2r+1)$-*zoned monarchaxines*.—No zoned polyarchaxine (XV., XVI., XVII.) has $(2r+1)$-gonal poles properly janal. Therefore every janal $(2r+1)$-zoned pole of Table B gives one of the F polyedra required, with its signatures.

TT'. 2r-*zoned and* $(2r+1)$-*zoned Homozones* (XXXI.).—No polyarchaxine has a $2r$-zoned homozone axis, if $r>1$. Therefore all the poles registered under the number g, Table B, give polyedra here required with all their signatures, of the number T, for $r>1$.

No polyarchaxine has a $(2r+1)$-zoned homozone axis, if $r>2$; therefore all the poles entered under the number h in Table B, give polyedra to be entered under T' in Table A, with their signatures, for $2r+1>5$. And the numbers TT' are thus completed for the above values of r.

Let $$A^{\alpha\beta\gamma}_{\alpha\beta\gamma}F\{^{\alpha}Z^{\zeta}_{\alpha}\}=c$$

be the number of 5-zoned janal poles entered under the number h in Table B, having

the zone ${}^{\sigma}Z$ of hexarchaxine signature (XVII.), y denoting the traces; and let ζ found in the c poles with ${}^{\sigma}Z$ have a zonoid pole of name not excluded from ${}^{\sigma}Z$. Then every principal polar A-gon of an hexarchaxine having those traces and the zone ${}^{\sigma}Z$ is included in the number c. The Tables of XXXI. give us the number c' of the hexarchaxines which have this pole and zone by our hypothesis (a) XXXVI. Wherefore for this value $r=2$,

$$c-c'=\mathrm{T}'=(\mathbf{PQ})\mathrm{A}^{x}_{\beta\gamma}\mathrm{Y}^{y}_{\mu\nu}\{{}^{\sigma}\mathrm{Z}\zeta\}$$

is the number T' required for these signatures.

U. (16. XXXI.). Let m_1 be the entire number of 3-zoned homozone polar faces $\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\ldots$ under the number h (XXXII.), which have the zonal signature Z_{μ} of the triarchaxine form (XV.) or of the hexarchaxine form (XVII.), or of both, a thing quite possible (XVII.); and let ζ in these homozone poles have a pole of name not excluded from Z_{μ}.

Let m_2 be the entire number of hexarchaxines having any secondary polar faces and this signature Z_{μ}, and let m_3 be the entire number of triarchaxines having any secondary polar faces and this signature Z_{μ}; then

$$m_1-m_2-m_3=\mathrm{U};$$

for all of the m_1 3-zoned homozone poles which are not secondary polar faces of hexarchaxines or triarchaxines, are to be enumerated as poles of 3-zoned homozone polyedra.

K. *r-ple monaxine contrajanals* (9, XXXI.).

XXXVIII. There can be no ambiguity in the enumeration of the r-ple monaxine contrajanals; for by its name and definition the solid can have no pole out of its single axis (XIX.). Wherefore every pole registered under the number k, Table B (XXIII.), gives a distinct polyedron of the number K under consideration.

MNN'N'' (11, XXXI.).

Zoneless monarchaxines.

No polyarchaxine has more than a 5-ple repetition. Hence for $(r+3)>5$, every pole registered under the number i in Table B (XXXIII.) gives one of the M solids with its signature when $r+3$ (Table B) is even, and one of the N solids if $r+3$ is odd.

M. For $r=4$, every janal pole registered in Table B under i, which is not archipolar on a zoneless triarchaxine, gives a 4-ple zoneless monarchaxine of the number M.

The only principal triarchaxine poles in the number i (XXXIII.) are among those 4-ple poles in which ζ has poles of one name only (XXI.). Let n_1 be the entire number of 4-ple janal A-gons (i, XXXIII.) having poles of one given name only; and let n_{11} be the number of zoneless A-gonal triarchipoles in Table A (5, XXXI.) which have tertiary poles of that name. Then n_1-n_{11} is the number of 4-ple zoneless monarchaxine A-gons having this zonoid signature ζ, and this gives the entry

$$n_1-n_{11}=(\mathbf{PQ})\mathrm{A}^{x}_{\mu}\mathrm{Y}^{y}_{\mu}\mathrm{Y}^{z}_{\mu}\{\zeta\}=\mathrm{M}\ (\mathrm{XXXI.}).$$

N. For $r=5$ (*i*, XXXIII.), every janal pole not archipolar on a zoneless hexarchaxine (XXXI.), or not having poles in ζ of one name only, gives a 5-ple zoneless monarchaxine of the number N (XXXI.). When 5-ple zoneless A-gons are constructed which have poles of one name only in ζ, and which are found also in hexarchaxines having tertiary poles of that name, we obtain N in the same way as above by subtraction for $2r+5=5$.

N′. For $r=3$, the Table of triple zoneless poles (XXXIII.) which have not amphiedral, amphigonal, or amphigrammic secondary poles described in ζ, and which axes therefore cannot be tertiary axes in any triarchaxine or hexarchaxine (for these have all janal tertiary axes), gives exactly the number of triple zoneless monarchaxines of the signature.

N″. Let p be the number of triple janal zoneless poles, of all terminating features, of an axis of given character, perpendicular to amphiedral secondary axes, *i. e.* showing in ζ only polar faces; and let p' be the number of zoneless hexarchaxines, and p'' that of zoneless triarchaxines whose secondary and tertiary axes have the same characters (given by the signatures under the numbers P and R (XXXI.)) with those p axes and these secondaries: we have

$$p-p'-p''=(\mathbf{PQ})\mathrm{Y}^3_{ja}\mathrm{Y}^3_{amed}\{\zeta\}=\mathrm{N}'',$$

where X^3_{ja} can be written for Y^3_{ja}, if $p'=p''=0$.

In the same manner we can obtain N''_2 for amphigonal and N''_3 for amphigrammic secondary axes. Hence

$$\mathrm{N}''=\mathrm{N}''_1+\mathrm{N}''_2+\mathrm{N}''_3 \text{ (11, XXXI.)}$$

is given, and all *zoneless monarchaxines* can be enumerated both for P-edra Q-acra, and for Q-edra and P-acra.

SS′, *monaxine monozones.*

XXXIX. S. Every $(3+r)$-ple pole (*j*, XXXIII.) of a monaxine monozone gives a polyedron of the number S (XXXI.), with the proper signature Z.

S′ (XXXI.). Let s be the number of 2-ple contrajanal poles (α) of given *character* (V.), under the number m (XXXIII.), and having the zonal signature Z″; and let s' be the entire number of the $(2r+1)$-zoned homozones (XXXII. *h*) which have the zone Z″, and in their zonoid signature ζ a pole of the name α. Then

$$s-s'=(\mathbf{PQ})_{mo.mo}\mathrm{Y}^3_{cojn}\{\mathrm{Z}''\}=\mathrm{S}' \text{ (XXXI.)};$$

which is thus known for every zone Z″ and for every character of axis.

D. *Zoned triaxines* (3, XXXI.).

XL. The Table B gives (*a*, *b*, *c*, XXXII.; *n*, XXXIII.) *l* for the number of 2-zoned heterozone poles of all names which are in zonal signatures ZZ′Z″, Z′Z″Z, or Z″ZZ′, which are all the same, the last-written zone in Table B being perpendicular to the axis carrying the janal pole recorded. If this be identical with one of the first two written zones, the pole *may be* tertiary on a zoned triarchaxine (XV.); if all the three signatures are alike, the pole *may be* tertiary on a hexarchaxine (XVII.); or the pole *may be* secondary on a $2r$-zoned monarchaxine (XII.), whatever be the signatures.

The number l comprises d tertiary poles of hexarchaxines, $d_{,}$ tertiary poles of triarchaxines, and $2d_{,,}$ secondary poles (XI.) of monarchaxines, which all have these zonal signatures. All other poles must be in zoned triaxines having these signatures; whence, as each (XII.) has three poles, we obtain, since d, $d_{,}$, and $d_{,,}$ are given,

$$D=\tfrac{1}{3}(l-d-d_{,}-2d_{,,}) \text{ (XXXI.)}.$$

W. *Homozone triaxines* (16, XXXI.).

There is never any ambiguity about the zoned poles of homozone triaxines, except when the zone, save in p subscribed, is of the form of a tetrarchaxine signature (^{4}Z) (XVI.), and when at the same time the pole named in ζ has the name of the secondary pole of the tetrarchaxine, named in (^{4}Z).

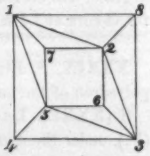
There are under the numbers g, p (XXXII., XXXIII.), $k_{,}$ 2-zoned homozone poles of all terminating features of a given name, which have the zone (^{4}Z) containing, though of course without p subscribed (XVII.), the feature named as secondary pole in ζ; and there are (6, XXXI.) $k_{,,}$ tetrarchaxines having this zonal signature, and the secondary pole named in ζ.

Wherefore $$k_{,}-k_{,,}=W$$

is the number of homozone triaxines which have these signatures $\{^{4}Z\zeta\}$.

It is worth while to show how the ambiguity spoken of can arise.

If we charge all the faces of a tetraedron P with tetraedra, and then efface two opposite edges of P, we obtain the solid here figured, where 1 2 3 5 are the summits of P, and 52 and 13 are the effaced edges. This is a homozone triaxine, in which the edges 32, 21, 15, 53 of P have become zoneless polar edges.



This homozone will be registered in our Table; and if, as will inevitably happen in our processes, we draw the polar edges 52, 13, we construct a homozone amphigrammic axis to which two 2-ple axes are perpendicular. But it would be an error to register the construction as a homozone triaxine; for it has a tetrarchaxine signature, and the same zonoid poles which the figured homozone has, and which can be made to appear by p subscribed in the zonal signature. We have simply completed a zoned tetrarchaxine. And this ambiguity may occur in far more complex constructions, in which the secondary poles may be faces or summits.

If, however, when we have completed a 2-zoned homozone by coronation either with lines as above, or with a pair of summits, we find that we have completed a zonal signature of tetrarchaxine form, which has no zonal faces, while our zonoid signature ζ shows a polar face, we have completed no tetrarchaxine.

L. *Zoneless triaxines* (10, XXXI.).

XLI. The Table B (XXXIII.) of janal poles (lmq) gives the entire number of 2-ple

janal and contrajanal zoneless poles. Let these be

$$\mu=\mu'+q,$$

μ' being the number of amphiedral and amphigonal together, and q that of the amphigrammic.

Every zoneless polyarchaxine has one double zoneless pole, secondary or tertiary; every homozone polyedron has one; every $2r$-ple monarchaxine has two (XX.), and every 2-ple monaxine contrajanal has one; and these, by what precedes, are all enumerated. The zonoid signatures show in all cases how many of these secondary and tertiary or sole axes are amphigrammic. Let the number be

m_1 of amphigonal and amphiedral together;
m_2 of amphigrammic poles.

All the μ poles, except m_1+m_2, are poles of zoneless triaxines; for they can be nothing else, these being the only janal polyedra not already disposed of.

As each triaxine (XX.) has three of these poles, then are

$$\tfrac{1}{3}(\mu-m_1-m_2)=\mathrm{L}\ (10,\ \mathrm{XXXI.})$$

zoneless triaxines, on which are $q-m_2$ amphigrammic axes. Thus *the entire number of polar edges on these* L *solids is known, and consequently the number of their different non-polar edges.*

We have thus demonstrated that we can enumerate by the data of art. XXXVI. all the janal polar P-edra Q-acra and Q-edra P-acra.

B, C. r-*zoned monaxine heteroids* (2. XXXI.).

XLII. Let δ be the number of different poles, in Table C (XXXIV.), terminating an r-zoned axis of given character and of given zonal signature $\{\mathrm{Z}\}$ or $\{\mathrm{ZZ}''\}$. Let δ' be the number of these δ poles found on polyarchaxine monarchaxine and triaxine, heterozone or homozone, polyedra (XXXII.), in the same signature. All the rest must be heteroid poles of which every monaxine heteroid has two. Hence

$$\tfrac{1}{2}(\delta-\delta')=\mathrm{B}\text{ or }=\mathrm{C}\ (\mathrm{XXXI.}),$$

according as r is even or odd, is the number of r-zoned monaxine heteroids having this signature.

J. r-*ple zoneless monaxine heteroids* (8, XXXI.).

XLIII. Let s be the number of zoneless poles terminating on r-ple axis of given character Y, in the Table C (XXXIV.). Let s_1 be the number of poles terminating r-ple zoneless axes of the character Y in the polyarchaxine monarchaxine triaxine and monaxine contrajanal, zoneless polyedra. This s_1 is known by what precedes, and $s_1=0$ of course, if $r>3$, and if the axis be gonoedral, edrogrammic, or gonogrammic. All the rest of the s poles are on r-ple monaxine heteroids, which contain each two of them. Hence

$$\tfrac{1}{2}(s-s_1)=\mathrm{J},$$

and J is given for all values of $r>1$, and for every character of axis.

Thus all polar polyedra, zoned or zoneless, can be exactly registered, and *as the signatures give the number of polar and zoned edges on them all, we know upon them all the number of different zoneless and non-polar edges.*

A. *Monozone* P-*edra* Q-*acra.*

XLIV. Let Z be any zonal signature having b epizonal (or zonal) edges, without p subscribed. Let E be the number of non-polar epizonal (or zonal) edges which are found in the zone under the number c' (or d') in Table D (XXXV.); and let E′ be the number of *different* non-polar epizonal (or zonal) edges read in the zone Z, which will of course here show p subscribed (IX.) upon all the zoned polar P-edra Q-acra. This E′ is given by inspection of signatures, and the instalment of it contributed by different polar polyedra will vary according to the manner in which Z is written in their signatures (IX.) (XVII.).

The number of epizonals (or zonals) in Z found upon monozones will be E—E′, and as each solid has (III.) b of these, we obtain

$$\frac{1}{b}(\mathrm{E}-\mathrm{E}')=\mathrm{A}\ (1,\ \mathrm{XXXI.})$$

for the number of monozone P-edra Q-acra having the signature Z.

But it may be that $b=0$, or that Z has neither zonal nor epizonal edges. In that case Z must have only diagonally traced summits. Let s be the number of these summits, which are by hypothesis non-polar. We have in the datum (f) (XXXVI.) the number d' of diagonally traced summits non-polar in the zone Z, that are found on the P-edra Q-acra which have or have not zoned axes. And inspection of the signatures of all zoned axes previously enumerated tells us how many different non-polar summits in Z these axial solids contain. Let this be d''. Then

$$\frac{d'-d''}{s}=\mathrm{A}$$

is the number of monozone P-edra Q-acra which have the zone Z.

Thus all monozone P-edra Q-acra and Q-edra P-acra of all zonal signatures can be registered, and *the number of their zoned edges being given by their signatures, that of their different zoneless edges is known.*

Ξ. *Janal anaxine* P-*edra* Q-*acra* (17, XXXI.).

XLV. Let G be the entire of janal anaxine pairs entered in Table D (XXXV.) of the P-edra Q-acra; and let G′ be the number of janal anaxine pairs on all the solids enumerated in art. XXX. G′ is known by inspection of signatures of polyedra already constructed. Then, as every janal anaxine polyedron has $\frac{1}{2}(\mathrm{P}+\mathrm{Q}-2)$ of these pairs,

$$\frac{2}{\mathrm{P}+\mathrm{Q}-2}(\mathrm{G}-\mathrm{G}')=\Xi\ (17,\ \mathrm{XXXI.})$$

is the number of those polyedra.

Θ. *Asymmetric polyedra* (18, XXXI.).

XLVI. Let J be the entire number of asymmetric edges on the P-edra Q-acra, which are found in Table D (XXXV.). We have shown that the number of repetitions of every zoneless non-polar feature is given for all the symmetrical polyedra; and as the signatures give the number of polar and zoned edges on them all, we know how many different zoneless and non-polar edges are on them all.

Let this number be J′; then

$$\frac{1}{P+Q-2}(J-J')=\Theta$$

is the number of the asymmetric P-edra Q-acra.

XLVII. e′, f′. *Monozone faces of* P-*edra* Q-*acra* (XXXIV.).

The number d′ (XXXIV.) is supposed already known (f) (XXXVI.).

Let h_{AB} be the number of non-polar edges epizonal in the zone Z (XXXV., *c′*) (g) (XXXIV.), which are the intersection of an A-gon and a B-gon.

Let

$$2h_{AA}+h_{AB}+h_{AC}+\ldots=h_{A(Z)}$$

be the entire number of different edges of A-gons, epizonal in Z, where we write $2h_{AA}$, because these edges are each in two different A-gons.

The Table of polar faces (XXXII.) gives us the entire number of different edges of A-gons that are epizonal non-polar in Z in polar faces. A polar face may have either one trace or two different traces, whose signature is Z; for the two zones, though of different configuration, may have the same signature. In all cases, the number of *different* epizonals in Z of that polar face is given by inspection.

Let $p_{A(Z)}$ be the whole number of non-polar epizonals in Z of polar A-gons. There remain

$$h_{A(Z)}-p_{A(Z)}$$

edges epizonal in Z of A-gons, which edges are in non-polar A-gons. When A is even, each A-gon has two of them; when A is odd, each A-gon has only one of them. Wherefore for A even,

$$\tfrac{1}{2}(h_{A(Z)}-p_{A(Z)})=A^{ev}\delta F\{Z\}=e' \text{ (XXXIV.)},$$

and for A odd,

$$h_{A(Z)}-p_{A(Z)}=A^{od}\delta F\{Z\}=f' \text{ (XXXIV.)}.$$

We can therefore enumerate all the monozone faces of the P-edra Q-acra, and of the Q-edra P-acra, and register each with its trace and zonal signature.

XLVIII. *Objanal monozone faces.*—When Z of the preceding article is the signature of a repeating zone, certain of the A-gons just enumerated will be the objanal monozone faces of (e), XXXVI. It is important that the objanal monozone faces should be separately registered, and it will be necessary to subtract these A-gons when they exist from the number of A-gons just found, in order that the *monozone* A-*gons*, which have

a trace of Z, may be understood as not being *objanal monozone*. We shall find that objanal monozone summits are obtained by construction, and by these we know the reciprocal faces.

h, XXXIV. *Janal anaxine* A-*gons of* the P-*edra* Q-*acra*.

XLIX. By (g) XXXVI. and *e'*, XXXV., we know the number

$$2(AA_{ja.m})+(AB_{ja.m})+(AC_{ja.m})+\ldots=(A_{j.a})$$

of janal anaxine edges of A-gons, where we write $2(AA_{ja.m})$, because these edges are each in two different A-gons.

Some of these A-gons are polar and some zoned non-polar, the rest being janal anaxine A-gons.

The *polar zoned* A-*gons* are exactly those found on the first five of the seven classes of polyedra named in art. XXX.; and they are all either heterozone $2r$-zoned A-gons ($r\overline{>}1$), or homozone $(2r+3)$-zoned A-gons ($r\overline{>}0$), or hexarchaxine 2-zoned A-gons, which can only improperly be called heterozone, since their (tertiary) axis is perpendicular to a zone identical with its own two zones (XXXII). Now these are precisely the polar A-gons enumerated under the numbers *a*, *b*, *c*, XXXII. ($r>0$), and under the number *h* (XXXII.) for $r\overline{>}0$.

Wherefore we know by inspection of signatures the number of *different* zoneless edges in these zoned polar A-gons.

The *polar zoneless* A-*gons* of the third and last but one of the seven classes of art. XXX. are all either principal poles of $(2r+4)$-ple monaxine monozones, which are found under the number *j* (XXXIII.), for all odd values of r; or they are 2-ple poles of monaxine monozones, or 2-ple poles of $(2r+3)$-zoned homozones, which 2-ple A-gons are all found entered under the number *m* (XXXIII.).

The polar zoneless A-gons of the last of the seven classes of XXX. are all entered under the number *k* (XXXIII.), for $r=2n+1$.

Inspection of the signatures of repetition in all these polar A-gons, zoned or zoneless, gives us the number of different zoneless non-polar edges, that is of janal anaxine edges, in them all. Let this number be $(A'_{j.a})$.

The *zoned non-polar* A-*gons* are all given by (e) XXXVI., under the numbers (j, k, l) XXXIV.; whence that of their *different* zoneless edges is given also. Let this be $(A''_{j.a})$. Then all the remaining

$$(A_{j.a})-(A'_{j.a})-(A''_{j.a})$$

janal anaxine edges of A-gons must be found in janal anaxine A-gons (XXVI.); and if we consider, as we may without confusion, the $2(AA)_{ja.m}$ above written as double edges, we can say that all these remaining edges are found in no A-gons which are not janal anaxine. Each A-gon has A of them; wherefore

$$\frac{(A_{j.a})-(A'_{j.a})-(A''_{j.a})}{A}=h \text{ (XXXIV.)}$$

is the number of the janal anaxine A-gons on all the P-edra Q-acra, or of janal anaxine A-aces in the Q-edra P-acra.

i. *Asymmetric* A-*gons of* P-*edra* Q-*acra* (XXXIV.).

L. We know by (g), XXXVI., the number of asymmetric edges of A-gons of P-edra Q-acra, which are all entered under the number f' (XXXV.). We have

$$2(AA_{\omega})+(AB_{\omega})+(AC_{\omega})+\ldots=A_{\omega}$$

for this entire number. By inspection of our Table C (XXXIV.), we see that there are in all the polar and zoned A-gons entered under a, b, c, d, e, f, d', e', f' (XXXIV.), which comprise all the symmetric A-gons of Table C, the number A'_{ω} of different zoneless non-polar edges in these A-gons. The remaining $A_{\omega}-A'_{\omega}$ must be all edges in asymmetric A-gons; wherefore

$$\frac{A_{\omega}-A'_{\omega}}{A}=i \text{ (XXXIV.)}$$

is the number of asymmetric A-gons of the P-edra Q-acra. If we would know the number i'' of these which are not janal anaxine A-gons, we have

$$i''=\frac{A_{\omega}-A'_{\omega}-A_{j.a}+A'_{j.a}+A''_{j.a}}{A}.$$

As the Tables A, B, C, D are supposed to contain the same account of the features both of P-edra Q-acra and of Q-edra P-acra, we can obtain all the results of XXXVII.... L. alike for both P-edra Q-acra and for Q-edra P-acra; and with every face enumerated, we have also its reciprocal summit.

Thus we have demonstrated, in this second section, that the data of art. XXXVI. are sufficient for the entire completion of the Tables A, B, C, D (XXXI.... XXXV.), for faces and for summits.

All that remains for the complete solution of our problem of classification and enumeration of the P-edra Q-acra and Q-edra P-acra, is that we show how these data (XXXVI.) can be obtained and registered without ambiguity or repetition. We shall consider first the reciprocals of the faces (d) (f) (XXXVI.), and the edges (g) (XXXVI.).

NOTE ON ARTICLES XLIX. AND L.—The number A_{ω} in art. L. is intended to comprise *all* edges of zoneless non-polar A-gons, and should have been defined as including the numbers b' and d' of XXXV.; for these, although symmetric edges, are often edges of asymmetric A-gons. The reader will therefore conceive that the numbers $(AA)_{ax.po.}=b'$ and $(AA)_{zo}=d'$, for every zonal signature, are included in the third line of L. And by the first-named edges in the seventh line of L. are meant edges not epizonal in those A-gons. In like manner the number $(A_{j.a.})$ in XLIX. is intended to comprise all zoneless polar edges, and all zonal edges in all zones, of A-gons on the solids of XXX.; for these are often edges of janal anaxine A-gons. The polars are in our signatures; the zonals ought to have been registered in a separate table in XXXV. as *objanal zonal edges*, *i. e.* edges $(AA)_{ob.zo.}$ (of A-gons) in a repeating zone (XXVII.). These are either diagonal traces drawn in objanal monozone (2A−2)-gons, or crowning edges of (2A−2)-gonal *objanal monozone penesolids*, of which we shall obtain in the sequel an accurate account. This will be all made clear in our applications.

VIII. *On a New Series of Organic Compounds containing Boron.*

By Dr. E. Frankland, *F.R.S.*

Received May 15,—Read May 22, 1862.

The substitution of a compound organic radical for an elementary constituent in inorganic compounds has proved itself to be one of the most important and fertile fields of modern chemical investigation. The application of this species of substitution to the inorganic compounds of metals has called into existence an entirely new and extensive family of organic substances—the organo-metallic compounds, bodies never met with in nature, distinguished by their well-marked affinities, and capable in some instances of effecting, in their turn, numerous substitutions of a like character. The realization of a similar substitution in the case of certain inorganic compounds of nitrogen and phosphorus has, in the hands of Hofmann, not only enriched the science with a host of new and interesting compounds, but has also brought our knowledge of the organic bases to a degree of completeness, which cannot be rivalled in any other class of organic compounds. Lastly, attempts have not been wanting to extend these reactions to the oxygen compounds of the metalloids; and although this portion of the field presents difficulties of a somewhat more formidable character, yet these attempts have not unfrequently been attended with success. Thus nitric oxide has been transformed into dinitroethylic and dinitromethylic acids*; sulphurous anhydride into ethylodithionic and methylotrithionic acids†; and carbonic anhydride into propionic and acetic acids‡.

The last-named reaction, confirming, as it did, the view previously expressed by Kolbe and myself§, that organic compounds in general are nothing more than substitutions of this nature effected in carbonic oxide, in carbonic acid, and possibly in other inorganic compounds of carbon, naturally awakened a desire to extend this inquiry to the oxygen compounds of boron and silicon, which are usually regarded as possessing certain important analogies with carbonic anhydride. With this end in view, boracic ether was submitted to the action of zincethyl by Mr. Duppa and myself. We found that the whole of the oxygen in boracic acid became replaced by ethyl, and in a short communication to the Royal Society‖, we described some of the properties of the remarkable body, boric ethide, thus formed. In the further study of this substance, and the exten-

* Philosophical Transactions for 1857, p. 59.

† Journal of Chemical Society, vol. x. p. 55, and p. 243.

‡ Ibid. vol. xi. p. 103; and Proceedings of the Royal Society, vol. x. p. 4.

§ Ann. der Chem. und Pharm. Bd. ci. s. 257. Proceedings of the Royal Institution of Great Britain for 1858.

‖ Proceedings of the Royal Society, vol. x. p. 568.

sion of the research to the homologous methyl compound, I much regret having been deprived of the cooperation of my friend and fellow-labourer who had rendered me such valuable assistance at the commencement of the investigation, but who was reluctantly compelled to abandon its further prosecution.

The first attempt to replace oxygen by ethyl in boracic anhydride was made by exposing the latter in a finely pulverized condition to the action of zincethyl at various temperatures, but it was found that the zincethyl was utterly powerless to effect the desired substitution; neither did the anhydrous acid yield in the slightest degree to WANKLYN'S compound of sodiumethyl and zincethyl, although it was digested and heated with it for several days. There could scarcely be a doubt that the intractability of the anhydride was due in great measure to its total insolubility in the surrounding liquid, and therefore, in order to place it under conditions more favourable for the action of the organo-metallic body, it was converted into boracic ether.

The ether was prepared by ROSE'S process*, which consists in distilling an intimate mixture of sulphovinate of potash and dried borax. The best proportions were found to be two parts by weight of borax, and three parts of the sulphovinate; but the yield of ether was very small, the greater part of the product consisting of alcohol. The removal of the latter by rectification, as recommended by ROSE, involved the loss of much ether; recourse was therefore had to chloride of calcium for its abstraction, a method which gave very satisfactory results, the product of pure ether being more than doubled. The following is a sketch of the process finally adopted. About 3 lbs. of the mixed borax and sulphovinate of potash were put into an ordinary PAPIN'S digester, which was placed in a sand-bath and exposed to a very gradually increasing heat so long as volatile products came over. The crude distillate obtained from several such operations was then treated with about one-fourth of its weight of fused chloride of calcium, and agitated until the latter was dissolved. The liquid now separated into two layers, a lower one consisting of an alcoholic solution of chloride of calcium, and an upper one containing nearly all the boracic ether, which retained only a small proportion of alcohol in solution. The upper layer was decanted and submitted to distillation. It began to boil at about 85° C., but the thermometer soon rose to 118° C., between which temperature and 125° C. the greater part of the remaining liquid passed over and was reserved for the purposes of the investigation. A thick oily liquid remained in the retort, and appeared to consist of boracic acid united with a smaller proportion of oxide of ethyl.

On adding zincethyl to the boracic ether thus prepared, a considerable elevation of temperature gradually occurred, whilst at the same time a most penetrating and peculiar odour was developed, due apparently to the vapour of some volatile body, that not unfrequently burst into flame, when the cork was removed from the flask in which the reaction took place. Some preliminary experiments showed that this volatile body could be distilled unchanged from the mixture, and that it was neither miscible with, nor apparently decomposed by, water. It was also spontaneously inflammable, and the

* POGGENDORFF'S 'Annalen,' Bd. xcviii. s. 245.

beautiful green flame with which it burnt demonstrated the presence of boron as one of its constituents.

In order to prepare this body in sufficient quantity, several ounces of boracic ether were placed in a capacious flask closed by a doubly perforated cork. Through one of these perforations passed a thermometer, and through the other a short glass tube, one-fourth of an inch in diameter, and open at both ends: the bulb of the thermometer dipped into the boracic ether. Successive quantities of pure zincethyl were introduced through the short glass tube by means of a pipette, the elevation of temperature after each addition being allowed to subside before the next portion was added. The failure of a further addition of zincethyl to produce any rise of temperature was regarded as evidence of the completion of the reaction, which was not attained until a comparatively very large amount of zincethyl had been added.

The liquid in the flask was now submitted to distillation in an oil-bath. It began to boil at 94° C., and between this temperature and 140° C. a considerable quantity of a colourless liquid distilled over. The distillation then suddenly stopped, and, to avoid secondary products of decomposition by the application of a greater heat, the operation was interrupted. On cooling, the materials remaining in the flask solidified to a mass of large crystals of ethylate of zinc and zincethyl. On rectification, the distillate began to boil at 70° C., but the thermometer rapidly rose to 95°, at which temperature the last two-thirds of the liquid passed over and were received apart. The product thus collected exhibited a constant boiling-point on redistillation.

The combustion with oxide of copper of this liquid and the remaining boron compounds described in this paper presented some difficulties; owing partly to the volatility of boracic acid in aqueous vapour, and partly to the tendency of that acid when fused to encase particles of carbon and prevent their oxidation. Fortunately the errors thus introduced were not so considerable as to throw any doubts upon the analytical results, although in many cases the excess in the percentage of hydrogen and the deficiency in that of carbon are somewhat greater than usual. To estimate the boron in the liquid obtained as above described, advantage was taken of the complete decomposition of the compound when heated to 100° with concentrated nitric acid in sealed tubes. The whole of the boron was in this way converted into boracic acid, but the latter could not be determined by the direct evaporation of the nitric acid solution, the loss of boracic acid amounting in such an operation to 15 or 20 per cent. of the whole amount. None of the known processes for estimating this acid appeared to be eligible in the present instance, and it therefore became necessary to seek for a new one. After the trial of various methods with but indifferent success, the following experiments showed that the evaporation of the acid solution of boracic acid with a known weight of magnesia in excess, the residue being then ignited, presented a process which, although far from rigidly accurate, could not, in the case of the boron compound to be analysed, diminish the amount of boron to a greater extent than about 0·2 per cent.

I. 1·4310 grm. of boracic acid, treated as above described, suffered a loss equal to 2·55 per cent. of boracic acid, or ·73 per cent. of boron.

II. ·5957 grm. boracic acid, similarly treated, lost ·56 per cent. of boron.

III. 2·2477 grm. boracic acid, similarly treated, lost ·35 per cent. of boron.

IV. 1·1845 grm. boracic acid lost ·51 per cent. of boron.

V. ·4125 grm. boracic acid experienced neither loss nor gain.

VI. ·8398 grm. boracic acid lost ·48 per cent. of boron.

VII. 1·1637 grm. boracic acid lost ·58 per cent. of boron.

VIII. 1·4601 grm. boracic acid lost ·57 per cent. of boron.

IX. 1·6307 grm. boracic acid lost ·47 per cent. of boron.

Submitted to analysis, the new boron compound yielded the following results:—

I. ·1307 grm. gave ·3506 grm. carbonic acid and ·1818 grm. water.

II. ·1144 grm. gave ·3068 grm. carbonic acid and ·1625 grm. water.

III. ·1071 grm. gave ·0380 grm. boracic acid.

These numbers agree with the formula

$$B\begin{cases}C_4H_5\\C_4H_5\\C_4H_5.\end{cases}$$

	Calculated.		Found.			
			I.	II.	III.	Mean.
C_{12} . . .	72	73·55	73·16	73·14	——	73·15
H_{15} . . .	15	15·42	15·45	15·78	——	15·61
B . . .	10·9	11·03	——	——	11·08	11·08
	97·9	100·00				99·84

The new body may be conveniently termed *boric ethide*. It is evidently formed by the replacement of the three atoms of oxygen in boracic acid by three atoms of ethyl, according to the following equation:—

$$2B\begin{cases}C_4H_5O_2\\C_4H_5O_2\\C_4H_5O_2\end{cases}+3Zn_2\begin{cases}C_4H_5\\C_4H_5\end{cases}=2B\begin{cases}C_4H_5\\C_4H_5\\C_4H_5\end{cases}+6\,\underbrace{\begin{matrix}C_4H_5\\Zn\end{matrix}}_{\text{Ethylate of zinc.}}O_2.$$

Boracic ether. Zincethyl. Boric ethide.

The ethylate of zinc thus produced combines with zincethyl to form the crystalline compound above alluded to. Hence the very large amount of zincethyl which was found necessary to complete the reaction.

Boric ethide possesses the following properties:—It is a colourless mobile liquid of a pungent odour; its vapour is very irritating to the mucous membrane, and provokes a copious flow of tears. The specific gravity of boric ethide at 25° C. is ·6961; it boils at 95° C. A determination of the specific gravity of its vapour by Gay-Lussac's method gave the following numbers:—

Weight of boric ethide	·2839 grm.
Observed volume of vapour	96·68 cub. centims.
Temperature of oil-bath	149° C.
Height of barometer	760·5 millims.
Height of mercury inside of tube above that outside	1·5 millim.
Height of column of oil	328·0 millims.

From these data the specific gravity of the vapour was calculated to be 3·4006. This number agrees very closely with that calculated upon the supposition that boric ethide is volumetrically composed like terchloride of boron, as is seen from the following calculation:—

1 vol. Boron vapour	·75319
3 vols. Ethyl	6·0117
The 4 vols. condensed to 2 vols.	2)6·76489
	3·38244

The density of boric ethide vapour increases considerably as the temperature approaches the boiling-point; thus a determination made at 132° gave the number 3·5979, whilst a second showed the specific gravity of the vapour at 101°·6 to be no less than 3·757.

Boric ethide is insoluble in water, and is very slowly decomposed by prolonged contact with it. Iodine has scarcely any action upon it even at 100° C. It floats upon concentrated nitric acid for several minutes without change; but suddenly a violent reaction takes place, and crystals of boracic acid separate. When boric ethide vapour comes in contact with air, it produces slight bluish-white fumes, which in the dark are seen to proceed from a lambent blue flame. The liquid is spontaneously inflammable in air, burning with a beautiful green and somewhat fuliginous flame. In contact with pure oxygen it explodes. Excluded from the air, boric ethide is quite a stable body; a quantity of it kept in a sealed tube for two years exhibited, on examination, no evidence of any alteration.

When boric ethide is heated to 99° C. with strong hydrochloric acid over mercury, a considerable quantity of hydride of ethyl is slowly evolved; ·0517 grm. of boric ethide, thus treated as long as gas was evolved, gave 35·33 cubic centimetres at 11°·1 C. and 248·4 millimetres mercurial pressure, corresponding to 11·11 cub. centims. at 0° C. and 760 millims. pressure. The reaction

$$B\begin{Bmatrix}C_4H_5\\C_4H_5\\C_4H_5\end{Bmatrix}+HCl=B\begin{Bmatrix}C_4H_5\\C_4H_5\\Cl\end{Bmatrix}+\begin{matrix}C_4H_5\\H\end{matrix}\Bigg\}$$

requires that 11·31 cub. centims. of hydride of ethyl at 0° C. and 760 millims. pressure should be liberated. That the gas thus evolved is hydride of ethyl is established by the following data:—

I.

	Millims.	Temp.
Pressure of gas used (dry)	225·3 at	13°·9 C.
Pressure after action of anhydrous sulphuric acid (dry) . .	223·9 at	13°·9 C.

II.

Pressure of gas used (dry)	23·7 at 14°·7 C.
Pressure after addition of oxygen (dry)	253·9 at 14°·7 C.
Pressure after explosion (dry)	195·6 at 14°·7 C.
Pressure after absorption of carbonic acid (dry)	147·7 at 14°·7 C.

No. I. proves the absence of members of the olefiant gas family. No. II. proves that the gas has the composition and condensation of hydride of ethyl, one volume of which consumes on combustion 3·5 volumes of oxygen, and generates twice its volume of carbonic acid; the following being the experimental numbers:—

Vol. of combustible gas.		Vol. of oxygen consumed.		Vol. of carbonic acid generated.
23·7	:	82·5	:	47·9
1	:	3·48	:	2·01

If boric ethide be heated with water to 99° C. for several hours, it also appears to suffer an analogous decomposition, although with extreme slowness; even with hydrochloric acid, the action is so tedious that I have not been able to prepare a sufficient quantity of boric chlorodiethide ($B(C_4H_5)_2Cl$) to examine its properties. In the cold, a strong solution of hydrofluoric acid has no action upon boric ethide, which also suffers scarcely any change by being heated to 99° C. for four hours with concentrated sulphuric acid. Gently heated for fourteen days with sodium in a sealed tube, boric ethide underwent no visible change.

Ammonia-Boric Ethide.

If a few drops of boric ethide be passed up into a dry eudiometer filled with mercury, and dry ammoniacal gas be then admitted into the same tube, each bubble of gas collapses with a shock, like that produced by a bubble of steam projected into cold water. A large quantity of ammonia is thus absorbed by boric ethide with extreme energy. To prepare the compound thus formed in larger quantity, several grammes of boric ethide were placed in a small flask filled with nitrogen and surrounded with ice: a current of dry ammoniacal gas was now passed into the flask so long as it was absorbed; finally, the product thus obtained was warmed to expel excess of ammonia, and then exposed *in vacuo* over sulphuric acid for twenty-four hours. It did not crystallize, and could not be distilled, except *in vacuo*, without decomposition. Submitted to analysis, it yielded 61·43 per cent. of carbon and 15·43 per cent. of hydrogen. The formula

$$NH_3+B(C_4H_5)_3$$

requires 62·66 per cent. of carbon and 15·66 per cent. of hydrogen. The unavoidable

slight oxidation of the boric ethide during the necessary manipulations affords a sufficient explanation of the deficiency in the amounts of carbon and hydrogen exhibited by the analysis. I should, however, have made renewed attempts to obtain this body in a state of greater purity, had not the investigation of the corresponding crystalline methyl compound described below, left no doubt that the formula above given expresses the composition of ammonia-boric ethide.

Ammonia-boric ethide is a somewhat oily liquid, possessing an aromatic odour and an alkaline reaction. Carbonic acid has no action upon it, even in the presence of water, but other acids decompose it instantly and liberate boric ethide. When it is exposed to a measured quantity of atmospheric air, there is scarcely any perceptible absorption of oxygen even after the lapse of several hours.

Boric Dioxyethide.

When boric ethide is placed in a flask and allowed to oxidize gradually, first in dry air and finally in dry oxygen, it forms a colourless liquid, which boils at 125° C., but cannot be distilled under atmospheric pressure without partial decomposition. At the ordinary temperature, this product of oxidation evaporates without residue in a stream of dry carbonic acid. It can be distilled *in vacuo* without decomposition, and a portion so rectified yielded on analysis the following results:—

I. ·2681 grm., burnt with oxide of copper and oxygen, gave ·5359 grm. carbonic acid and ·2720 grm. water.

II. ·2246 grm., similarly treated, gave ·4376 grm. carbonic acid and ·2303 grm. water.

Owing to the causes already mentioned, the complete combustion of this body was very difficult, nevertheless the above numbers agree sufficiently well with the formula

$$B(C_4H_5)_3O_4,$$

as is evident from the following comparison:—

	Calculated.		Found.	
			I.	II.
C_{12} . . .	72·0	55·42	54·52	55·10
H_{15} . . .	15·0	11·54	11·27	11·92
B . . .	10·9	8·39	——	——
O_4 . . .	32·0	24·65	——	——
	129·0	100·00		

I regard this liquid as a compound of vinic ether, with a body having the formula

$$B\begin{cases} C_4H_5 \\ O \\ O \end{cases}$$

and derived from boracic acid by the substitution of one equivalent of ethyl for one of oxygen. For this body the name *boric dioxyethide* is appropriate, whilst its ethereal

compound may be conveniently termed diethylate of boric dioxyethide. The formula of the latter will therefore be

$$B\begin{cases}C_4H_5\\O\\O\end{cases} + \begin{matrix}C_4H_5\\C_4H_5\end{matrix}\Big\}O_2, \text{ or } B\begin{cases}C_4H_5\\C_4H_5O_2\\C_4H_5O_2.\end{cases}$$

The formation of diethylate of boric dioxyethide from boric ethide may be thus represented:—

$$B\begin{cases}C_4H_5\\C_4H_5\\C_4H_5\end{cases} + O_4 = B\begin{cases}C_4H_5\\C_4H_5O_2\\C_4H_5O_2.\end{cases}$$

This view of the constitution and mode of formation of the oxidized product is supported by its behaviour with water; for when diethylate of boric dioxyethide is placed in contact with water it is instantly decomposed, alcohol and *dihydrate of boric dioxyethide* being formed, according to the following equation:—

$$\underbrace{B\begin{cases}C_4H_5\\C_4H_5O_2\\C_4H_5O_2\end{cases} + 2\begin{matrix}H\\H\end{matrix}\Big\}O_2}_{\text{Diethylate of boric dioxyethide.}} = \underbrace{B\begin{cases}C_4H_5\\HO_2\\HO_2\end{cases} + 2\begin{matrix}C_4H_5\\H\end{matrix}\Big\}O_2.}_{\text{Dihydrate of boric dioxyethide.}}$$

Dihydrate of boric dioxyethide may be conveniently prepared in a state of purity by agitating its aqueous solution with ether, which dissolves the boric compound. The ethereal solution must then be decanted, and on evaporation at common temperatures in a stream of dry carbonic acid, the new compound is left behind as a white and very volatile crystalline mass. The latter was sublimed at a gentle heat in a current of dry carbonic acid, and was made to condense in weighed tubes for analysis.

I. ·3870 grm., burnt with oxide of copper in a stream of air, gave ·4652 grm. carbonic acid and ·3285 grm. water.

II. ·5087 grm., oxidized with nitric acid in a sealed tube, the liquid supersaturated with a known weight of magnesia and then evaporated to dryness, gave ·2387 grm. boracic acid.

These numbers agree closely with the formula of dihydrate of boric dioxyethide:—

	Calculated.		Found.	
			I.	II.
C_4 . . .	24	32·47	32·78	——
H_7 . . .	7	9·47	9·43	——
B . . .	10·9	14·75	——	14·66
O_4 . . .	32	43·31	——	——
	73·9	100·00		

Dihydrate of boric dioxyethide is a colourless, volatile, crystalline body, very soluble

in water, alcohol, and ether. It possesses an agreeable ethereal odour, and a most intensely sweet taste. Exposed to the air, it evaporates at ordinary temperatures, undergoing at the same time partial decomposition, and invariably leaving a slight residue of boracic acid. It may be sublimed without change at about 40° C. in a current of dry carbonic acid, and it then condenses in magnificent crystalline plates resembling naphthaline. It fuses at a gentle heat, and at a higher temperature boils with partial decomposition. Its vapour tastes intensely sweet.

Boric dioxyethide might be regarded as the anhydride of a bibasic acid: the diethylate of boric dioxyethide would then be the ether of this acid, whilst the volatile crystalline body just described would be the hydrated acid itself. The latter does in fact redden litmus paper, but in other respects its acid qualities are very obscure, and I have not been able to form definite salts with it. It therefore scarcely possesses a valid claim to a place amongst the acids.

Considering boric ethide to be formed by the substitution of the ethyl in zincethyl for the oxygen in boracic acid, Mr. DUPPA and myself expressed the reaction as follows:—

$$\underbrace{2B\begin{Bmatrix}C_4H_5O_2\\C_4H_5O_2\\C_4H_5O_2\end{Bmatrix}}_{\text{Boracic ether.}} + \underbrace{3Zn_2\begin{Bmatrix}C_4H_5\\C_4H_5\end{Bmatrix}}_{\text{Zincethyl.}} = \underbrace{2B\begin{Bmatrix}C_4H_5\\C_4H_5\\C_4H_5\end{Bmatrix}}_{\text{Boric ethide.}} + \underbrace{6\,\begin{matrix}C_4H_5\\Zn\end{matrix}\Big\}O_2}_{\text{Ethylate of zinc.}}$$

Another but less probable view of the change presents itself in the supposition that the three atoms of ethyl in boric ethide were already present in the boracic ether, the action of the zincethyl being simply to remove the whole of the oxygen from the boracic ether. KEKULÉ† has in fact adopted this latter view of the reaction.

So long as the organic radical of the zinc compound and that of the boracic ether are identical, it is impossible to prove whether the three individual atoms of ethyl in boric ethide were originally present in the boracic ether, or have been derived from the zincethyl. Indicating by an asterisk the atoms of ethyl which finally become part of the boric ethide, it is impossible to prove conclusively whether the reaction takes place according to the first or second of the following equations:—

$$(1)\quad 2B\begin{Bmatrix}C_4H_5O_2\\C_4H_5O_2\\C_4H_5O_2\end{Bmatrix} + 3Zn_2\begin{Bmatrix}C_4H_5^*\\C_4H_5^*\end{Bmatrix} = 2B\begin{Bmatrix}C_4H_5^*\\C_4H_5^*\\C_4H_5^*\end{Bmatrix} + 6\,\begin{matrix}C_4H_5\\Zn\end{matrix}\Big\}O_2.$$

$$(2)\quad 2B\begin{Bmatrix}C_4H_5^*O_2\\C_4H_5^*O_2\\C_4H_5^*O_2\end{Bmatrix} + 3Zn_2\begin{Bmatrix}C_4H_5\\C_4H_5\end{Bmatrix} = 2B\begin{Bmatrix}C_4H_5^*\\C_4H_5^*\\C_4H_5^*\end{Bmatrix} + 6\,\begin{matrix}C_4H_5\\Zn\end{matrix}\Big\}O_2.$$

Although we cannot thus label, as it were, the atoms taking part in the reaction, we can unerringly trace the movements of the alcohol radicals, if we secure their identification by varying their composition in the two compounds used in the process. The

† Lehrbuch der org. Chemie, p. 489.

study of the action of zincmethyl upon boracic ether would obviously decide between these views. If boric ethide were produced from these materials, KEKULÉ's hypothesis would be established; but if, on the other hand, boric methide were the result of the reaction, then the correctness of the view originally taken by Mr. DUPPA and myself would be proved to be correct. The following are the results obtained in pursuing this inquiry.

Boric Methide.

When a strong ethereal solution of zincmethyl is added to boracic ether, an elevation of temperature to the extent of 8° or 10° C. is observed, whilst at the same time a most intensely pungent odour is developed; this odour, although it resembles that of boric ethide, is far more powerful, and more persistently irritating to the mucous membrane. A slow evolution of a spontaneously inflammable gas, burning with a splendid green flame, was also noticed; and this evolution of gas became more rapid when the warmth of the hand was applied to the flask containing the ingredients. Preliminary experiments proved that this gas was nearly insoluble in water, but almost completely soluble in alcohol, the residue remaining undissolved being marsh-gas derived from the action of the alcohol upon traces of zincmethyl vapour with which the gas was contaminated. The gas was not condensed by a freezing mixture of ice and salt. It was, with the exception of a small percentage of marsh-gas, instantaneously dissolved by solution of ammonia, which yielded the gas again unchanged when neutralized by an acid. Concentrated sulphuric acid was without action upon the gas.

These data led to the following plan for collecting the gas in a state of purity. About two ounces of boracic ether were mixed in a small flask with rather more than their own bulk of an ethereal solution of zincmethyl, of such strength as to be spontaneously inflammable in a high degree. The flask, loosely corked, was placed in ice-cold water, and allowed to stand for a couple of hours until the reaction was complete: it was then furnished with a bent tube passing through a cork, and designed to conduct the gas into a second flask placed in a freezing mixture of ice and salt; from this flask the gas passed into a third containing about half an ounce of strong solution of ammonia. The air in the whole of the apparatus was now displaced by nitrogen, and the flask containing the boracic ether and zincmethyl removed from the ice-cold water. A slow evolution of gas immediately commenced, and was kept up at a convenient speed by plunging the generating flask into cold water, to which heat was very slowly applied. The gas, in passing through the freezing mixture, deposited nearly the whole of the ether and zincmethyl vapour with which it was contaminated; and on reaching the solution of ammonia, the boron compound was instantaneously absorbed, whilst other gases, if present, passed through the ammonia unacted upon, and escaped into the atmosphere. The solution of ammonia soon became covered with a stratum of a lighter liquid, which increased in quantity until the stream of gas ceased to pass through. The ammonia-flask was now disconnected with the rest of the apparatus, and reserved for the next operation. The residue in the generating flask solidified to a crystalline mass on cooling.

It now only remained to disengage the gaseous boron compound from its combination with ammonia. For this purpose the ammonia-flask was fitted with a funnel-tube terminating beneath the surface of the liquid, and a gas-delivery tube, the latter leading to a LIEBIG'S potash apparatus charged with concentrated sulphuric acid; finally, the opposite extremity of the latter apparatus was connected with a mercurial gas-holder. To prevent dangerous explosions, on the elimination of the spontaneously inflammable gas from its ammonia compound, the whole of the air-spaces of the apparatus were filled with nitrogen. Everything being thus prepared, dilute sulphuric acid was gradually poured into the ammonia-flask through the funnel-tube, the contents of the flask being frequently agitated. No gas was evolved until the excess of ammonia was saturated; then, however, it was given off abundantly, and the addition of a few drops of sulphuric acid, from time to time, through the funnel-tube served to keep up a convenient current. The gas was allowed to pass freely through the depressed mercurial gas-holder until a sample of it proved, by its perfect solution in ammonia, that all nitrogen had been swept from the apparatus. The exit-tube of the gas-holder was now closed, and the gas collected in sufficient quantity for subsequent experiments.

The following determinations, together with the analysis of its ammonia compound, prove that this gas is *boric methide*, and that its formula is

$$B\begin{cases}C_2H_3\\C_2H_3\\C_2H_3.\end{cases}$$

I. An indefinite quantity of the gas was cautiously led over ignited oxide of copper, the carbonic acid and water produced being collected and weighed in the ordinary manner. 0·5875 grm. of carbonic acid and 0·3664 grm. of water were obtained. These numbers show that the atomic relation of carbon to hydrogen is as 2 : 3.

$$C : H = 2 : 3{\cdot}04.$$

II. A determination of the specific gravity of the gas gave the following results:—

Temperature of room at the moment of closing the flask	12°·2 C.
Height of barometer	759·4 millims.
Height of column of mercury in neck of flask above outer level of metal	4·5 millims.
Weight of flask and gas	28·7500 grms.
Temperature in balance-case	16°·1 C.
Weight of flask filled with dry air*	28·5882 grms.
Temperature in balance-case	15°·0 C.
Capacity of flask	141·7 cub. centims.

* The gas in the flask was replaced with dry air, by plunging the flask into a large beaker filled with carbonic acid, and then directing a rapid stream of the dry acid gas into the flask, so as to remove the spontaneously inflammable gas, which, rushing out of the neck of the flask with considerable velocity, reached the surface of the carbonic acid, and there formed beautiful wreaths of green flame. The flask was now several times exhausted and filled with dry air.

From these data the specific gravity of the gas was calculated to be 1·9108—a number which closely coincides with the calculated specific gravity of boric methide, which contains 1 volume of boron vapour and 3 volumes of methyl, the four volumes being condensed to two.

1 vol. Boron vapour	·75319
3 vols. Methyl	3·10956
2)	3·86275
	1·93137

Boric methide is produced from boracic ether and zincmethyl by the following reaction:—

$$2B\begin{Bmatrix}C_4H_5O_2\\C_4H_5O_2\\C_4H_5O_2\end{Bmatrix} + 3Zn_2\begin{Bmatrix}C_2H_3\\C_2H_3\end{Bmatrix} = 2B\begin{Bmatrix}C_2H_3\\C_2H_3\\C_2H_3\end{Bmatrix} + 6\begin{Bmatrix}C_4H_5\\Zn\end{Bmatrix}O_2.$$

Boracic ether. Zincmethyl. Boric methide. Ethylate of zinc.

The formation of boric methide under these circumstances proves conclusively that the corresponding ethyl compound is formed, not by the removal of the whole of the oxygen from boracic ether, but by the actual substitution of the three atoms of oxygen in boracic acid by three atoms of ethyl, whilst boric methide is in like manner produced by the similar substitution of methyl for oxygen,—a kind of substitution which is quite in harmony with the mode of formation of very numerous compounds in the organo-metallic family.

Boric methide exists at ordinary temperatures as a colourless and transparent gas, possessing a peculiar and intolerably pungent odour, irritating the mucous membrane, and provoking a copious flow of tears. Its specific gravity is 1·93137. It retains its gaseous condition when exposed to a cold of −16° C.; but at 10° C., and under a pressure of three atmospheres, it condenses to a colourless, transparent, and very mobile liquid. It is very sparingly soluble in water, but very soluble in alcohol and in ether. In contact with atmospheric air it takes fire spontaneously, burning with a bright green flame, which is very fuliginous if the volume of the flame be considerable. If the gas issue into the air through a tube $\frac{1}{10}$th of an inch in diameter, the amount of smoke is surprisingly great, two or three cubic inches of gas, when consumed in this way, filling the atmosphere of a capacious room with large comet-like flocks of carbonaceous matter. This curious phenomenon is probably due, in part at least, to the formation of a superficial coating of boracic acid, which envelopes the particles of carbon and prevents their combustion. Suddenly mixed with atmospheric air or oxygen, boric methide explodes with great violence. In contact with air, both boric methide and the vapour of boric ethide exhibit two distinct kinds of spontaneous combustion. Thus, when these bodies issue very slowly from a glass tube into the air, they burn with a lambent blue flame invisible in daylight, and the temperature of which is so low that a

finger may be held in it for some time without much inconvenience. Under these circumstances partial oxidation only takes place, and it is to the products thus formed that the peculiar pungent odour of boric ethide and boric methide is due. When, on the other hand, these bodies issue into the air more rapidly, the lambent blue and nearly cold flame changes to the green and hot flame above mentioned. I have not examined the spectra of the two differently coloured flames from the same compound, but they doubtless present a widely different appearance, thus affording another instance of the dependence of the spectra of bodies upon temperature,—a phenomenon to which Dr. TYNDALL and myself recently called attention in the case of lithium*.

Boric methide is not acted upon by binoxide of nitrogen or by iodine. Solution of bichromate of potash scarcely affects it, but the addition of concentrated sulphuric acid at once determines the reduction of the chromic acid. When boric methide is allowed to bubble through water into chlorine, each bubble burns explosively with a bright flash of light and the separation of carbon. It has no tendency to unite with acids. Concentrated sulphuric acid has no action upon it; when mixed with hydriodic acid gas, it suffers no change; but, on the other hand, it is freely absorbed by solutions of the fixed alkalies, and by ammonia. If a very rapid current of the gas, mixed with half its volume of marsh-gas, be passed through a stratum of strong solution of ammonia only half an inch deep, not a trace of boric methide escapes absorption.

Ammonia-Boric Methide.

When dry ammoniacal gas is mixed with an equal volume of dry boric methide, both gases instantly disappear with the evolution of a considerable amount of heat, and the production of a white, volatile, crystalline compound. The latter is also formed when boric methide is passed into solution of ammonia. The colourless liquid stratum which forms upon the surface soon solidifies when it is placed over sulphuric acid *in vacuo*. A quantity of the compound obtained by this latter process was purified by solution in ether and subsequent recrystallization. On being submitted to analysis, it yielded the following results:—

I. ·2652 grm., burnt with oxide of copper, gave ·4862 grm. carbonic acid and ·401 grm. water.

II. ·2809 grm. gave ·5105 grm. carbonic acid and ·4217 grm. water.

III. ·2124 grm., decomposed by dilute hydrochloric acid in a stream of carbonic acid, gave a solution of chloride of ammonium, which, treated in the usual manner, yielded ·6450 grm. of chloride of platinum and ammonium.

These numbers lead to the formula

$$NH_3+B(C_2H_3)_3,$$

as is seen from the following comparison of calculated numbers with experimental results:—

* Philosophical Magazine, S. 4. vol. xxii. p. 472.

	Calculated.		Found.			
			I.	II.	III.	Mean.
C_6 .	36	49·39	49·99	49·56	——	49·77
H_{12} .	12	16·46	16·80	16·68	——	16·74
N . .	14	19·20	——	——	19·07	19·07
B . .	10·9	14·95	——	——	——	14·42
	72·9	100·00				100·00

The excess of carbon, and to some extent also that of hydrogen, in these analyses is doubtless due to the retention of a small amount of ether by the crystals, the great volatility of the latter preventing any protracted exposure over sulphuric acid; but the substance used in No. II. was exposed for a much longer time than that employed in No. I.

Ammonia-boric methide is deposited from its ethereal solution in magnificent arborescent crystals, which rapidly volatilize without residue when exposed to the air. They possess a caustic and bitter taste, and a very peculiar odour, in which both the smell of ammonia and of boric methide can be recognized. Ammonia-boric methide fuses at 56° C. and boils at about 110° C. In a current of air, or better, of carbonic acid, it sublimes at a very gentle heat, and condenses in magnificent arborescent crystals. Determinations of the specific gravity of its vapour, at three different temperatures, gave the following results:—

Weight of ammonia-boric methide used . . . . . ·0580 grm.

	I.	II.	III.
Observed volume of vapour . .	57·9 cub. cent.	58·6 cub. cent.	59·9 cub. cent.
Temperature of oil-bath . . .	119°·0 C.	130°·0 C.	139° C.
Height of mercury inside of tube above that outside	99·9 millims.	93·1 millims.	88·5 millims.

Height of barometer . . . . . 751·8 millims.
Height of column of cold oil . . . . . 307 millims.

From these data the specific gravity of the vapour was calculated to be

I.	II.	III.
1·251	1·258	1·250

These numbers indicate that the vapour of ammonia-boric methide consists of equal volumes of boric methide and ammonia united without condensation:—

1 vol. Boric methide . . .	1·93137
1 vol. Ammonia	·5873
	2)2·51867
	1·25933

Thus the formula of ammonia-boric methide is a four-volume formula*, a state of

* H_3O_3 = 2 vols.

condensation which is usually considered to be abnormal, and which, where it occurs, is generally explained by the assumption of a decomposition of the body at the moment of conversion into vapour. The proof of the disunion or integrity of the vaporous molecule of ammonia-boric methide would be interesting in connexion with these so-called anomalous vapour-densities, but I have to regret my inability to offer any sufficiently decisive solution of this problem. The difficulty to be overcome is the finding of a reagent that will not decompose ammonia-boric methide at elevated temperatures, but which would absorb ammonia only, out of a mixture of this gas with boric methide, at a temperature above the boiling-point of ammonia-boric methide. Chloride of calcium does not decompose ammonia-boric methide; but although it readily absorbs ammonia at ordinary temperatures, yet it allows the whole of it to escape again at 110° C. Chloride of zinc decomposes ammonia-boric methide before the latter volatilizes. The same effect is produced by all the strong acids, which are therefore also inadmissible, whilst dry boracic acid does not absorb ammonia even at ordinary temperatures. The substance which appeared to be best adapted for this reaction was dry and recently fused chloride of copper. This salt does not decompose ammonia-boric methide below the boiling-point of the latter, whilst it readily absorbs ammonia, and retains it at a temperature of 160° C. I will now describe the mode in which an experiment with this substance was conducted, and the results which were obtained. A quantity of ammonia-boric methide was introduced into a graduated tube filled with mercury, and inverted in a vessel containing the same metal. The whole was now immersed in an oil-bath, and heat applied until the boron compound was converted into vapour, the volume of which, at a known temperature and pressure, was then observed. After the apparatus had been allowed to cool, a fragment of chloride of copper was passed up into the tube, and heat again applied. The boron compound soon melted and enveloped the fragment of chloride of copper: as the temperature approached the boiling-point of ammonia-boric methide, the latter slowly boiled off from the chloride of copper, and the vapour then occupied the same volume as that read off before the introduction of the chloride of copper. The mercury in the tube remained steady for two or three minutes; it then gradually ascended, and the contraction of the vapour-volume continued until it was reduced to exactly one-half, as indicated by the following numbers:—

Corrected volume of vapour before treatment with chloride of copper	35·67 cub. centims.
Ditto after treatment with chloride of copper	17·85 cub. centims.

By treatment with chloride of copper, 100 volumes of vapour were therefore reduced to 50·04 vols., the residue consisting of pure boric methide gas. It is obvious that this absorption may be due either to decomposition of the vapour of ammonia-boric methide *by chloride of copper* at an elevated temperature, or to the decomposition *by heat* of the boric compound into equal volumes of boric methide and ammonia, the latter being then absorbed by the chloride of copper. Unfortunately, the result of the experiment

is not sufficiently decisive to compel the adoption of either of these hypotheses, although the formation of the vapour and its existence for a few minutes in contact with chloride of copper favour the first more than the second; thus indicating that the vapour of ammonia-boric methide consists of equal volumes of ammonia and boric methide united without condensation, a result which would harmonize with the very generally observed rule, that when two gases or vapours unite in equal volumes, the volume of the compound is equal to that of its constituents.

Ammonia-boric methide scarcely absorbs a perceptible amount of oxygen at ordinary temperatures, even after several days' exposure to the gas; but it takes fire below 100° C. when heated in contact with the air. Its vapour is also very inflammable; thus, when ammonia-boric methide is placed under the receiver of an air-pump, and the air is being withdrawn, the explosion of the mixture of air and vapour in the cylinders of the pump is frequently determined by the rise of temperature consequent upon the depression of the pistons when the rarefaction has become considerable.

Boric methide is also absorbed by aniline with great avidity. Acids expel the gas from this compound unchanged.

Terhydride of phosphorus has no action upon boric methide. A mixture of equal volumes of the two gases is spontaneously inflammable, burning with a yellowish-white flame, in which the characteristic green tinge attending the combustion of boric methide is no longer perceptible.

Compounds of Boric Methide with Potash, Soda, Lime, and Baryta.

Solution of caustic potash absorbs boric methide with great energy. The saturated solution, exposed over sulphuric acid *in vacuo*, dries down to a gummy mass, which scarcely exhibits signs of crystallization. The same body may be more conveniently formed by decomposing ammonia-boric methide with alcoholic solution of potash, taking care to employ an excess of the former. On evaporation over sulphuric acid *in vacuo*, the excess of the ammonia compound volatilizes, and is decomposed by the sulphuric acid with the elimination of boric methide: thus the potash compound evaporates in an atmosphere of boric methide. Nevertheless even by this method I did not succeed in obtaining the potash compound in a state of purity; potash-boric methide thus prepared yielding on analysis 47·93 per cent. of potash, and 42·86 per cent of boric methide, numbers only very remotely indicating the formula

$$KO\,B(C_2H_3)_3,$$

which requires 45·67 per cent. of potash and 54·33 per cent. of boric methide. The appearance of the compound, even after exposure to gentle heat *in vacuo*, suggested the presence of water, which could not, however, be expelled at a temperature below that at which potash-boric methide itself is decomposed.

Boric methide is also readily absorbed by solution of neutral carbonate of potash, bicarbonate of potash and potash-boric methide being apparently formed. Although

boric methide and potash unite with remarkable energy, yet they are separated by acids with the greatest readiness; even carbonic acid in the presence of water can expel boric methide from its potash compound; thus, if an aqueous solution of potash-boric methide be passed into carbonic acid standing over mercury, the acid gas soon becomes replaced by pure boric methide.

Soda-boric methide, baryta-boric methide, and lime-boric methide are similar bodies, produced by the absorption of boric methide gas by caustic solutions of soda, baryta, and lime; they are all readily soluble in water and react alkaline.

Boric methide in combination with the alkalies and alkaline earths has almost entirely lost its powerful affinity for oxygen; nevertheless, when these bodies are placed in contact with a known quantity of oxygen over mercury for several days, the volume of the gas perceptibly diminishes.

The great difficulty, not to say danger, attending the gradual oxidation of considerable quantities of a gaseous and spontaneously inflammable body like boric methide, has prevented me from following this compound into its products of oxidation, as was done in the case of boric ethide. With a graduated supply of oxygen, however, boric methide appears to comport itself like boric ethide, and the compounds formed are probably homologous with diethylate and dihydrate of boric dioxyethide.

In conclusion, it can scarcely be doubted that the action upon boracic ether of the zinc compounds of the remaining alcohol radicals would produce the homologues of the bodies described in the foregoing pages. It may also be remarked, that the existence of bodies like boric dioxyethide, in which one-third of the oxygen in boracic anhydride is replaced by ethyl, altogether abolishes any supposed analogy between carbonic and boracic acids, whilst it proves that the composition of the latter acid is expressed by the formula BO_3, or some multiple of that formula. I am at present engaged in studying the action of zincethyl and sodiumethyl upon the ethers of silicic, carbonic, oxalic, and acetic acids.

IX. *On the Posterior Lobes of the Cerebrum of the Quadrumana.* By William Henry Flower, *F.R.C.S., Assistant-Surgeon to, and Demonstrator of Anatomy at, the Middlesex Hospital. Communicated by Dr.* Sharpey, *Sec. R.S.*

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Tiedemann states that the hippocampus minor is not found in the brain of Monkeys, or of any other animals which he had examined, but is peculiar to Man*. In his figure of a horizontal section of the brain of *Simia nemestrina*, a part is described as "scrobiculus parvus loco cornu posterioris," and the drawing corresponds with the description. Many writers on human anatomy have followed Tiedemann's statement; thus Cruveilhier observes, "Du reste, l'ergot [hippocampus minor] de même que la cavité digitale [posterior cornu of the lateral ventricle] n'existe guères que chez l'homme, sans doute parce que l'homme seul présente un grand développement de la partie occipitale du cerveau†."

More recently, the presence of a "posterior lobe" of the cerebrum, a "posterior horn of the lateral ventricle," and a "hippocampus minor" have been affirmed by an eminent authority in this country, to be the distinguishing characteristics of the human brain‡.

On the other hand, according to Cuvier, "les ventricules antérieurs ou latéraux n'ont de cavité digitale que dans l'homme et *dans les singes*. Cette partie n'existe dans aucun autre mammifère. Sa presence dépend de celle des lobes postérieurs§." M. Serres, in his well-known work on the comparative anatomy of the brain, has the following passage:—"Le petit pied d'hippocampe, ou le relief de l'anse d'une anfractuosité dans la corne postérieure du grand ventricule, n'a encore été aperçu que dans l'homme; on l'efface en le déplissant par le procédé que l'on met en usage pour développer le grand ventricule latéral; je l'avais moi-même méconnu en procédant à sa recherche de cette manière. Je l'ai découvert au contraire chez les singes et les phoques, en pratiquant sur le lobe postérieur une section verticale au niveau du genou postérieur du corps calleux: on enlève de cette manière toute la cavité anciroïde, et en l'entr'ouvrant, en comprimant légèrement le lobe postérieur, on voit le petit pied d'hippocampe. Dans les cerveaux qui ont été durcis par l'alcohol, on le met à découvert par cette section et

* The passage in full is, "Pedes hippocampi minores vel ungues, vel calcaria avis, quæ a posteriore corporis callosi margine tanquam processus duo medullares proficiscuntur, inque fundo cornu posterioris plicas graciles et retroflexas formant, in cerebro Simiarum desunt, nec in cerebro aliorum a me examinatorum mammalium occurrunt; homini ergo proprii sunt."—Icones Cerèbri Simiarum et quorundam Mammalium rariorum. Heidelberg, 1821, p. 51.

† Anatomie Descriptive. Paris, 1836, tome iv. p. 697.

‡ Professor Owen, Proc. Linn. Soc. 1858, and Annals and Mag. of Nat. Hist. June 1861.

§ Leçons d'Anatomie Comparée (3rd edit.), tome iii. p. 103.

par le procédé ordinaire. Je l'ai maintenant sous les yeux, chez le papion (*Simia sphynx*), chez le rhésus, chez une autre espèce de macaque, et chez le mandrill (*S. maimon*)*."

In LEURET'S 'Anatomie Comparée du Système Nerveux†' is an observation to the same effect; and in the continuation of the work by M. GRATIOLET, a passage occurs, which is so important in connexion with what has been written upon the subject by other authors, that I shall quote it in full‡:—

"Les ventricules latéraux s'enroulent, disons-nous sur les deux corps striés interventriculaires. Celle de leurs extrémités, qui répond à la massue antérieure du corps strié, est la *corne antérieure* ou *frontale* du ventricule latéral. L'autre extrémité, enroulée comme le corps strié lui-même, répond à sa massue inférieure. Nous la nommons *corne inférieure* ou *sphénoïdale* du ventricule latéral. Tel est le ventricule dans la plupart des animaux mammifères. Mais dans l'*homme* et dans les *singes*, de la partie postérieure de l'arc du ventricule latéral se detache un prolongement un peu recourbé en dedans comme la corne d'un rhinoceros, ou comme une griffe. Ce prolongement est la *corne postérieure* ou *occipitale* du ventricule latéral. On le désigne encore sous le nom de *cavité ancyroïde*.

"Ce prolongement est fort remarquable; dans les *singes*, il a une grandeur énorme en égard à l'ensemble du ventricule latéral dont l'arc est fort petit. Dans l'homme, la prédominance passe à celui-ci. Cette observation est importante, parce qu'elle coïncide avec des observations faites sur la périphérie des hémisphères."

In a foot-note he adds, "Ce prolongement occipital du ventricule est particulier aux primates (singes) et à l'homme, et par conséquent, il caractérise fort bien le type d'organisation de ces êtres. Toutefois, il ne peut-être considéré comme un signe d'élévation, car il est beaucoup plus grand en égard à la partie enroulée du ventricule dans les singes, où son développement est énorme, que dans l'homme, où la partie enroulée l'emporte évidemment sur lui. Cette remarque est d'une haute importance, et fait voir que des dispositions, qui caractérisent un groupe élevé, ne peuvent toujours être choisies comme *critérium* des dispositions sériales à l'aide desquelles l'ensemble de ce groupe est zoologiquement conçu. Si l'on attachait à la considération de prolongement occipital une importance absolue, l'homme serait inférieur au singe. C'est une preuve entre mille que la faute dont les zoologistes doivent le mieux se garantir, c'est de prendre dans leurs raisonnements la partie pour le tout.

"On pourrait supposer, en considérant la grandeur des ventricules latéraux dans le fœtus, que cette grandeur de la cavité ancyroïde chez les singes résulte d'un arrêt de développement. Mais cette conclusion serait loin d'être exacte; en effet, aux lobes antérieurs qui, chez les singes, sont extrêmement réduits, correspond un ventricule très-réduit dans toutes ses parties, tandis que le lobe postérieur, malgré la grandeur de son ventricule, a un développement relatif énorme."

It was not until the greater number of the observations related in the following paper

* Anatomie Comparée du Cerveau. Paris, 1826, tome ii. p. 470.

† Tome i. 1839, p. 402. ‡ Tome ii. 1857, p. 74.

were made, and the principal conclusions arrived at, that I met with this passage, which affords from an independent source a confirmation of their general accuracy, but at the same time leaves untouched the necessity for detailed descriptions of the condition of this portion of the brain in particular species. I also think it not improbable that renewed investigations into the development of the human brain may necessitate some modification in the conclusions contained in the last paragraph.

Some of the special monographs upon the brains of Quadrumana exhibit discrepancies almost equal to those existing in the more general statements upon the subject. Thus Schroeder van der Kolk and Vrolik found in the Chimpanzee the "lateral ventricle distinguished from that of Man by the very defective proportions of the posterior cornu, wherein only a stripe is visible as an indication of a hippocampus minor*," while in a memoir on the brain of the same animal by Mr. Marshall, these parts are shown to be as fully developed as in an average example in the human brain†. Dr. Rolleston has recently described and figured a well-marked posterior cornu and hippocampus minor in the Orang‡; and Mr. Huxley, in an admirable memoir to which I shall have occasion hereafter to refer, has given a full account of the structure of the posterior lobes in *Ateles*, a monkey of the Platyrrhine or New World group§.

Such being the present state of the literature of a subject which is evidently within the reach of ordinary observation, it will be seen that its full elucidation must be based upon an appeal to nature, and not to authority. Therefore, with a desire to contribute some reliable data to an interesting branch of anatomical science, I have taken the opportunity recently afforded by the examination of the brains of numerous examples belonging to three families of Quadrumana of making the following observations. The greater number of the specimens described are from animals which have died during the past summer in the Gardens of the Zoological Society. For their transmission into my hands while in that perfectly fresh condition so essential to the success of cerebral investigations, I am indebted to the attention of Mr. Bartlett, the able Superintendent of the Society's Collection.

As other considerations besides those of a purely anatomical character have been supposed to be involved in inquiries of this nature, I must state at the outset that these have been undertaken without reference to any theory as to the transmutation of species, or origin of the human race; whatever inferences others may draw from the facts related, for my own part I see no reason to assign any special importance, in determining the value of such a theory, to the condition of the particular portion of the cerebral organization now under consideration, especially as the general close resemblance between the physical structure of Man and the Quadrumana has long been a matter of common observation.

As it is to the brain of Man that the comparisons instituted in this paper chiefly refer, it will be necessary in the first instance to call attention to certain points in the struc-

* Nieuwe Verhandlingen der erste Klasse van het Koningl. Nederlandsche Instituut. Amsterdam, 1849.
† Nat. Hist. Review, July 1861. ‡ Ibid. April 1861. § Proc. Zool. Soc., June 11, 1861.

ture of that organ; and I must premise that in all the ensuing descriptions the nomenclature used is the one proposed by Mr. HUXLEY*, founded on that of M. GRATIOLET, to whom the merit of reducing to an intelligible and harmonious system the apparently confused and intricate surface-markings of the brain of the Primates is chiefly due†.

Plate II. fig. 1 is a sketch of the internal surface of the right hemisphere of the brain of an adult European. It has been carefully drawn from a specimen which had been prepared for the purpose, by injecting the carotid arteries with strong spirit, and hardening within the cranium, so that the general form and relative situation of the different parts have been accurately preserved. Five principal sulci are seen upon this face of the cerebral hemisphere. 1. The *calloso-marginal* (*i*, *i*), running lengthwise, parallel to, and between, the upper border of the corpus callosum and the superior margin of the hemisphere. 2. A nearly vertical fissure (*k*, *k*) placed about midway between the posterior end of the corpus callosum and the apex of the hemisphere. As it forms the line of demarcation between the parietal and occipital lobes, it has received the name of *occipito-parietal* ("scissure perpendiculaire interne" of GRATIOLET). 3. A deep and distinctly marked sulcus (*l*, *l*) commencing just below the posterior extremity of the corpus callosum, and running backwards and slightly upwards almost to the apex of the hemisphere, where it divides into an ascending and descending branch. This is the *calcarine* sulcus, so named from its relation to the part called, when seen from the interior of the ventricle, "calcar avis," or "hippocampus minor" ("anfractuosité de la cavité digitale," CRUVEILHIER; posterior portion of the "scissure des hippocampes," GRATIOLET). 4. The *dentate* sulcus, in which the small gyrus called "corpus dentatum" (corps godronné) is situated. This is the anterior portion of the "scissure des hippocampes" of GRATIOLET, and holds the same relation to the hippocampus major as the calcarine fissure does to the hippocampus minor. 5. A fissure (*n*, *n*) running more or less parallel to the last, but at a lower level, named from its connexion with the eminentia collateralis, the *collateral* sulcus. The relations of the last three sulci have been pointed out in detail by Mr. HUXLEY in the paper referred to.

The gyri on the inner surface have their boundaries defined by the above-named sulci, and are of a very simple character. They will be better understood by a reference to the figure, than by a verbal description. 17 is the marginal gyrus; 18 the callosal; 18′ the quadrate lobule; 19 the uncinate gyrus; 20 the dentate; and 25 the internal occipital lobule.

In the hemisphere of the hardened brain (Plate II. fig. 1) the space between the posterior edge of the corpus callosum and the extremity of the lobe was divided into four equal parts, and the sections indicated by the lines A, B, and C were made at the points thus determined. For accuracy of comparison, the same rule has been observed in all the sections of the Simian brains which follow. The figures A, B, and C represent the surface of such sections, and exhibit the distance to which the different sulci penetrate

* On the Brain of *Ateles paniscus*, Proc. Zool. Soc. June 11, 1861.

† Mémoire sur les Plis Cérébraux de l'Homme et des Primatès. Paris, 1854.

into the substance of the hemisphere. It must be observed that in the human brain there is some variation in this respect, but the average depth of the calcarine fissure is well illustrated in this example, as also the usual size of the posterior cornu. This cavity appears in section as a mere fissure curved round a delicate stratum of white substance, which covers internally the grey matter of the calcarine sulcus, and which constitutes the ventricular surface of the hippocampus minor. In section C no trace of the opening can be discovered.

Such is the appearance presented by this portion of the brain, in the highest state of perfection which that organ attains; a condition, however, always the least favourable for studying the morphological significance and true relations of its several parts. The original design, now obscured by special adaptive modifications, can only be traced either by observing the gradual evolution of the same parts from their most rudimentary condition, or by a comparison of similar structures in other animals of simpler organization.

By the first method we learn that while the hemispheres of the brain are mere sacs, and perfectly devoid of convolutions, the calcarine is one of the first of the sulci which appears on the surface (about the fifteenth week, according to TIEDEMANN), and coincident with it, an elevation is seen upon the interior of the ventricle, the future hippocampus minor. The eminence is, in fact, formed simply by an involution of the wall of the original ventricular cavity, and such portion of this cavity, situated in the posterior lobe, as has escaped being closed in by the growth of the surrounding cerebral substance, constitutes the posterior cornu. The variability of its extent in the human subject, and consequent apparent variability of the hippocampus minor, is well known*. Little physiological importance can, however, be attached to the size of the latter as commonly estimated by the projection into the ventricle. The real amount of cortical or ganglionic neurine surrounding the calcarine sulcus can only be ascertained by an examination of the length, depth, and complexity of that sulcus, and remains unaltered whatever may be the extent to which the cavity of the ventricle is closed: just as the size and form of the corpus striatum would be unaffected by the absence or closure of the portion of the lateral ventricle which lies in contact with it. Such an examination, both in different individuals of the human race, and in the brains of various animals, may supply important data in future investigations concerning the functions of special portions of the grey matter on the surface of the cerebrum.

* Many of the variations in the condition of these parts must be ascribed rather to pathological changes than to original conformation. Thus, in aged and debilitated subjects, the posterior cornu is often enlarged, and somewhat funnel-shaped, the calcarine projection being also more or less obliterated, a circumstance arising apparently from gravitation of the intra-ventricular fluid during long continuance of the recumbent posture. In atrophy of the cerebral substance, from whatever cause, attended by an increased size of the ventricular cavities, the change is usually most strikingly seen in the posterior cornu.

Order QUADRUMANA.

Family 1. *Catarrhina.*

Orang-Utang (*Pithecus satyrus*).—The brain of a young female of this species, which died in the Gardens of the Zoological Society, May 1850, is preserved in the Museum of the Middlesex Hospital. Although this specimen has retained its form and general characters unusually well, I do not propose to give any detailed account of it, being unwilling to multiply the too numerous imperfect descriptions which already exist of the cerebral anatomy of this animal—imperfect inasmuch as they are mostly taken from specimens the form of which has been more or less altered by preservation in spirit. A general flattening of the cerebral mass, contraction of the hemispheres, with loss of characteristic outline, obliteration of distinction between white and grey substance, and adherence of contiguous walls of sulci and cavities, render such brains ill adapted for the successful study either of the external characters or internal structure.

The following points relating to the posterior lobes of the cerebrum are, however, to be noted in this specimen. On looking directly down upon the centre of the upper surface of the brain, no part of the cerebellum is visible, either laterally or posteriorly. When viewed from one side, the posterior lobes are seen to project exactly as far backwards as completely to cover the cerebellum, but not to extend beyond it. But an examination of the interior of the cranium from which the brain was taken, shows that the shrinking of the hemispheres has reduced their dimensions in this direction. It is therefore perfectly evident that the posterior lobes of the cerebrum, according to any definition taken from external characters, exist in a very well-developed condition, although not prolonged backwards to quite so great an extent as they usually are in the human brain.

To examine the interior, the upper portions of the hemispheres were removed to the level of the inferior surface of the corpus callosum, and then further portions were carefully dissected away so as to expose the lateral ventricles with their three cornua. The general form of the cavity presents almost the exact counterpart of that in the human subject. The posterior cornu extends as far backwards as an average example in Man, being $\frac{1}{4}$ths of an inch long, and its apex being but $\frac{3}{4}$ths of an inch ($= \frac{1}{4}$th of the entire length of the hemisphere) from the surface of the posterior lobe. The projection of the hippocampus minor bears comparison with a very well-developed specimen of this structure as met with in the human brain. Its length is $\frac{5}{8}$ths of an inch, its breadth at the base $\frac{3}{16}$ths of an inch. The eminentia collateralis is more prominent than in many human brains. The hippocampus major has no distinct digital marks, but the convex border of its expanded termination has a slightly nodulated appearance. There is a complete correspondence of form in the ventricles of the two hemispheres, the posterior cornu extending backwards to a similar extent in both.

As it seemed desirable to possess an exact means of estimating the length of the posterior lobes in different animals by a criterion derived from internal structure, I have

taken the most prominent part of the convex border of the hippocampus major as the limit between the antero-median and the posterior portions of the cerebrum. The former includes the anterior and middle cornua of the ventricle, the corpus striatum, thalamus opticus, and the hippocampus major; the latter, the posterior cornu and hippocampus minor, where these exist. In Man and the Quadrumana, the angle formed at the junction of the hippocampus major and minor readily indicates the exact spot on which to place the compasses (see Plate III. fig. 7). Such measurements should, if possible, be taken before the brain is removed from the cranial cavity. Let the length of the first part be called A, and that of the second B. In Man the average proportion in several examples of A to B is as 100 to 53. In this Orang's brain A measures exactly 2 inches, and B 1 inch, or as 100 to 50; so that the posterior lobe, as defined by internal structure, exhibits the same slight diminution upon that of the human brain, as was already estimated by the amount of covering of the cerebellum. It must be remembered, however, that this observation is taken from a spirit preparation.

Presbytes leucoprymnus (Otto).—An adult female died in the Zoological Society's Gardens, August 20, 1861, and the brain was examined while in a perfectly fresh condition. The posterior lobes completely covered, and projected beyond, the cerebellum. The sulci on the internal face of the cerebral hemisphere (Plate II. fig. 2) bear a very close resemblance in general arrangement (though of course far less complex) to those of the human brain. The occipito-parietal (*k*, *k*) runs downwards and forwards, and almost meets the calcarine (*l*, *l*), the second internal annectent gyrus scarcely appearing on the surface. The calcarine sulcus differs from that, not only of Man, but of nearly all Apes, in extending to the extremity of the lobe, and even turning round to the outer surface, without dividing into branches. The three sections show that this sulcus penetrates more deeply into the substance of the hemisphere than in Man, but it is tolerably simple in its course, passing nearly directly inwards. The posterior cornu of the ventricle is distinctly open to within a very short distance of the hinder extremity of the lobe, and is of considerable vertical depth, being curled round the very prominent calcarine involution, or hippocampus minor.

In the left hemisphere a horizontal section was made, so as to expose the lateral ventricle (Plate III. fig. 5, *a*). The posterior cornu was seen, when opened from above, to take the same general course as in the brain of Man, viz. outwards, backwards, and finally somewhat inwards, but, owing to the depth of the calcarine fissure, and consequent great projection of the hippocampus minor, it is placed nearer to the external wall of the hemisphere than it is in Man. The measurements of the antero-median (A) and posterior (B) portions of the hemisphere are 1·5 inch and ·7 inch—A to B as 100 to 47. The latter is therefore proportionally less than in the human brain.

Cercopithecus.—Several examples of this genus have come under examination, including *C. sabæus*, *C. ruber*, and *C. mona*, but a description of *C. pygerythrus* (F. Cuv.), the common Vervet Monkey, will suffice. In order to obtain a side view of the brain *in situ*, the right half of the cranium of an adult animal of this species was carefully

removed with the saw and bone forceps, then the dura mater was taken away, and the contents of the cranial cavity exposed. While still in an undisturbed condition, the extent to which the posterior lobes of the cerebrum projected beyond the cerebellum was ascertained to be fully $\frac{1}{4}$th of an inch. On comparing the form of the brain with that of the human subject, very great similarity is seen in the contour of the posterior half of the cerebrum, but the anterior lobes in the Monkey are much reduced, being narrowed almost to a point, flattened, and largely excavated in the orbital regions. Fig. 6 (Plate III.) is a profile view of this brain, and shows accurately the relative form and situation of the different parts while *in situ*, and the arrangement of the sulci upon the outer face of the hemisphere.

Fig. 3 (Plate II.) is a sketch of the convolutions of the inner face of the right hemisphere. The calcarine sulcus is very strongly marked, and describes a curve having the concavity upwards; it bifurcates as usual at the posterior end. The occipito-parietal sulcus, instead of running downwards and forwards as in Man and *Presbytes*, is directed somewhat backwards, and does not join the calcarine sulcus, but terminates at the upper margin of a prominent gyrus (the second internal annectent) which borders the last-named fissure superiorly. The sections show that the calcarine fissure extends to a greater depth than in any other genus (yet examined), and has a singular complexity of form, as there is concealed within it, and attached to its floor, a small but distinct gyrus (Plate II. fig. 3, and Plate III. fig. 7, 26). This convolution, which may be called from its position "calcarine," commences anteriorly by a slight elevation of the floor of the calcarine fissure, increases as it proceeds backwards, and comes to the surface where the sulcus turns up towards its termination; then, bounded superiorly by the lower branch of the sulcus, sweeps round the inferior border of the lobe, and becomes continuous with the infero-occipital gyrus of the outer side. It appears to be always present in *Cercopithecus*, *Macacus*, and *Cebus*, and probably in all the allied genera, but is absent in the highest and lowest members of the order. The consequence of the depth and complexity of the calcarine sulcus is that the involution of grey matter forming its walls is much increased, and bears a very large proportion to the mass of the lobe, and the cornu of the ventricle is thrown quite to the outer side of the hemisphere, being at its termination only separated by a thin stratum of white matter from the cortical layer of its external face. The walls of the cornu are in such close apposition that I have not been able to satisfy myself that it is in the adult *Cercopithecus* an actual cavity in the same sense as the remaining portion of the ventricle, especially as the staining of the lining membrane found in other parts rarely extends more than a quarter of an inch upon the surface of the hippocampus minor. But as fine sections of hardened brains show a line in which the cerebral substance is absent, having always the same definite extent, form, and direction,—as the slightest touch with the handle of the scalpel will separate the walls,—as in some genera it is undoubtedly as distinctly open as any other part of the cavity, and as it has been so considered by Gratiolet, Huxley, and other competent observers, I have no hesitation in looking upon it as homologous to the posterior cornu

in Man. Whether it is an actual or potential cavity is, however, of very little consequence, as there can be no question that the portion of the brain answering to that which in Man is called hippocampus minor, attains in *Cercopithecus* a really prodigious development in comparison with the size of the cerebral hemisphere.

The proportionate length of the posterior lobe, as measured upon a section made while the base of the brain still remained within the cranium, slightly exceeds that of Man, being to the antero-median portion as 54 to 100.

Macacus.—The brains of *M. silenus*, *M. nemestrinus*, *M. cynomolgus*, *M. sinicus*, *M. radiatus*, and *M. erythræus* have all been examined. They resemble one another so closely that the latter (the Rhesus Monkey) alone need be described. The length of the posterior lobes, both as to the extent to which the cerebellum is covered, and as ascertained by internal measurement, is slightly inferior to that of *Cercopithecus*. The convolutions of the inner face of the hemisphere (Plate II. fig. 4) have the same general arrangement as in that genus. The calcarine sulcus does not extend to quite so great a depth, but it conceals within it, though on rather a smaller scale, a similar gyrus (Calcarine, No. 26 B, Plate II. fig. 4). In adult examples the walls of the posterior cornu adhere very closely, but in a new-born Rhesus they were distinctly separate almost to the very end of the lobe. In this specimen the hemispheres were so elongated backwards as to project by nearly one-fourth of their length beyond the cerebellum.

Cynocephalus.—Gratiolet has demonstrated that the principal cerebral characteristic of this genus is the great development of the occipital lobes. In a nearly full-grown example of *C. porcarius*, I find that they project $\frac{4}{10}$ths of an inch beyond the cerebellum, or rather more than $\frac{1}{3}$th of the entire length of the hemisphere; proportionally more, therefore, than in Man. Measured internally, the proportion of the posterior to the antero-median portion of the cerebrum is as 57 to 100—greater than in any other of the Catarrhine Apes. The calcarine sulcus is very deep and complex, as in the last two genera, but the specimen did not reach my hands in time to give any drawings.

Family 2. *Platyrrhina.*

Cebus apella.—The brain of this species presents a different form from that of *Cercopithecus*, or any of the Old World Apes. The hemispheres are much elongated and compressed laterally, so as to give a regularly oval outline to the entire cerebrum when seen from above. The cerebellum, though large, is entirely covered, and the posterior lobes are of great proportional length, being to the antero-median as 59 to 100.

The sulci upon the inner face of the cerebral hemisphere are shown in Plate III. fig. 5. The calcarine fissure at its anterior extremity joins the dentate, so that the callosal gyrus is not actually continuous with the uncinate as in Man and most of the Quadrumana; but at the point of union the sulcus is extremely shallow, or, in other words, the band which connects the two above-named gyri does not quite reach the surface. At first the sulcus runs somewhat in a downward direction, but ultimately takes a considerable

sweep upwards. It penetrates deeply into the lobe, not quite to the same extent as in *Cercopithecus*, but is so disposed as to contain a similar convolution.

Hapale jacchus, the Common Marmoset.—In an adult specimen which died at the Zoological Society's Gardens, September 25, 1861, the brain was exposed *in situ* by clipping away different portions of the cranial bones, and the drawings (Plate III. figs. 8 & 9) made before its outline had been altered by removal from its bed in the skull. Seen from above, the two hemispheres form an elongated oval, slightly narrowed anteriorly. The olfactory lobes extend forwards beyond the cerebrum, but no part of the cerebellum is seen. The side view shows the elongation and flattening of the whole hemisphere, and the extent to which the posterior lobes project beyond the cerebellum. The orbital region of the anterior lobe is greatly excavated. The fissure of SYLVIUS is well marked, but on separating its lips no distinct median lobe is seen, the only indication of it being a very slight elevation of the floor of the middle third of the fissure. The outer surface of the hemisphere is perfectly smooth and free from sulci, a faint depression only occupying the situation of the antero-temporal, the most persistent of all the sulci of the outer face in the Quadrumana.

The inner surface of the hemisphere (Plate III. fig. 10), quite smooth in its anterior and superior portions, presents, nevertheless, three distinctly marked sulci, the dentate (*m*), the collateral (*n*), and, occupying exactly the same situation as in the higher Primates, the calcarine sulcus (*l*). This is quite simple, not bifurcated at the end, or joined by any other fissure, and describes a curve with the convexity upwards. There is no trace either of the occipito-parietal or the calloso-marginal sulcus. A section made at the middle of the posterior lobe (corresponding to the section B in the larger brains) shows that the calcarine fissure is of great depth and has a downward curve. The grey matter surrounding it occupies nearly the whole of the interior space of the lobe, forming a hippocampus minor of simple construction, but very great relative size. The very narrow stripe of medullary white matter between this involution and the external surface contains a distinctly marked crescentic opening, the section of the posterior cornu.

A horizontal section through the left hemisphere (Plate III. fig. 9) exhibits the great length of the posterior lobe as compared with the antero-median, viz. as 62 to 100. The posterior cornu in this view describes a regular curve, with the concavity inwards, and is seen to extend to within $\frac{1}{16}$th of an inch of the apex of the hemisphere. Its walls fell apart directly the section was made, and there appeared to be a distinct lining membrane, on which fine blood-vessels were seen to ramify.

The above description would apply almost equally well to the brain of *H. œdipus*, two examples of which have been dissected.

Thus, in the brain of these diminutive creatures and in that of Man, placed at opposite ends of an extensive series, and in many respects so widely removed from each other, are found certain well-marked common characters in the posterior lobes; and the principal distinction that we can draw between them, with respect to this portion of the brain, is, that in the Marmoset the whole lobe is more elongated, the calcarine fissure

more deeply cut, the hippocampus minor more prominent, and the posterior cornu patent to a greater extent.

As has been already pointed out, from an examination of vertical sections of skulls, it is among members of this family that the occipital region of the cerebrum attains its greatest, and also, as far as the Apes properly so called are concerned, its least development; the first in *Chrysothrix*, one of the lowest, and the last in *Mycetes*, the genus usually placed at the head of the family. It is remarkable, also, that among the Catarrhina, as estimated by the same means, the backward development of the cerebrum, in relation to the cerebellum, appears to coincide with the order in which these animals are zoologically arranged, being least among the anthropoid Apes, and attaining its maximum in the *Cynocephali*.

Family 3. *Strepsirrhina*.

Most of the descriptions and figures of the brains of members of this family hitherto published are unsatisfactory. For example, that given by TIEDEMANN* of *Lemur mongoz*, though evidently drawn with great accuracy and care, represents a brain the form of which has been considerably altered by hardening in spirit. The same objection applies, but even more strongly, to the delineations of the brain of *Stenops* given by VROLIK† and by SCHROEDER VAN DER KOLK‡. That of *Tarsius*, in the admirable memoir of BURMEISTER §, may perhaps be excepted.

Having, therefore, lately had an opportunity of dissecting a Lemur in a fresh condition, I have thought it desirable to give a new figure of the external characters of the brain. The view of the upper surface (Plate III. fig. 11) was drawn after the removal of the skull-cap, while the brain was still in the head; the other two (Plate III. figs. 12 & 13) immediately after it was taken out, and with the assistance of a cast of the interior of the cranium. After the brain had been a fortnight in spirit, the hemispheres had lost one-fifth of their length, together with their characteristic outline, and had left about half of the cerebellum uncovered.

Lemur nigrifrons, GEOFF. (*L. mongoz*, LINN.?).—After removing the upper portion of the skull by a horizontal incision, and then taking away the dura mater, the surface of the encephalon was exposed. The part brought into view consisted of the cerebral hemispheres, with a small portion of the olfactory lobes projecting in front, and of the cerebellum behind. The general outline of the two cerebral hemispheres presented an oval figure, very narrow in front and broad behind, where it was deeply indented in the middle line. On looking directly down upon the centre of this oval, the portion of the cerebellum visible was part of the upper surface of the superior vermis, chiefly exposed by the divergence of the posterior apices of the cerebral hemispheres, and a very narrow

* *Op. cit.* tab. iv. figs. 1, 2, 3, 4.

† Bijdrage tot de Anatomie van den *Stenops Kukang*. Leiden, 1841.

‡ Nieuwe Verhand. der 1ste Klasse v. h. Kon. Nederlandsche Inst. 1843.

§ Beiträge zur näheren Kenntniss der Gattung *Tarsius*. Berlin, 1846.

border of each lateral lobe. The extreme projection was ·15 of an inch behind the cerebrum. In front the olfactory lobes extended $\frac{1}{10}$th of an inch beyond the cerebrum. The hemispheres were 1·65 inch in length, and 1·3 inch across their broadest part. On gently separating the edges of the longitudinal fissure, the corpus callosum was seen to cover completely the corpora quadrigemina; its length was ·85 of an inch.

The brain was now removed. The general surface of the cerebrum is smooth, but marked with strongly defined, deeply cut, regular, and almost symmetrical sulci. The anterior or frontal lobes are attenuated, being flattened above, compressed laterally, and excavated below for the orbital plates. They are distinctly marked off from the temporal lobe by the fissure of Sylvius, which runs upwards and backwards to the parietal region, and has an abrupt and slightly bifurcated termination. The average depth of this fissure is nearly one-fourth of an inch, and on separating its lips, a small, smooth, but distinctly defined *insula* or median lobe of oval form was disclosed. This observation is important, as Gratiolet says, "Le lobe central [insula] paraît particulier à l'homme et aux singes; peut-être voit-on quelque chose d'analogue dans les makis, mais on ne voit rien de semblable chez les autres mammifères." The temporal lobe is full and deep, and terminates posteriorly without any definite boundary in the posterior or occipital lobe. This last is shallow, and excavated on its under and inner surface for the cerebellum.

The sulci on the outer face of the hemisphere are—1. The fissure of Sylvius (*e*). 2. A well-marked longitudinal sulcus on the upper surface of the frontal lobe, inclining outwards posteriorly, probably corresponding with the infero-frontal (*a*). 3. A slight longitudinal indentation on the orbital surface of the same lobe. 4. A very distinct sulcus on the temporal lobe, parallel to, but extending rather higher than, the fissure of Sylvius, and curving forwards at its upper end (*f*): this is the antero-temporal (scissure parallèle). 5. A well-marked longitudinal sulcus on the upper surface of the parietal, and extending into the occipital lobe, marking off the upper limit of the angular gyrus. 6. A slight longitudinal indentation on the outer side of the occipital lobe. There is no trace of the temporo-occipital sulcus (scissure perpendiculaire externe), so well marked in the higher Apes, or of either of the parietal fissures; indeed the region on which they should be placed is very greatly reduced. In number, extent, and situation the sulci above described nearly correspond to those of *Callithrix moloch* (as figured by Gratiolet), a Platyrrhine Ape about the same size as the Lemur.

Upon the internal face of the hemisphere (Plate III. fig. 14) are seen—1. The callosomarginal (*i*), distinct only in the middle third of the hemisphere. 2. A very deeply marked calcarine sulcus (*l*, *l*), extending from below the posterior end of the corpus callosum, backwards and slightly upwards, to near the extremity of the hemisphere, where it ends abruptly without bifurcation. 3. Joining this, almost at a right angle, is the occipito-parietal (*k*), which does not quite reach the upper margin of the hemisphere. 4. A slight indication of the collateral sulcus (*n*). 5. A well-marked dentate sulcus (*m*).

The olfactory bulbs are in size intermediate between those of the lower Apes and

those of the Carnivora. The corpora albicantia are represented by a single mass, which however is cleft posteriorly, indicating its separation into two portions. The corpora geniculata form distinct white nodules on the sides of the crura, but not visible until the edge of the temporal lobe is slightly lifted up. The pons is but little elevated. The medulla oblongata is very wide, and the tracts called corpora trapezoidea clearly marked out. The corpora quadrigemina resemble those of other Quadrumana, the anterior being larger and of more rounded form than the others. The cerebellum shows a marked inferiority to that of the true Monkeys. The median vermis, especially the inferior portion, is very large. The lateral vermis (flocculus) is also greatly developed, and forms the principal part of the lateral mass of the cerebellum. The body of the lobe is, however, not so much reduced as in the Carnivora.

To return to the cerebral hemispheres. A section was made through the right posterior lobe at the point B (Plate III. fig. 14). The calcarine sulcus is now seen to extend to about the middle of the section, but to be of the simplest form. The cortical layer which it carries with it (hippocampus minor) is bordered by a thin stratum of white substance, which is separated from the contiguous medullary cerebral matter, as in the other Quadrumana, by a fine crescentic line, indicating the presence of a posterior cornu of the lateral ventricle. In a horizontal section of the left hemisphere this cornu appeared as a mere fissure, with walls in close apposition, but traceable nearly to the termination of the hemisphere. In this view, the most marked difference between the parts displayed, and those of the ordinary Quadrumana, consisted in the comparative shortness of the posterior lobe, this being, as compared with the antero-median portion, only as 35 to 100.

None of the authors who have written upon the brains of the Lemuridæ, whose works I have been able to consult, describe a hippocampus minor. Vrolik expressly states that it is absent in *Stenops* ("L'éminence digitale, l'éminence collatérale de Meckel manquent," *op. cit.* p. 79), and Burmeister alone mentions a posterior cornu to the ventricle in *Tarsius*, the only observation upon it being, that it is "very long." There can be no doubt, however, of the strict homology of the calcarine fissure, and its surrounding grey matter (hippocampus minor), in the Lemur, to that of the parts so described in Man and all the intermediate forms; and that in this low and almost aberrant member of the order, although of reduced length, corresponding with that of the hemisphere, it extends more deeply into, and bears a greater ratio to the surrounding mass of the lobe than it does either in Man or in the anthropoid Quadrumana.

The presence of the same parts is shown even more distinctly in the brain of a Galago (*Otolicnus*) preserved in the Museum of the Middlesex Hospital. The animal to which it belonged died in the Zoological Society's Gardens in 1852*. While alive it was referred to *O. Garnettii* (Ogilby), but its dimensions did not agree with those of the type specimen of that species in the British Museum; its generic determination is, however, sufficient for the present purpose. For reasons given in the case of the Orang, I do not

* See Proc. Zool. Soc., March 23, 1852, p. 73.

purpose to give a description of the external characters of this brain, but only an account of such parts of the internal structure as have special reference to the subject of this paper. A horizontal section of both hemispheres has been made at the level of the corpus callosum, and the lateral ventricles are laid open. Fig. 15 (Plate III.) represents this dissection. A broad and very distinct posterior cornu extends backwards almost to the extremity of the hemisphere. Its floor and inner wall are raised into a prominence, having distinctly the characters of the hippocampus minor as found in Man and the higher Quadrumana, and corresponding with the deeply marked calcarine sulcus on the inner face of the lobe. The form of the eminence is somewhat triangular, the apex being directed backwards; but its surface is convex, both from above downwards and in the antero-posterior direction, so that the axis of the cavity into which it projects, though directed generally backwards, has first an outward inclination, and finally turns somewhat inwards. The anterior or broad end of the eminence is concave, being adapted to the curved posterior margin of the hippocampus major, from which it is separated by a deep groove. The length of the hippocampus minor is one-fourth of an inch; its breadth at the base almost as much. The part of the outer wall of the ventricle which projects into the angle between the hippocampi may be compared with the "eminentia collateralis" of the human brain*.

On comparing the posterior lobe in *Galago* with the same part in the true Apes, it is seen that there is, as in *Lemur*, a very marked reduction in length. This abbreviation is the more remarkable as there is no approach to it in the lowest of the Platyrrhine Monkeys. In the possession of a well-defined Sylvian fissure, a median lobe, and a calcarine sulcus, and in the general characters of the convolutions of the hemispheres, the brain of the Lemuridæ follows precisely the same type as that upon which the brain of Man and the other Quadrumana is formed, and differs essentially from that of the Carnivora and all other orders of Mammalia. But while the gradations of the brains of this type are tolerably regular and unbroken between the largest and the smallest of the series (i. e. *Homo* and *Hapale*), the Lemurs do not continue in precisely the same line of degradation, but rather should be placed as a small subseries parallel to the lower part of the large series, and distinguished from it by the shortness of the posterior lobes, the large size of the olfactory bulbs, and the inferior condition of the cerebellum.

With regard to the general characters of the posterior lobes throughout the series, although the examination of all the forms is not yet complete, the facts which have already been brought together are sufficient to justify the following conclusions:—

* A further examination of this specimen, and of the brains of some allied genera, leads me to doubt whether the above-described 'cavity' in the posterior lobe existed before dissection, the length of time that it had been in spirit having greatly facilitated this process. If it did not, it will justify the statement of the absence of the hippocampus minor by anatomists who have looked at this structure only in its relation to the posterior cornu, but at the same time will afford a further illustration of what I have endeavoured to show throughout this paper, viz. that the part of the brain to which this term has been applied can exist independently of the ventricular cavity.

1. That the posterior lobes, whether we understand by the term that portion of the cerebrum which lies over the cerebellum, or, taking our definition from internal structure, that part which is situated behind the hippocampus major, exist in all the Quadrumana, and are characterized in all by the presence of a deep longitudinal sulcus (calcarine) on their inner surface.

2. That the length of this part of the brain, in relation to that of the antero-median portion, varies in different members of the series, but is greater in many of the Apes than it is in Man, and attains its maximum in the smaller members of the family Platyrrhina.

3. That the depth and complexity of the characteristic involution of the cortical grey matter surrounding the calcarine sulcus (or, in other words, that part which, according to its homology with the structure so named in the human brain, must be called "hippocampus minor") is one of the most striking characteristics of the typical Simian brain, as it is greatest in *Cercopithecus*, *Macacus*, *Cynocephalus*, and *Cebus*, less in the anthropoid Apes, and least of all, in proportion to the mass of cerebral substance contained in the lobe, in Man.

Appendix.

Table showing the comparative length of the posterior lobes of the cerebrum in certain Quadrumana, and other Mammalia, measured upon the plan described at p. 191.

	Actual length in inches.		Proportion.	
	Antero-median portion.	Posterior portion.	Antero-median portion.	Posterior portion.
Homo (average)	4·40	2·35	100	53
Pithecus satyrus	2·00	1·00	100	50
Presbytes leucoprymnus	1·50	·70	100	47
Cercopithecus pygerythrus	1·65	·90	100	54
C. sabæus	1·65	·90	100	54
C. mona	1·20	·65	100	54
Macacus silenus	1·75	·95	100	54
M. erythræus	1·75	·90	100	52
Cynocephalus porcarius	2·20	1·25	100	57
Cebus apella	1·60	·95	100	59
Nyctipithecus felinus	1·20	·70	100	58
Hapale œdipus	·85	·50	100	59
H. jacchus	·80	·50	100	62
Lemur nigrifrons	1·00	·35	100	35
Stenops Javanicus	·90	·40	100	44
Otolicnus ——?	·85	·35	100	41
Pteropus Edwardsii	·79	·18	100	24
Erinaceus Europæus	·60	·07	100	11
Cercoleptes caudivolvulus	1·20	·40	100	33
Felis domesticus	1·20	·25	100	21
Canis familiaris	1·55	·55	100	35
Equus caballus	3·50	1·25	100	35
Sus scrofa	2·15	·55	100	25
Dicotyles torquatus	2·00	·40	100	20
Lepus cuniculus	1·00	·20	100	20

Explanation of the Plates.

PLATE II.

All the figures are of the natural size, except the first.

Fig. 1. Inner face of the right cerebral hemisphere of human brain, reduced one-half in linear dimensions. A, B, C. Sections of the posterior lobe at the points indicated by the lines so lettered.

Fig. 2. The same part of *Presbytes leucoprymnus*.

Fig. 3. The same part of *Cercopithecus pygerythrus*.

Fig. 4. The same part of *Macacus erythræus*.

PLATE III.

Fig. 5. The same part of *Cebus apella*.

Fig. 5 *a*. Horizontal section of left hemisphere of *Presbytes leucoprymnus*. The transverse lines show the mode of estimating the length of the posterior portion of the brain (B) as compared with the antero-median (A).

Fig. 6. Side view of the brain of *Cercopithecus pygerythrus*, showing the exact form when *in situ*. The outline is drawn from a cast of the interior of the cranium.

Fig. 7. Horizontal section of the brain of the same animal. On the right side the middle and posterior cornua are completely opened, so as to exhibit the relative size and situation of the two hippocampi. In exposing the hippocampus minor to this extent, the limits of the cornu (as seen in the sections A, B, C, fig. 3) have not been exceeded; but as the walls are more or less adherent, this must be regarded partly as a dissection. On the left side the walls of the cornu remain undisturbed, part of the brain only having been cut away to expose the commencement of the hippocampus major.

Fig. 8. Side view of the brain of *Hapale jacchus*.

Fig. 9. Upper surface of the same brain, the left hemisphere in section.

Fig. 10. Inner face of the right hemisphere of the same brain. B. Section of the posterior lobe.

Fig. 11. Upper surface of the brain of *Lemur nigrifrons*.

Fig. 12. Base of the same brain.

Fig. 13. Side view of the same.

Fig. 14. Inner face of the same. B. Section of the posterior lobe.

Fig. 15. Horizontal section of the brain of *Galago*. On the right side the section is carried rather low, and the hippocampus minor cut through; on the left the whole surface of this structure is exposed by opening the posterior cornu from above. The form of the brain is somewhat altered by keeping in spirit.

Nomenclature and lettering of all the Figures.

Gyri of the outer face:—

1. Infero-frontal.
2. Medio-frontal.
3. Supero-frontal.
1′. Supra-orbital.
4. Antero-parietal.
5. Postero-parietal.
5′. Postero-parietal lobule.
6. Angular.
7. Antero-temporal.
8. Medio-temporal.
9. Postero-temporal.
10. Supero-occipital.
11. Medio-occipital.
12. Infero-occipital.
13. First external annectent.
14. Second external annectent.
15. Third external annectent.
16. Fourth external annectent.

Gyri of the inner face:—

17. Marginal.
18. Callosal.
18′. Quadrate lobule.
19. Uncinate.
20. Dentate.
21–24. Internal annectent.
25. Internal occipital lobule.
26. Calcarine.

Sulci of the outer face:—

a. Infero-frontal.
b. Supero-frontal.
c. Antero-parietal.
d. Postero-parietal.
e. Sylvian.
f. Antero-temporal.
g. Postero-temporal.
h. Temporo-occipital.

Sulci of the inner face:—

i. Calloso-marginal.
k. Occipito-parietal.
l. Calcarine.
m. Dentate.
n. Collateral.
** Hippocampus major.
* Hippocampus minor.

X. *On Magnetic Calms and Earth-Currents.*

By CHARLES V. WALKER, *Esq., F.R.S., F.R.A.S.*

Received February 3,—Read February 13, 1862.

I USE the word "calm" in a purely negative sense—not storm. For many months the earth has shown few marked signs of activity. Very few notable earth-currents have attracted attention since those of 1861, January 22 to 26, referred to in the last paragraph but one of my paper "On Magnetic Storms, &c.," read February 14*.

I had not proceeded far in the discussion of the results which form the subject of that communication, without discovering that I possessed close at hand, and under my immediate control, the means of verifying the conclusions to which I had arrived as to the general direction of earth-currents; of extending the observations to periods when the earth is free of signs from extraordinary activity; and of further pursuing my inquiries.

In sections 3 and 4 of Table XII.† I grouped together a series of observations; and in fig. 5 of Plate III. gave a graphic illustration of the same; and from those data deduced the approximate Direction of Earth-Currents; and determined the azimuth of the drift to be in turn about N.E. and S.W.

I was unable to include many cases in these sections of the Table. Simultaneous observations including both limiting lines were few; I was therefore glad to have so ready at hand the means of multiplying and modifying them. The groups of observations in the Table referred to were made on eight or nine different lines of telegraph, making various angles with the magnetic meridian; and were bounded on the one hand by the London-Tonbridge line, making an angle with the magnetic meridian of 13° W., and on the other hand by the Dover-London line, making an angle of 136° E., the two lines making with each other an angle of 149°; from which it could be deduced, and was shown, that the direction in which the currents moved was included within an arc of (180°−149°=)31°; and that this arc was situated midway between the lines in question (which are those set off on fig. 5, Plate III., and numbered there, as well as in Tables XI. and XII., 23 and 26 respectively), extending from 46° to 77° E. of magnetic north, or W. of magnetic south.

The Dover-London telegraph wires pass Tonbridge, where they enter my private office, being attached to a telegraph instrument placed there. This gives me the immediate command of the Dover-London line, numbered 26—25 in Table XI. p. 130, and in figs. 1 and 5, Plate III., and which is one of the limiting or boundary lines. By cutting off the communication with Dover, that is to say, by connecting the wire with

* Philosophical Transactions, 1861, p. 113. † Ibid. 1861, p. 131.

the earth on the Dover side of Tonbridge, I obtain the other limiting line, London-Tonbridge, Nos. 23—24; or if the wires are connected with the earth on the London side of Tonbridge, I obtain the Dover-Tonbridge line. This is not given in my Table XI., but is very nearly identical with the Ashford-Tonbridge line there given, and numbered 32—31. It makes an angle of 118° E. with the magnetic meridian, and therefore falls, as shown in fig. 1, Plate III., intermediate between the other two, and incidentally is very useful in this investigation.

The earth-currents which form the subject of the present communication are not detected by the ordinary telegraph galvanometer. I have therefore prepared a small horizontal galvanometer. The coil is $2\frac{1}{4}$ inches long, $\frac{3}{4}$ of an inch wide, and $\frac{3}{4}$ of an inch high, and is filled with silk-covered copper wire, one yard of which weighs 5 grains; it is No. 35 of the Birmingham iron-wire gauge, corresponding to a diameter of $\frac{5}{440}$ inch. It is placed in the magnetic meridian. The needle is 1 inch in length, and carries a light index projecting from the E. side of the coil. The range of the index is about 55° on either side of 0°. The whole is covered by a glass shade. Earth-currents that attract no attention on the telegraph needles, produce on this instrument a deflection of 40° or 50°. It is placed in proper connexion with one of the Dover-London wires, and can be at any instant placed in circuit by merely pressing a spring, and thrown out of circuit by removing the pressure. Possession is obtained with equal facility of the London-Tonbridge or the Dover-Tonbridge section, by inserting a brass plug in one or other of two holes made in a divided brass disc. This is all the apparatus required. It is fixed on a slab within arm's reach from my chair. At any moment, when I see by the telegraph needles that the wires are unoccupied by telegrams, I can take the three complete observations in a few seconds. The word "up" is engraved in face of the N.E. quadrant, and the word "down" in face of the S.E.

The arrangement of this miniature observatory will be readily understood from the following diagram (fig. 9).

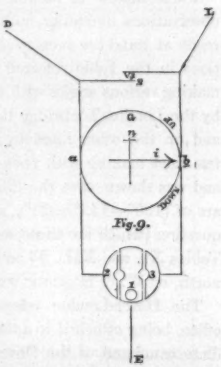


Fig. 9.

L is the telegraph wire, entering the office from London; D the wire from Dover; S the contact spring, and s the stud on which it rests. When the spring and stud are in contact, the ordinary telegraph signals pass along them between D and L without being visible on the galvanometer G. The needle n, with its index i, and the circle of the galvanometer G, are given in one-third size; a and b are the terminals of the galvanometer wire. When the spring S is depressed or removed from contact with the stud s, as shown in the diagram, no current can pass between D and L without passing through the galvanometer G. 1, 2, and 3 show in one-third size a brass disc divided into three parts, and fixed upon a block of mahogany, insulated each from the others. The middle piece of the disc is connected by a wire with the gas- and water-pipes, and is therefore in con-

nexion with the earth E. A brass plug is provided, which fits into the respective holes 1, 2, and 3. When in the centre hole 1, this instrument is not in action; when in the hole 2, it connects 1 and 2 together, so that the wire on the *a* or Dover side of the galvanometer is in direct communication with the earth E, and the galvanometer is in circuit between London and Tonbridge; when the plug is transferred to the hole 3, for like reasons the galvanometer is in circuit between Tonbridge and Dover. Observations on the whole line, or on either of the two sections, are readily and rapidly obtained by this arrangement.

The coil of the galvanometer is for convenience so wound, that when it is traversed by a positive current of electricity travelling *towards* London, or, to use railway language, "up" the line, the index moves in the right quadrant, or toward the word "up" engraved there; when, on the other hand, it is travelling *from* London, or "down" the line, the index moves to the left quadrant, or toward the word "down."

The following is a Table of Observations made during October 1861. They are taken in the order of the columns 1, 2, and 3. The letters *u* or *d* are entered against each observation, according as the index moves to the word "up" or "down." The time is taken at the end of the third observation. It is given in "Greenwich Mean Time," fractions of minutes being rejected; the local clock error being known by means of the "Time Signals" that come two or three times a day from Greenwich.

Table XIII.

Directions and Values of Earth-currents, collected at Tonbridge, 1861, October; from the London-Dover;—London-Tonbridge;—and Tonbridge-Dover lines.

Date.	Time.	Column 1. Dover-London ..... / London-Dover ..... s. d.	Column 2. Tonbridge-London ... / London-Tonbridge ... s. d.	Column 3. Dover-Tonbridge ..... / Tonbridge-Dover ..... s. d.	Date.	Time.	Column 1. Dover-London ..... / London-Dover ..... s. d.	Column 2. Tonbridge-London ... / London-Tonbridge ... s. d.	Column 3. Dover-Tonbridge ..... / Tonbridge-Dover ... s. d.
1861.	h m				1861.	h m			
Oct. 1.	9.31 A.M.	16 *u*	6 *d*	14 *u*	Oct. 4.	12.26 P.M.	40 *u*	55 *u*	50 *u*
	11.42 A.M.	15 *u*	35 *d*	30 *u*		4.50 P.M.	10 *d*	20 *d*	2 *d*
	2.20 P.M.	4 *u*	36 *d*		Oct. 5.	7.44 A.M.	35 *d*	50 *d*	45 *d*
	2.35 P.M.	3 *u*	28 *d*	7 *u*	Oct. 8.	11 A.M.	20 *d*	10 *d*	31 *d*
Oct. 2.	9.34 A.M.	9 *u*	4 *d*	11 *u*		12.56 P.M.	20 *u*	15 *u*	15 *u*
	12.24 P.M.	23 *u*	26 *d*	42 *u*		1.47 P.M.	45 *u*	10 *d*	50 *u*
	2.15 P.M.	17 *u*	35 *d*	20 *u*		3.12 P.M.	48 *u*	50 *d*	55 *u*
	2.47 P.M.	0	0	0	Oct. 9.	6.20 A.M.	12 *u*	20 *d*	25 *u*
	3.20 P.M.	6 *u*	28 *d*	18 *u*		9.20 A.M.	35 *d*	25 *u*	35 *d*
	3.35 P.M.	0	5 *d*	0	Oct. 10.	6.58 A.M.	10 *u*	15 *d*	20 *u*
Oct. 3.	7.2 A.M.	35 *d*	5 *d*	20 *d*		3.14 P.M.	30 *u*	50 *d*	50 *u*
	11.44 A.M.	32 *u*	14 *d*	38 *u*		10.0 P.M.	55 *d*	5 *u*	10 *d*
	5.20 P.M.	19 *d*	3 *d*	20 *d*	Oct. 11.	6.25 A.M.	55 *u*	50 *d*	55 *u*
	10.12 P.M.	20 *u*	0	20 *u*		9.6 A.M.	55 *d*	35 *u*	50 *d*
Oct. 4.	6.25 A.M.	33 *d*	3 *u*	45 *d*		10.25 A.M.	0	30 *u*	0
	12.24 P.M.	35 *u*	29 *d*	50 *u*	Oct. 12.	7.8 A.M.	5 *u*	40 *d*	40 *u*

TABLE XIII. (continued).

Date.	Time.	Column 1. Dover-London ...s. / London-Dover ...s.	Column 2. Tonbridge-London ...s. / London-Tonbridge ...s.	Column 3. Dover-Tonbridge ...s. / Tonbridge-Dover ...s.	Date.	Time.	Column 1. Dover-London ...s. / London-Dover ...s.	Column 2. Tonbridge-London ...s. / London-Tonbridge ...s.	Column 3. Dover-Tonbridge ...s. / Tonbridge-Dover ...s.
1861.	h m	s	s	s	1861.	h m	s	s	s
Oct. 12.	9.10 A.M.	20 *d*	20 *u*	40 *d*	Oct. 24.	7.41 A.M.	20 *d*	20 *u*	20 *u*
Oct. 14.	11.59 A.M.	10 *d*	10 *d*	10 *d*		7.47 A.M.	10 *d*	20 *d*	5 *u*
	12.52 P.M.	25 *u*	30 *d*	35 *u*		7.54 A.M.	15 *u*	22 *d*	35 *u*
	2.16 P.M.	10 *u*	12 *d*	11 *d*		9.20 A.M.	45 *d*	25 *u*	50 *d*
	2.45 P.M.	5 *u*	25 *d*	15 *u*		9.37 A.M.	55 *d*	45 *u*	55 *d*
Oct. 15.	11.34 A.M.	20 *u*	40 *d*	32 *u*		10.25 A.M.	30 *d*	15 *u*	43 *d*
	12.19 P.M.	18 *u*	16 *d*	25 *u*		11.24 A.M.	0	12 *d*	8 *u*
	1.23 P.M.	8 *u*	30 *d*	20 *u*		11.47 A.M.	13 *d*	25 *u*	38 *d*
	2.30 P.M.	2 *u*	15 *d*	7 *u*		12.29 P.M.	0	15 *d*	0
	3.25 P.M.	10 *d*	4 *u*	15 *d*		2.40 P.M.	22 *u*	55 *u*	15 *u*
	7.16 P.M.	20 *u* & *d*	5 *u*	40 to 60 *d*		2.44 P.M.	55 *u*	50 *u*	50 *u*
Oct. 16.	6.37 A.M.	10 *d*	5 *u*	35 *d*		3.32 P.M.	55 *u*	13 *d*	55 *u*
	7.44 A.M.	5 *d*	35 *u*	35 *d*		4.23 P.M.	7 *u*	35 *u*	0
	12.18 P.M.	40 *u*	50 *d*	45 *u*		5.3 P.M.	3 *d*	32 *u*	40 *d*
Oct. 17.	10.52 A.M.	23 *u*	10 *d*	30 *u*	Oct. 25.	6.14 A.M.	15 *d*	5 *u*	30 *d*
	11.25 A.M.	20 *u*	25 *d*	38 *u*		6.42 A.M.	50 *d*	20 *u*	55 *d*
	11.43 A.M.	26 *u*	26 *d*	40 *u*		10.21 A.M.	15 *d*	15 *u*	20 *d*
	2.13 P.M.	20 *u*	18 *d*	35 *u*		10.29 A.M.	40 *d*	10 *u*	33 *d*
	3.41 P.M.	10 *u*	20 *d*	15 *u*		11.10 A.M.	16 *d*	13 *u*	30 *d*
	5.6 P.M.	10 *d*	15 *d*	0		4.50 P.M.	20 *u*	20 *d*	30 *u*
Oct. 18.	6.35 A.M.	30 *d*	0	35 *d*		9.25 P.M.	25 *d*	15 *u*	40 *d*
	7.17 A.M.	35 *d*	25 *u*	45 *d*	Oct. 26.	6.10 A.M.	20 *u*	25 *d*	30 *u*
	8.36 A.M.	35 *d*	30 *u*	32 *d*		7.32 A.M.	20 *d*	35 *u*	20 *d*
	10.58 A.M.	14 *u*	10 *d*	28 *u*		7.44 A.M.	25 *d*	40 *u*	40 *d*
	11.37 A.M.	24 *u*	10 *d*	34 *u*		8.50 A.M.	0	5 *d*	0
	12.30 P.M.	40 *u*	30 *d*	48 *u*		10.54 A.M.	21 *d*	15 *u*	40 *d*
	12.57 P.M.	22 *u*	15 *d*	30 *u*		12.19 P.M.	0	10 *u*	0
	1.36 P.M.	20 *u*	25 *d*	40 *u*		12.57 P.M.	0	0	0
	2.44 P.M.	22 *u*	18 *d*	34 *u*		1.10 P.M.	20 *u*	20 *d*	26 *u*
	3.39 P.M.	15 *u*	12 *d*	16 *u*		1.26 P.M.	0	8 *u*	0
Oct. 19.	7.12 A.M.	25 *d*	10 *u*	38 *d*		4.1 P.M.	0	5 *d*	0
	12.1 P.M.	12 *u*	40 *d*	45 *u*		9.45 P.M.	0	10 *d*	0
Oct. 21.	7.7 A.M.	5 *d*	0	35 *d*		10.45 P.M.	10 *d*	10 *d*	10 *d*
	10.10 A.M.	28 *d*	15 *u*	32 *d*	Oct. 28.	7.15 A.M.	0	15 *u*	0
	1.24 P.M.	50 *u*	22 *d*			10.19 A.M.	0	20 *d*	8 *u*
	1.42 P.M.	50 *u*	10 *d*	50 *u*		10.35 A.M.	23 *d*	15 *u*	32 *d*
	2.21 P.M.	55 *u*	23 *d*	52 *u*		11.2 A.M.	5 *d*	24 *d*	0
	2.56 P.M.	43 *u*	15 *d*	47 *u*		11.18 A.M.	16 *d*	18 *d*	12 *d*
	3.35 P.M.	45 *u*	22 *d*	50 *u*		11.37 A.M.	20 *d*	10 *u*	35 *d*
Oct. 22.	6.15 A.M.	0	5 *d*	2 *u*		12.13 P.M.	20 *d*	4 *u*	27 *d*
	7.15 A.M.	5 *d*	0	25 *d*		12.57 P.M.	23 *d*	0	26 *d*
	7.57 P.M.	5 *d*	0	5 *d*		1.9 P.M.	30 *d*	14 *u*	40 *d*
Oct. 23.	6.13 A.M.	0	5 *u*	0		2.50 P.M.	10 *d*	3 *u*	14 *d*
	7.0 A.M.	5 *d*	0	0		3.21 P.M.	8 *u*	40 *d*	28 *u*
	7.31 A.M.	5 *d*	10 *u*	30 *d*					
	10.13 A.M.	18 *d*	35 *u*	32 *d*		10.10 P.M.	20 *u*	15 *d*	15 *u*
	1.1 P.M.	16 *u*	16 *d*	38 *u*		10.20 P.M.	5 *d*	10 *d*	10 *d*
	1.45 P.M.	40 *u*	0	45 *u*		10.30 P.M.	5 *d*	10 *d*	5 *d*
	3.9 P.M.	45 *u*	0	50 *u*		19.40 P.M.	5 *u*	25 *d*	20 *u*
Oct. 24.	6.20 A.M.	0	7 *d*	7 *u*		10.50 P.M.	15 *d*	5 *u*	5 to 20 *d*
	7.5 A.M.	0	0	2 *u*		11.0 P.M.	20 *d*	5 to 15 *u*	20 *d*

TABLE XIII. (continued).

Date.	Time.	Column 1. Dover-London ..... s. d. / London-Dover ..... s. d.	Column 2. Tonbridge-London ... s. d. / London-Tonbridge ... s. d.	Column 3. Dover-Tonbridge ..... s. d. / Tonbridge-Dover ..... s. d.	Date.	Time.	Column 1. Dover-London ..... s. d. / London-Dover ..... s. d.	Column 2. Tonbridge-London ... s. d. / London-Tonbridge ... s. d.	Column 3. Dover-Tonbridge ..... s. d. / Tonbridge-Dover ..... s. d.
1861.	h m				1861.	h m			
Oct. 28.	11.10 P.M.	0	10 *d*	3 *u*	Oct. 29.	4.43 A.M.	10 *d*	0 to 5 *u*	15 *d*
	11.20 P.M.	15 *u*	18 *d*	20 *u*		4.53 A.M.	10 *d*	0 to 5 *u*	15 *d*
	11.30 P.M.	5 *u*		5 *u*		5.0 A.M.	8 *d*	0 to 10 *u*	14 *d*
	11.40 P.M.	36 *d*	42 *u*	52 *d*		5.10 A.M.	8 *d*	8 *u*	18 *d*
	11.50 P.M.	25 *d*	22 *u*	35 *d*		5.20 A.M.	4 *d*	0 to 10 *u*	5 *d*
	12.0 P.M.	18 *d*	0	20 *d*		5.30 A.M.	0	0 to 15 *u*	0
Oct. 29.	12.10 A.M.	22 *d*	22 *d*	22 *d*		5.40 A.M.	4 *d*	0 to 4 *u*	5 *d*
	12.20 A.M.	10 *d*	0	15 *d*		5.50 A.M.	0	10 *u*	2 *d*
	12.30 A.M.	18 *d*	8 *d*	20 *d*		6.0 A.M.	0	5 *u* to 5 *d*	0
	12.40 A.M.	22 *d*	8 *u*	36 *d*		6.10 A.M.	0	0 to 5 *u*	0
	12.50 A.M.	10 *d*	7 *d*	13 *d*		6.20 A.M.	0	5 *d*	0
	1.0 A.M.	10 *d*	5 *d*	12 *d*		6.30 A.M.	0	0	0
	1.10 A.M.	15 *d*	6 *d*	20 *d*		6.40 A.M.	5 *u*	0	8 *u*
	1.20 A.M.	12 *d*	8 *d*	20 *d*		6.50 A.M.	5 *u*	10 *d*	25 *u*
	1.30 A.M.	18 *d*	6 *d*	20 *d*		7.7 A.M.	10 *u*	7 *u*	10 *u*
	1.40 A.M.	12 *d*	5 *d*	19 *d*					
	1.50 A.M.	15 *d*	5 *d*	21 *d*		7.30 A.M.	15 *u*	0	20 *u*
	2.0 A.M.	18 *d*	10 *u*	13 *d*		7.47 A.M.	15 *u*	0	15 *u*
	2.10 A.M.	18 *d*	0	20 *d*		10.5 A.M.	10 *u*	0 to 20 *d*	16 *u*
	2.20 A.M.	23 *d*	19 *u*	13 *d*		12.2 P.M.	8 *u*	40 *d*	28 *u*
	2.30 A.M.	18 *d*		20 *d*		12.25 P.M.	0	10 *d*	0
	2.40 A.M.	10 *d*	15 *u*	12 *d*		3.13 P.M.	10 *d*	12 *u*	16 *d*
	2.50 A.M.	5 *d*	0	10 *d*		3.29 P.M.	10 *d*	12 *u*	18 *d*
	3.0 A.M.	15 *d*	10 *d*	15 *d*	Oct. 30.	6.10 A.M.	0	10 *u*	10 *d*
	3.10 A.M.	13 *d*	5 *d*	19 *d*		6.30 A.M.	0	10 *u*	5 *d*
	3.20 A.M.	13 *d*	0	16 *d*		6.41 A.M.	0	20 *u*	8 *d*
	3.30 A.M.	15 *d*	0	20 *d*		7.42 A.M.	0	15 *u*	5 *d*
	3.40 A.M.	12 *d*	4 *d*	5 *d*		9.20 A.M.	17 *u*	1 to 10 *d*	20 *u*
	3.50 A.M.	10 *d*	0	12 *d*	Oct. 31.	6.12 A.M.	10 *d*	10 *u*	20 *d*
	4.0 A.M.	12 *d*	2 *d*	19 *d*		6.30 A.M.	0	10 *u*	0
	4.10 A.M.	14 *d*	10 *u*	20 *d*		9.43 A.M.	18 *u*	8 *u* to 12 *d*	21 *u*
	4.20 A.M.	10 *d*	5 *u*	20 *d*		11.59 A.M.	14 *u*	42 *d*	38 *u*
	4.30 A.M.	9 *d*	0 to 5 *u*	11 *d*		6.54 P.M.	0	0 to 15 *u*	4 *d*

Column 1, *u* is the direction 26—25 of Table XI.* and Plate III. figs. 1 & 5.

„ *d* „ 25—26 „ „ „

Column 2, *u* „ 24—23 „ „ „

„ *d* „ 23—24 „ „ „

Column 3, *u* „ 32—31 „ „ fig. 1.

„ *d* „ 31—32 „ „ „

When column 1 contains an entry of an "up" current *u*, and column 2 of a "down"

* Philosophical Transactions, 1861, p. 130.

current *d*, we have the result of which a few cases were collected and given in Table XII. section 3, and shown graphically in Plate III. fig. 5. The result is further confirmed when the entry in column 3 corresponds in direction with that in column 1.

When the directions recorded in columns 1, 2, and 3 are *d*, *u*, and *d* respectively, we have the results given in Table XII. section 4.

As I shall have occasion to refer incidentally to the possible influence of heat or cold, &c. over the relative values of the currents registered in the three columns of Table XIII., I have given in Table XIV. the Meteorological Register taken at Tonbridge, and kindly furnished to me by Dr. FIELDING. The barometer readings are corrected and reduced to sea-level at 32°.

TABLE XIV.—Meteorological Register taken at Tonbridge, October 1861.

	Barometer, 9 A.M.	Degree of moisture, 9 A.M.	Shade, maximum.	Shade, minimum.	Mean temperature.	Wind at 9 A.M.	Pluviometer, 9 A.M.
	in.	°	°	°	°		
October 1.	29·827	94	70·5	55·3	62·90	S.E.	0·015
2.	29·967	88	65	48	56·50	N.	0·080
3.	30·217	93	66	52	59	N.	0·010
4.	30·223	79	65·8	48	56·75	S.E.	0·005
5.	30·089	96	66·8	55·5	61·15	S.	0·000
6.	30·227	91	61	57	59	E.	0·010
7.	30·153	97	69·8	53·5	61·55	N.E.	0·080
8.	29·975	97	71·6	57	64·30	N.	0·010
9.	29·973	94	65	48	56·50	S.W.	0·040
10.	30·045	96	67·5	51·5	59·50	N.E.	0·030
11.	29·559	88	69	49·8	59·40	S.S.W.	0·070
12.	30·039	79	62	55	58·50	S.W.	0·320
13.	30·105	79	65·8	55·6	60·70	S.W.	0·005
14.	30·133	92	70·4	47·2	58·80	E.	0·000
15.	30·181	81	68·4	48·2	58·30	E.	0·000
16.	30·265	90	57	43·2	50·10	N.	0·000
17.	30·351	74	58·6	49·2	53·90	E.	0·000
18.	30·221	80	59	42	50·50	S.E.	0·000
19.	30·065	93	60	43	51·50	E.	0·000
20.	29·933	96	62	48	55	E.	0·000
21.	29·863	80	59	50·5	54·75	E.	0·020
22.	29·881	93	60·3	48·5	54·40	S.E.	0·030
23.	30·103	89	57·4	47	52·20	S.W.	0·310
24.	30·135	84	61·3	56	58·65	S.	0·010
25.	30·167	91	62	50·5	56·25	W.S.W.	0·030
26.	30·217	84	57	43	50	N.E.	0·000
27.	30·143	80	56	46·5	51·25	E.	0·000
28.	30·143	72	53·4	43	48·20	E.	0·000
29.	30·065	74	52·2	43	47·60	E.	0·000
30.	30·983	76	53·3	43	48·15	N.	0·015
31.	30·921	89	53	44	48·50	N.	0·000

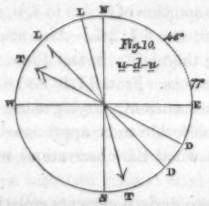
In Table XV. is given an analysis of the contents of Table XIII. Observations of a like character (as well as others to be hereafter referred to) were taken during the month of November. I have not thought it necessary to give them in detail, but have included a summary of them in this Table together with those of October.

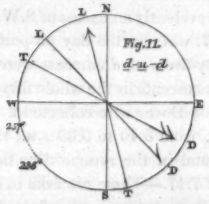
TABLE XV.—Analysis of Observations of Earth-Currents collected 1861, October and November.

	Col. 1.	Col. 2.	Col. 3.	October.	November.		
	0	0	0	3	2	5	
	u	*d*	*u*	78	37	115	
	d	*u*	*d*	83	27	110	
Normal	.....	.....	.....	164	66		230
	u	*u*	*u*	1	10	11	
	d	*d*	*d*	25	6	31	
Abnormal	.....	.....	.....	26	16		42
	u	*d*	*d*	1		1	
	d	*u*	*u*	1		1	
	d	*d*	*u*	1		1	
	u	*u*	*d*		1	1	
Exceptional	.....	.....	.....	3	1		4
							276

In October there are three cases (000), and in November two, in which no deflection of the needle occurred in either of the three observations; and many individual cases are recorded. No particular stress is laid on these cases of no action. They merely indicate that the current, if any, was too small to affect the particular instrument used; a more delicate instrument might doubtless have given signs.

Of the 276 complete observations, 230, or five-sixths, are in accordance with the conclusions already arrived at—that the general direction of the drift of earth-currents is approximately N.E. or S.W. And the numbers of each kind come out nearly the same: 115 N.E. *u*, *d*, *u*;—and 110 S.W. *d*, *u*, *d*. The conditions of this group of results, which for convenience may be called *normal*, given graphically in Plate III. fig. 5, may be gathered in detail more readily from figs. 10 and 11.





N is the magnetic north; L—D, London-Dover; L—T, London-Tonbridge; T—D, Tonbridge-Dover. The respective lines of direction are further shown by the arrow-

heads,—one semi-barb indicating column 1; two, column 2; three, column 3. The arrow-heads pointing upwards all apply to "up" currents *u*; those pointing downwards, to "down" currents *d*. The dotted portion of the circumference of the circle is the arc of the horizon, within which the resultant is to be found. The degrees are given reckoned from the north, eastward round the circle. Fig. 10 represents the N.E. *normal*, or *u*, *d*, *u*, in Tables XIII. and XV.; fig. 11 the S.W. *normal*, or *d*, *u*, *d*.

From these Tables it appears that the *prevailing* currents, or those of most frequency, are from the N.E. or S.W.; this as well in calm periods as in periods of magnetic storms. In the absence of long-continued and consecutive observations, it is not easy to form an opinion as to whether the N.E. or S.W. currents prevail more or less at one part of the day than at another; or to what extent, if at all, the directions or alternations are influenced by local meteorological changes or conditions. In 'Les Archives des Sciences Physiques,' vol. xi. pp. 110–136, is a memoir by Father Secchi, "On the Connexion of Meteorological Phenomena and Variations of the Intensity of Terrestrial Magnetism," in which he expresses very strongly his opinion that every rupture of meteorological equilibrium produces a rupture of electrical equilibrium, which can only be re-established by means of a current which discharges itself from place to place, which current cannot fail to act upon the magnetometers. Here is a wide field for research. Although I am not engaged in investigating the *origin* of the currents, I cannot avoid expressing my opinion that the value of existing currents, if not their direction (I speak locally), may be more or less influenced by meteorological changes, especially cloud, sunshine, or temperature. The currents at calm periods are at best but feeble. The resistances of the various parts of the telegraph wire through which they pass vary with the varying temperatures, so that it is quite reasonable to expect that, even when no change is taking place in the absolute value of the current travelling in the earth, the needle of the galvanometer may move forward or backward according as sunshine or cloud, heat or cold prevail here or there in the district under examination. This opinion is sanctioned in some degree by the result of some night observations made on October 28—29. My original observations were almost wholly made by day. On the night in question observations were made every ten minutes from 10.10 P.M. to 6.50 A.M. During this period there was evidently an excess of S.W. currents; the proportion of *u*, *d*, *u* to *d*, *u*, *d* currents was 1 : 2·7, whereas the day proportion for the month was 3·7 : 2·7. Also fifteen out of the twenty-five *d*, *d*, *d* currents were collected during the night; in fact there were only ten *u*, *d*, *u* currents in the whole fifty-six night observations. From 11.40 P.M. to 5.40 A.M. the London-Dover wire collected a continuous down *d* current, varying more or less in intensity; from 5.40 to 6.30 A.M., the current was too feeble to be appreciated; at 6.40 it was found in the reverse direction *u*, and was so when the observations were interrupted at 7.47.—These remarks in passing.

There is no consistency in the relations between two derived currents collected at the same time from the same earth-drift of electricity: this may be in great measure due, as I have hinted, to local meteorological interference, if not to the absolute differences

of value in the different sections of the drift itself. I have taken at random, from the October observations, a few cases for illustration:—

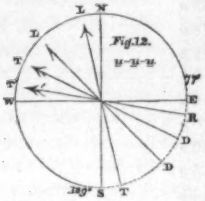
13 *u*	38 *u*
14 *u*	28 *u*
„	31 *u*
15 *u*	15 *u*
„	16 *u*
„	20 *u*
„	30 *u*
„	35 *u*
16 *u*	14 *u*
„	38 *u*
17 *u*	20 *u*
18 *u*	21 *u*
„	25 *u*

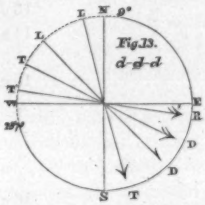
The first column are Dover-London currents; the second, Dover-Tonbridge. Being read off on the same instrument, and under circumstances so favourable, they are strictly comparable. There are no relative values to be traced. For instance, 13° corresponds with 38°; 18° with 21° and 25°; 15° is coupled with various values—15°, 16°, 20°, 30°, and 35°, and so on. These illustrations may be extended at pleasure.

Since my original communication to the Royal Society, an "eighth article," by Professor Loomis, has appeared in the 'American Journal,' vol. xxxii., November 1861. On discussing the results accumulated in America, he infers that all the facts are consistent with the supposition of electric currents moving to and fro on the earth's surface, the average direction of which, on that continent, is from about N. 45° E. to S. 45° W., a result remarkably in accordance with the conclusions to which we have arrived by a somewhat different process for the S.E. part of England. He has also discussed, in quite another way, the magnetic disturbances in Europe; and he obtains a direction for the electric wave, connected with those disturbances, from N. 28° E. to S. 28° W. over the surface of Europe. These approximations are noteworthy.

It is plain, however, from Tables XIII. and XV., that currents from time to time flow from some point in the S.E. and in the N.W. quadrants. These directions may for the present be called *abnormal*. In the existing state of our knowledge, it is impossible to say whether they indicate the state of *transition* between the two *normal* quadrants. The cases are few in number. In October, 26 were recorded; in November, 16; or $\frac{3}{13}$ of the whole number. In December also I noticed a few. Like the *normal*, they are subdivided into two classes: *u*, *u*, *u*, 11; *d*, *d*, *d*, 31. The current was constant in the *d*, *d*, *d* direction, with a single interruption, from midnight October 28 to 1.50 A.M. October 29. Currents of this kind, as may be seen in Table XIII., are as definitely

marked as the others; their values are equally large. They are expressed graphically in figs. 12 and 13.





The references and letters are the same as in figs. 10 and 11. The lines of direction are not favourable for arriving at an approximate place for the resultant. We have to seek it within the large arc of 131°, which includes respectively the whole of the S.E. or N.W. quadrant. I have the command at Tonbridge of the RamsgateHarbour-Tonbridge line, to which I shall have occasion hereafter to refer. It corresponds nearly with the Shalford-Red Hill, 34—33 of Table XI. and Plate III. fig. 1; it makes an angle with the magnetic meridian of 99°, reckoned eastward. Many observations on this line are given in Table XVI. column 3, and in all cases will be found to coincide in direction with those on the Tonbridge-Dover line. I have therefore been able to lay down this line in figs. 12 and 13. It is indicated by the letters R T', and by three semibarbs. It reduces the arc to 112°, which, however, is still large, and which I cannot further reduce by the means at my command.

Since I wrote the note (1861, July) which appears in the 'Philosophical Transactions' for that year, Part I. p. 96, the Astronomer Royal has laid his proposition for erecting earth-current wires before the Board of Directors of the South-Eastern Railway, which has been referred to me; and it is needless to say has had my favourable report. The Directors have entertained the proposition most cordially; and have approved of the erection of the wires at cost price, and conceded the right of way and maintenance on the payment for each of a nominal sum of a few shillings annually.

The Greenwich-Dartford wire will make an angle of about 60° W. of N. (magnetic), nearly coincident with my Tonbridge-Ashford, or with the Tonbridge-Dover line of the present communication. The Greenwich-Croydon wire will make an angle of about 47° E. of N., or not far from the direction of my Ashford-Hastings line. By combining the wires at Greenwich, the Dartford-Croydon line may be obtained, which is 84° E. of N., or nearly my Ramsgate-Ashford direction. Treating these lines in the usual way, we shall have an arc of 36° or 107°. The former will be between N. and N.E., and out of the range. The latter, which however is very large, will include the range. I look forward with great interest to the completion of these wires, in order that we may

see the result of uninterrupted observations, which cannot be obtained from our highly occupied telegraph wires.

I purpose erecting these wires as soon as the insulators are made; which will be constructed with great care, of a double concentric cup of ebonite, with an outer cup of French porcelain. The ebonite cups will be turned in a lathe before fitting, so as to present a perfectly smooth surface. They will be placed on the apex of our telegraph poles*.

Four *exceptional cases* presented themselves among the 276 observations. Their character is given in Table XV. They are beyond the reach of the system of analysis we have adopted. Whether the general direction of the current-drift was changing at the moment of observation, and was complete in one part of the district and not so in the other, I cannot, with these very rare and isolated data, pretend to say. The results are given precisely as read off. There was no reason to suspect any interferences from artificial currents; a few more cases may throw more light on this exceptional class.

A few entries will be found in the Table XIII. as thus: 5° to 15°; 0° to 10°; 5° *u* to 5° *d*, &c. The derived currents in these cases were unstable. A few of the earlier cases that occurred were questioned, and I was disposed to reject them, the impression being that they were due to interference from strong telegraph currents entering into the observing-wire. But instances occurred in which there was no reason to suspect interferences, they are therefore placed on record.

We speak of electric *currents* in this inquiry; the word conveys the idea of length without width. The currents in question necessarily and evidently cover large areas, presenting as it were an *electric plane*. Passing on from the determination of mere direction, I was able to survey the two sides of the same plane. By reference to the map (vol. 159, Plate II.), the RamsgateHarbour-London and the Dover-Tonbridge lines are not many degrees from being parallel. They are about 20 miles apart; the former is 67 miles, and the latter 45 miles in length. I have at Ashford junction a turn-plate or switch. When desiring to make the observations on the Ramsgate-London line, I call Ashford and give the word "branch;" the reply is "yes" or "no," according as it is at liberty or not. If at liberty, the switch is turned, and I have the command of the wire from Ramsgate to London, the telegraph length of which is 97½ miles; and then, by placing the plug in the hole 3 of fig. 9, for reasons already explained, the command is obtained of the Ramsgate-Tonbridge line.

Observations of this kind have been made from time to time; the results are given in Table XVI. The letters in this and subsequent Tables have the same references as those in Table XIII.

* I completed the two wires, and handed them over to the Astronomer Royal, June 30, 1862.

TABLE XVI.—Directions and Values of Earth-Currents collected at Tonbridge, 1861, November and December; from the Tonbridge-Dover, London-Ramsgate, and Tonbridge-Ramsgate lines.

Date.	Time.	Column 1. Dover-Tonbridge ..... s. d. Tonbridge-Dover ..... s. d.	Column 2. Ramsgate-London ..... s. d. London-Ramsgate ..... s. d.	Column 3. Ramsgate-Tonbridge, s. d. Tonbridge-Ramsgate, d.	Date.	Time.	Column 1. Dover-Tonbridge ..... s. d. Tonbridge-Dover ..... s. d.	Column 2. Ramsgate-London ..... s. d. London-Ramsgate ..... s. d.	Column 3. Ramsgate-Tonbridge, s. d. Tonbridge-Ramsgate, d.
1861.	h m				1861.	h m			
Nov. 17.	12.35 P.M.	28 *u*	10 *u*	14 *u*	Dec. 3.	6.13 A.M.	18 *d*	9 *d*	9 *d*
Nov. 20.	7.34 A.M.	25 *d*	15 *d*	20 *d*	Dec. 4.	6.20 A.M.	0	5 *d*	0
Nov. 21.	7.7 A.M.	0	0	0		1.25 P.M.	40 *u*	27 *u*	32 *u*
Nov. 22.	7.1 A.M.	15 *u*	12 *u*	0	Dec. 6.	6.39 A.M.	5 *u*	0	0
Nov. 25.	3.25 P.M.	14 *u*	0	7 *u*	Dec. 7.	6.24 A.M.	25 *u*	35 *u*	42 *u*
Nov. 26.	6.35 A.M.	35 *u*	20 *u*	15 *u*	Dec. 9.	7.18 A.M.	20 *u*	0	0
	3.24 P.M.	10 *u*	15 *u*	15 *u*	Dec. 13.	6.49 A.M.	10 *u*	20 *u*	20 *u*
Nov. 27.	6.19 A.M.	10 *u*	10 *u*	15 *u*	Dec. 14.	6.35 A.M.	15 *d*	10 *d*	20 *d*
	11.47 A.M.	15 *d*	11 *d*	10 *d*	Dec. 17.	6.30 A.M.	10 *d*	5 *d*	5 *d*
	12.32 P.M.	16 *d*	0	0	Dec. 19.	6.23 A.M.	0	15 *u*	20 *u*
	2.18 P.M.	28 *d*	8 *d*	20 *d*		8.18 P.M.	55 *d*	55 *d*	55 *d*
	2.38 P.M.	26 *d*	24 *d*	29 *d*	Dec. 20.	6.30 A.M.	5 *u*(?)	15 *d*	10 *d*
Nov. 28.	6.20 A.M.	0	0	2 *u*	Dec. 21.	6.24 A.M.	10 *u*	20 *u*	30 *u*
Nov. 29.	6.33 A.M.	5 *d*	2 *d*	0	Dec. 26.	6.34 A.M.	25 *u*	20 *u*	25 *u*
	1.13 P.M.	17 *d*	23 *d*	35 *d*	Dec. 27.	6.35 A.M.	15 *u*	15 *u*	20 *u*
Nov. 30.	6.22 A.M.	10 *d*	0	5 *d*	Dec. 28.	6.23 A.M.	0	2 *u*	2 *u*
Dec. 2.	2.27 P.M.	22 *u*	7 *d*	5 *d*	Dec. 31.	6.26 A.M.	30 *d*	25 *d*	30 *d*

Column 2 gives the Ramsgate-London results, or the survey of the north side of the parallelogram; and column 1 the Dover-Tonbridge results, or survey of the south side of the parallelogram. In every instance, with a solitary exception during the two months of observation, the directions coincide; the current or drift or electric plane is at least 20 miles wide, and the behaviour of its two limits is consistent. The proportion between the values of the currents on the two sides of the plane is not constant, but is a little better maintained than that of Table XIII. before referred to. The Ramsgate-London, or 67-mile line collected by 97½ miles of telegraph wire, gives in the majority of cases a less value than the Dover-Tonbridge, or 45-mile line, collected in 46¾ miles of wire.

Column 3 of Table XVI., already noticed, is the Ramsgate-Tonbridge line. It makes a diagonal across the plane. The directions in all cases coincide with those of the other two lines, and so give a further evidence of consistency.

Tonbridge is almost in a direct line between London and Hastings, and very nearly equidistant. Lines 21—22, and 23—24, Plate III. fig. 1, give the true readings. I have a switch or turn-plate in the Telegraph Office at the Tonbridge junction, by means of which the Tonbridge-Hastings wire can be placed at my request in connexion with the

Tonbridge-London wire. The direct line between London and Hastings is 53 miles. I have thus an opportunity of making observations on the whole of this line, or on either half, the direction of all three being the same. The results of these observations are given in Table XVII. Column 1 contains the value of currents collected on the whole line of 53 miles; column 2, those on the London half of 27 miles; and column 3, those on the Hastings half of 26 miles.

TABLE XVII.—Directions and Values of Earth-Currents collected at Tonbridge, 1861, November and December; from the London-Hastings, London-Tonbridge, and Tonbridge-Hastings lines.

Date.	Time.	Column 1. Hastings-London ..... *u*. London-Hastings ..... *d*.	Column 2. Tonbridge-London ... *u*. London-Tonbridge ... *d*.	Column 3. Hastings-Tonbridge... *u*. Tonbridge-Hastings... *d*.	Date.	Time.	Column 1. Hastings-London ..... *u*. London-Hastings ..... *d*.	Column 2. Tonbridge-London ... *u*. London-Tonbridge ... *d*.	Column 3. Hastings-Tonbridge... *u*. Tonbridge-Hastings... *d*.
1861.	h m				1861.	h m			
Nov. 15.	12.34 P.M.	8 *d*	18 *d*	1 *d*	Nov. 29.	6.24 A.M.	2 *u*	5 *u*	0
	1.56 P.M.	8 *d*	24 *d*		Nov. 30.	6.17 A.M.	5 *u*	10 *u*	0
Nov. 16.	10.19 A.M.	4 *u*	44 *u*	6 *u*	Dec. 2.	11.48 A.M.	0	8 *d*	0
	10.53 A.M.	20 *d*	50 *d*	0	Dec. 3.	6.13 A.M.	5 *d*	0	5 *d*
Nov. 17.	12.35 P.M.	5 *u*	10 *u*	4 *u*	Dec. 4.	6.17 A.M.	10 *d*	15 *d*	5 *d*
Nov. 20.	7.37 A.M.	5 *d*	10 *d*	0		1.29 P.M.	0	16 *d*	1 *d*
	10.50 A.M.	7 *u*	8 *u*	4 *u*	Dec. 5.	6.24 A.M.	5 *d*	0	5 *d*
Nov. 21.	7.10 A.M.	0	5 *d*	0	Dec. 6.	6.37 A.M.	0		0
	9.35 A.M.	12 *u*	18 *u*	0	Dec. 7.	6.22 A.M.	0	5 *d*	0
	11.46 A.M.	4 *u*	6 *u*	3 *u*	Dec. 9.	7.16 A.M.	5 *d*	0	5 *d*
	12.49 P.M.	0	0	5 *u*	Dec. 13.	6.43 A.M.	0	0	0
Nov. 23.	6.22 A.M.	5 *d*	10 *d*	0	Dec. 14.	6.33 A.M.	5 *u*	10 *u*	0
	9.1 A.M.	0	10 *u*	0	Dec. 17.	6.29 A.M.	0	5 *d*	0
Nov. 25.	1.34 P.M.	6 *u*	12 *u*	4 *u*	Dec. 18.	6.20 A.M.	5 *d*	10 *d*	5 *d*
	2.47 P.M.	8 *u*	5 *u*	9 *u*	Dec. 19.	6.21 A.M.	5 *d*	10 *d*	5 *d*
	3.24 P.M.	8 *u*	23 *u*	7 *u*	Dec. 20.	6.20 A.M.	10 *d*	25 *d*	5 *d*
Nov. 26.	6.34 A.M.	0	10 *d*	0	Dec. 21.	6.22 A.M.	15 *d*	28 *d*	10 *d*
Nov. 27.	6.17 A.M.	0	0	0	Dec. 26.	6.30 A.M.	0	10 *d*	0
	12.32 P.M.	7 *d*	5 *d*	5 *d*	Dec. 27.	6.34 A.M.	0	0	0
	2.18 P.M.	0	3 *u*	0	Dec. 28.	6.21 A.M.	0	0	0
Nov. 28.	6.17 A.M.	0	5 *u*	2 *d*(?)	Dec. 31.	6.23 A.M.	0	0	0

If the value of these derived currents depended simply on the mere distance between the earth-plates or observing-stations, and their bearing each on the other, it is obvious that the values in columns 2 and 3 would be identical, or in this case nearly so. The London-Tonbridge wire-length is 41 miles; the Tonbridge-Hastings 33; so that the value on the latter length should be a little higher if anything, the resistance being less. But, with very rare exceptions, the Tonbridge-Hastings values are seen to be greatly below the Tonbridge-London. The contrast is remarkable. In some cases the differences are very conspicuous. I have made a sufficient number of observations, extended

over two months, to satisfy myself that the one section is under all circumstances less active in derived currents than the other. This difference can only be attributed, as already suggested in my former communication (p. 109), to the different geological conditions of the two sections of country, a difference which may operate in two ways: either the resistance of the section may be relatively great, so that the earth-plates penetrate into a portion of the electric plane that is traversed by a current of low value, and hence the derived current is comparatively low; or the resistance of the section may be relatively small, so that, although the earth-plate may penetrate into a portion of the electric plane that is traversed by a current of higher value, yet the wire resistance, in contrast with the high conducting power of the earth section, may cause the derived current to have a relatively low value.

The London-Hastings line makes an angle of possibly 70° with the N.E. or S.W. resultant, that is with the *u*, *d*, *u* or the *d*, *u*, *d* currents; so that even with a good geological section and with a perfect knowledge of the relative resistances of its various parts, it would be no easy matter even to hint at the precise relation between the value of the current and the structure and arrangement of the strata. Mr. Robert Hunt has kindly furnished me with a geological section of the country between London and Hastings drawn by Mr. F. Drew. Between London and Tonbridge are included, London clay; Woolwich beds (sand); chalk; gault (clay); lower greensand, sand and a little limestone; clay; sand and sandstone; clay. Between Tonbridge and Hastings, sand and sandstone; clay and a little limestone. There are many faults also between Tonbridge and Hastings.

Column I of Table XVII. contains the values given by the whole length, from London to Hastings. They differ but little, save in one or two instances, from the Tonbridge-Hastings values; and are consequently very low in comparison with the Tonbridge-London values. These facts all indicate the very notable influence of local conditions, other than the meteorological variation already noted, over the relative value of the current in different parts of the plane.

It was a matter of considerable importance to determine with certainty whether the currents with which I was dealing were in whole, or only in part, earth-currents; whether, that is, any portion of the observed deflections of the galvanometer needle were due to electricity collected from the atmosphere by the suspended telegraph wires that were employed in these observations. To determine this I availed myself of the assistance of Ashford. After taking observations in the usual way between London and Dover, and between Tonbridge and Dover, I desired the Ashford clerk to detach the wire from his instrument and leave the end insulated. I thus had a wire of 67 miles connected with the earth at London, or one of 25 miles connected with the earth at Tonbridge, the end in either case being insulated at Ashford. It was desirable to make a considerable number of observations, at various hours of the day and under all conditions of weather, in order to test this question rigidly. The results are given in Table XVIII.

TABLE XVIII.—Observations showing, by detaching one end of the Telegraph wire from the earth, that currents collected at Tonbridge, 1861, October, November, and December, from the London-Dover and Tonbridge-Dover lines, were true and proper Earth-currents.

Date.	Time.	Column 1. Dover-London ..... *u*. / London-Dover ..... *d*.	Column 2. Dover-Tonbridge ..... *u*. / Tonbridge-Dover ..... *d*.	Column 3. Columns 1 and 2 with wire off earth at Ashford.	Date.	Time.	Column 1. Dover-London ..... *u*. / London-Dover ..... *d*.	Column 2. Dover-Tonbridge ..... *u*. / Tonbridge-Dover ..... *d*.	Column 3. Columns 1 and 2 with wire off earth at Ashford.
1861.	h m	°	°	°	1861.	h m	°	°	°
Oct. 18.	11.37 A.M.	24 *u*	34 *u*	0	Nov. 27.	11.47 A.M.	10 *d*	15 *d*	0
	1.36 P.M.	20 *u*	40 *u*	0		2.18 P.M.	17 *d*	28 *d*	0
	2.44 P.M.	22 *u*	34 *u*	0		2.38 P.M.	18 *d*	26 *d*	0
	3.39 P.M.	15 *u*	16 *u*	0	Nov. 28.	6.15 A.M.	0	0	0
Oct. 21.	10.10 A.M.	28 *d*	32 *d*	0	Nov. 29.	6.20 A.M.	0	5 *d*	0
Oct. 24.	11.47 A.M.	13 *d*	38 *d*	0	Nov. 30.	6.21 A.M.	5 *d*	10 *d*	0
	2.44 P.M.	55 *u*	50 *u*	0	Dec. 2.	11.49 A.M.	23 *u*	34 *u*	0
	4.23 P.M.	7 *u*	0	0		2.24 P.M.	15 *u*	22 *u*	0
Oct. 28.	11.18 A.M.	16 *d*	12 *d*	0	Dec. 3.	6.24 A.M.	15 *d*	18 *d*	0
	10.30 P.M.	5 *d*	5 *d*	0	Dec. 4.	6.15 A.M.	5 *d*	0	0
	11.0 P.M.	20 *d*	20 *d*	0		12.9 P.M.	20 *u*	13 *u*	0
	11.30 P.M.	5 *u*	5 *u*	0		1.24 P.M.	34 *u*	40 *u*	0
	12.0 P.M.	18 *d*	20 *d*	0		2.40 P.M.	20 *u*	16 *u*	0
Oct. 29.	0.30 A.M.	18 *d*	20 *d*	0	Dec. 5.	6.20 A.M.	0	0	0
	1.0 A.M.	10 *d*	12 *d*	0		12.34 P.M.	17 *u*	40 *u*	0
	1.30 A.M.	18 *d*	20 *d*	0	Dec. 6.	6.20 A.M.	0	0	0
	2.0 A.M.	18 *d*	13 *d*	0	Dec. 7.	6.20 A.M.	16 *u*	25 *u*	0
	2.30 A.M.	18 *d*	20 *d*	0	Dec. 9.	7.12 A.M.	15 *u*	20 *u*	0
	3.0 A.M.	15 *d*	15 *d*	0	Dec. 13.	6.40 A.M.	10 *u*	10 *u*	0
	3.30 A.M.	15 *d*	20 *d*	0	Dec. 14.	6.30 A.M.	5 *d*	15 *d*	0
	4.0 A.M.	12 *d*	19 *d*	0	Dec. 17.	6.17 A.M.	10 *d*	10 *d*	0
	4.30 A.M.	9 *d*	11 *d*	0	Dec. 18.	6.18 A.M.	15 *d*	10 *d*	0
	5.0 A.M.	8 *d*	14 *d*	0	Dec. 19.	6.20 A.M.	0	0	0
	5.30 A.M.	0	0	0	Dec. 20.	6.20 A.M.	0	5 *u*	0
	6.0 A.M.	0	0	0	Dec. 21.	6.18 A.M.	0	10 *u*	0
	6.30 A.M.	0	0	0	Dec. 26.	6.25 A.M.	20 *u*	25 *u*	0
Nov. 21.	11.46 A.M.	10 *d*	20 *d*	0	Dec. 27.	6.30 A.M.	10 *u*	15 *u*	0
Nov. 25.	1.31 P.M.	0	0	0	Dec. 28.	6.17 A.M.	0	0	0
	3.23 P.M.	16 *u*	14 *u*	0	Dec. 31.	6.19 A.M.	25 *d*	30 *d*	0
Nov. 26.	3.24 P.M.	5 *u*	10 *u*	0					

It will be seen that in no single instance was any deflection of the needle obtained when the wire was off at Ashford; so that we are right in regarding all the currents with which we have been dealing, as far as atmospheric electricity is concerned, as true and proper earth-currents.

Columns 1 and 2 are the deflections obtained on the London-Dover and Tonbridge-Dover lines in the usual way. Column 3 are the 0° resulting from repeating each of the two previous observations with the Ashford wire detached.

I have also made repeated observations on the effect of polarizing the earth-plates by

the passage of a powerful current; and all with the same negative result. An observation is made, say on the Tonbridge-London line, and the deflection noted. A powerful current from the ordinary telegraph battery (a current which it would be imprudent to receive on the observing galvanometer) is transmitted for a given time, say half a minute, through the Tonbridge-London circuit, in the direction which would have produced the deflection previously observed. After this the observation is repeated. No appreciable difference has in any instance been found in the deflection. The direction remains the same; and the value unaltered. So that the currents, as far as the polarizing of the plates is concerned, may be regarded as true and proper earth-currents.

The alterations in direction of the currents and the varying values in either direction indicate that they are in nowise due to the mere electromotive force of the earth-plates employed.

I have mentioned in my former communication (pp. 94, 95) that the connexion with the earth is very frequently made by means of the fish-jointed rails. Some misgivings might arise on a first glance as to the influence of this arrangement over the results. The current collected under such circumstances, however, would be true in direction, but reduced in value. We need not enter into the discussion of this question from the fact that no such earth-connexion has been required for our present purposes.

To prevent misapprehension, I have thought it better to give a list of the earth-connexion employed at all the stations concerned in this investigation.

London	Gas-pipes; and a lightning conductor terminating in a wet pit of coke.
Tonbridge	Gas- and water-pipes.
Hastings	Gas- and water-pipes.
Ramsgate Harbour . . .	Water-pipes.
Dover	Gas-pipes.

I still continue taking observations; but they are all of the same character as those now placed on record, and merely afford further illustrations of the points that have been discussed. It would be premature to say that the subject is tolerably exhausted, as far as the means at my command are concerned; but I do not at this moment see any salient point that is within my reach. The steady daily and hourly march of these phenomena, and their relation, if any, to the like march of magnetometers, will soon be within the reach of Mr. AIRY; and we may be well assured that these and the other collateral questions will be ably discussed by him.

The results arrived at in these two communications may be briefly summed up as follows:—

1st. That currents of electricity are at all times moving in definite directions in the earth.

2nd. That their direction is not determined by local causes.

3rd. That there is no apparent difference, except in degree, between the currents

collected in times of great magnetic disturbance, and those collected during the ordinary calm periods.

4th. That the prevailing directions of earth-currents, or the currents of most frequent occurrence, are approximately N.E. and S.W. respectively.

5th. That there is no marked difference in frequency, duration, or value, between the N.E. and the S.W. currents.

6th. That (at least during calm periods) there are definite currents of less frequency from some place in the S.E. and N.W. quadrants respectively.

7th. That the direction of the current in one part of a plane on the earth's surface (at least as far as the S.E. district of England is concerned) coincides with the direction in another part of the plane; and if the direction changes in one part, it changes in all parts of the plane.

8th. That the relation in value between currents in a given part of the plane and currents in another given part is not constant, but is influenced by local meteorological conditions, and varies from time to time.

9th. That the value of the current of a given length, moving in a given line of direction, is not necessarily the same as that of a current of the same length on the same line of direction produced, and that their relative value depends on the physical character of the earth interposed between the respective points of observation, and is tolerably constant.

10th. That the currents which have formed the bases of these investigations are derived currents from true and proper earth-currents, and neither in whole nor in any appreciable part have been collected from the atmosphere, nor are due either in whole or in any appreciable part to polarization imparted to earth-plates by the previous passage of earth-currents or of powerful telegraphic currents; nor are they due to any electromotive force in the earth-plates themselves.

11th. That the earth-currents in question (at least the powerful currents present at all times of great magnetic disturbance) exercise a *direct* action upon magnetometers, just as artificial currents confined to a wire exercise a direct action upon a magnet.

XI. *On the Spectrum of Carbon.* *By* JOHN ATTFIELD, *Esq., F.C.S., Director of the Laboratory of the Pharmaceutical Society; lately Demonstrator of Chemistry at St. Bartholomew's Hospital. Communicated by Dr.* FRANKLAND, *F.R.S.*

Received June 19,—Read June 19, 1862.

IT is well known that a mixture of coal-gas and air burns with a flame of slight luminosity. When such a flame is prismatically examined under favourable circumstances, as by the ordinary spectroscope, the light it emits is found to consist of four groups of rays of different refrangibility. These rays appear in the field of the instrument as faint yellow, light green, bright blue, and rich violet bands of light.

In 1856 SWAN* found that the spectrum thus obtained was common to all hydrocarbon flames. He showed that they were best seen in an olefiant gas-flame fed with air by a blowpipe jet, measured and recorded their distances from each other, searched for, but did not find, corresponding dark bands in the solar spectrum, and gave no theory in explanation of their origin.

On recently reading SWAN'S paper by the light that Professors BUNSEN and KIRCHHOFF have thrown on the subject, I came to the conclusion that these bands must be due to incandescent carbon vapour; that, if so, they must be absent from flames in which carbon is absent, and present in flames in which carbon is present; that they must be observable equally in the flames of the oxide, sulphide, and nitride as in that of the hydride of carbon; and, finally, that they must be present whether the incandescence be produced by the chemical force, as in burning jets of the gases in the open air, or by the electric force, as when hermetically sealed tubes of the gases are exposed to the discharge from a powerful induction coil.

Experiment has fully confirmed the truth of this theory, and the following are the details of the investigation:—

To obtain intimate acquaintance with the spectrum in question SWAN'S experiments were repeated. On feeding the flames with undiluted oxygen, instead of with air, still brighter spectra than he describes were obtained. The heat thus produced, however, volatilized potassium, sodium, lead, &c. from glass jets, and zinc and copper from brass jets; and this result ensued whether the oxygen was directed into the centre of, or made to surround the hydrocarbon flame. Finally, a mixture of olefiant gas and oxygen, or of coal-gas, saturated with benzole, and oxygen, was burned at an ordinary platinum oxyhydrogen safety-jet. In this way a cylindrical flame, half an inch in length and one-

* Edinb. Phil. Trans., vol. xxi. p. 411.

tenth of an inch in diameter, was obtained, and gave, on examination by the spectroscope, a brilliant well-defined spectrum.

The spectrum thus produced corresponds in appearance with the description of that observed by SWAN, excepting in the number of fine lines in each band of light. The yellow-green band, composed, according to the drawing accompanying SWAN's paper, of four lines, I find to contain six; the green band to have five instead of two; the blue five, that is one more than SWAN noticed; and the violet, beside being distinctly double, to have a faint hair-line between its two halves. Indeed, in this as in other spectra, the reduction of bands into groups of lines seems simply dependent on the refractive power of the spectroscope, an increased number of prisms causing greater dispersion of the spectrum, and, consequently, a division in a line or band that otherwise would appear to be single.

Having thus reproduced a satisfactory spectrum of the flame of a hydrocarbon, I next turned my attention to that of a nitrocarbon. Rejecting prussic acid vapour, on account of its containing hydrogen, I chose cyanogen. Cyanide of mercury was heated in a retort, and the cyanogen thus produced cooled and dried by passing over fragments of fused chloride of calcium contained in the neck of the retort. Ignited and examined by the spectroscope, this cyanogen flame gave a splendid series of bands, and these became still more distinct and brilliant on feeding the flame with oxygen by the platinum safety-tube already mentioned. Familiarity with the spectrum of hydrocarbon flames enabled me to detect it in this nitrocarbon light, other lines present being afterwards proved to be due to incandescent nitrogen.

But to establish the absolute identity of the hydro- and nitro-carbon spectra, excluding of course the lines due to nitrogen, they were simultaneously brought into the field of the spectroscope, one occupying the upper, the other the lower half of the field. This was readily effected after fixing the small prism, usually supplied with spectroscopes, over half of the narrow slit at the further end of the object-tube of the instrument. The light from the oxy-hydrocarbon flame was now directed up the axis of the tube by reflexion from the little prism, while that from the oxy-nitrocarbon flame passed directly through the uncovered half of the slit. A glance through the eye-tube was sufficient to show that the characteristic lines of the hydrocarbon spectrum were perfectly continued in the nitrocarbon spectrum. A similar arrangement of apparatus, in which the hydrocarbon light was replaced by that of pure nitrogen, showed that the remaining lines of the nitrocarbon spectrum were identical with those of the nitrogen spectrum. In this last experiment the source of the pure nitrogen light was the electric discharge through the rarefied gas.

The above experiments certainly seemed to go far towards proving the spectrum in question to be that of the element carbon. Nevertheless, the ignition of the gases having been effected in air, it was conceivable that hydrogen, nitrogen, or oxygen had influenced the phenomena. To eliminate this possible source of error, the experiments were repeated out of contact with air. A thin glass tube, 1 inch in diameter and

3 inches long, with platinum wires fused into its sides, and its ends prolonged by glass quills having a capillary bore, was filled with pure dry cyanogen, and the greater portion of this gas then removed by a good air-pump. Another tube was similarly prepared with olefiant gas. The platinum wires in these tubes were then so connected with each other that the electric discharge from a powerful induction coil could pass through both at the same time. On now observing the spectra of these two lights, in the simultaneous manner previously described, the characteristic lines of the hydrocarbon spectrum were found to be rigidly continued in that of the nitrocarbon. Moreover, by the same method of simultaneous observation, the spectrum of each of these electric flames, as they may be termed, was compared with the corresponding chemical flames, that is, with the spectra of the oxyhydro- and oxynitro-carbon jets of gas burning in air. The characteristic lines were present in every case. Lastly, by similar interobservation, a few other lines in the electric spectrum of the hydrocarbon were proved to be due to the presence of hydrogen, and several others in the electric spectrum of the nitrocarbon to be caused by the presence of nitrogen.

The electric discharge through cyanogen rapidly causes decomposition. The characteristic spectrum soon disappears, a black deposit is formed on the sides of the tube, and the spectrum of nitrogen alone remains. Olefiant gas is similarly affected, but not to the same extent. The glass tube is much blackened, but the spectrum is constant. BERTHELOT has, in fact, already shown that olefiant gas is decomposed by the electric current, acetylene being at the same time produced. Indeed, as acetylene may, according to BERTHELOT, be formed from its elements under the influence of the electric discharge, it is inconceivable that a hydrocarbon gas could be perfectly decomposed in such a tube as I have described.

The spectrum under investigation having then been obtained in one case when only carbon and hydrogen were present, and in another when all elements but carbon and nitrogen were absent, furnishes, to my mind, sufficient evidence that the spectrum is that of carbon.

But an interesting confirmation of the conclusion just stated is found in the fact, that the same spectrum is obtained when no other elements but carbon and oxygen are present, and also when carbon and sulphur are the only elements under examination. And first with regard to carbon and oxygen. Carbonic oxide burned in air gives a flame possessing a continuous spectrum. A mixture of carbonic oxide and oxygen burned from a platinum-tipped safety-jet also gives a more or less continuous spectrum, but the light of the spectrum has a tendency to group itself in ill-defined ridges. Carbonic oxide, however, ignited by the electric discharge in a semivacuous tube, gives a bright sharp spectrum. This spectrum was proved, by the simultaneous method of observation, to be that of carbon plus the spectrum of oxygen. With regard to carbon and sulphur almost the same remarks may be made. Bisulphide-of-carbon vapour burns in air with a bluish flame. Its spectrum is continuous. Mixed with oxygen and burned at the safety-jet, its flame still gives a continuous spectrum, though more distinctly

furrowed than in the case of carbonic oxide; but when ignited by the electric current, its spectrum is well defined, and is that of carbon plus that of sulphur. That is to say, it is the spectrum of carbon plus the spectrum that is obtained from vapour of sulphur when ignited by the electric discharge in an otherwise vacuous tube.

Having thus demonstrated that dissimilar compounds containing carbon emit, when sufficiently ignited, similar rays of light, I come to the conclusion that those rays are characteristic of ignited carbon vapour, and that the phenomenon they give rise to on being refracted by a prism is the spectrum of carbon.

The spectrum of carbon is a very beautiful one. The lines composing each band of light regularly diminish in brightness in the direction of greatest refraction, and appear to retreat from the observer like pillars of a portico seen in perspective. It differs greatly from that of every other element that I am acquainted with; and though, in each of the experiments described, it was of course accompanied by the spectrum of either nitrogen, hydrogen, sulphur, or oxygen, its diagnosis was not thereby interfered with; it is, in fact, most widely different from, and cannot possibly be confounded with, either of them.

The brightest band of the carbon spectrum being blue, and its other constituents being on the one hand light green and on the other violet, the associated rays of ignited carbon vapour, as indeed seen by the naked eye in carbon flames, I conceive to be of a light-blue colour. The tint may be observed in the flame of a spirit-lamp, in a burning jet of carbonic oxide, in the blowpipe flame of any hydrocarbon, and at the base of a common candle flame. I have no hesitation in saying that should a source of heat be found of sufficient intensity to volatilize the diamond and ignite its vapour, blue will be the colour of the light emitted.

The subject of the *emission* of carbon light by carbon vapour naturally leads to the consideration of the *absorption* of carbon light by carbon vapour. This latter research I am compelled to defer for a time.

The investigation also suggests the important question, *Is the spectrum of a compound simply the sum of the spectra of its constituents?* I have made several experiments tending to confirm such a law, but must perform many more before coming to a decided conclusion. Some observations made by Professor PLÜCKER in the course of an examination of the effects of the electric discharge on rarefied gases* seem to indicate that a compound has a spectrum different from that of the superposed spectra of its constituents.

Finally, I beg to offer my best thanks to Dr. FRANKLAND for allowing me the use of his laboratory and apparatus in making this research, and to J. P. GASSIOT, Esq., and Captain G. W. PUGET (34th Regt.) for the loan of induction coils.

* POGGENDORFF's 'Annalen,' Bd. cv. S. 77, and Bd. cvii. S. 538.

XII. *On the Theory of Probabilities.*

By George Boole, *F.R.S., Professor of Mathematics in Queen's College, Cork.*

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This paper has for its object the investigation of the general analytical conditions of a Method for the solution of Questions in the Theory of Probabilities, which was proposed by me in a work entitled "An Investigation of the Laws of Thought" (London, Walton and Maberly, 1854).

The application of this method to particular problems has been illustrated in the work referred to, and yet more fully in a 'Memoir on the Combination of Testimonies and of Judgments' published in the Transactions of the Royal Society of Edinburgh (vol. xxi. Part 4). Some observations, too, on the general character of the solutions to which the method leads, founded upon induction from particular cases, were contained in the original treatise, and the outlines, still in some measure conjectural, of their general theory were given in an Appendix to the Memoir. But the complete development of that theory was attended with analytical difficulties which I have only lately succeeded in overcoming. It involves discussions relating to the properties of a certain functional determinant, and to the possible solutions of a system of algebraic equations of peculiar form—discussions which will, I trust, be thought to possess a value, as contributions to Mathematical Analysis, independent of their present application.

As concerns the nature of the problems to which the method is applicable, it may be stated that they are such that the numerical elements, both given and sought, are the probabilities of events or states of things the definitions of which, and the connexions of which, are capable of expression by logical propositions. There is ground for believing that all questions whatever involving probability are ultimately reducible to this general form. This point, however, I do not purpose to discuss here. It has been already in some degree considered in the Memoir referred to.

In order to explain more fully the necessity for the present investigation, it will be requisite to state the fundamental principles upon which the method in question rests. There are only two of them which can possibly afford matter for discussion.

1st. The expression in language of the data of a problem in the Theory of Probabilities is to a certain extent arbitrary, because it depends upon the extent of meaning of the primary simple terms employed to express the events the conceptions of which it involves. But the choice of simple terms is, if we consider it with respect to our absolute power of choice, arbitrary. Any complex combination of events can be contemplated as a single whole in thought, and expressed by a single term. The invention of new simple terms to express what was before expressed by a combination of terms is a normal phenomenon in the growth of language.

Now the first principle upon which the method rests is the following:—

Principle I.—The different forms which a problem may be made to assume by different elections with respect to the simple terms of its expression are mutually equivalent.

For instance, if the following data were given,

The probability of rain is p,
The probability of rain with snow is q,

the form which the problem would assume in a language in which there was no word for snow, but in which the combination of snow with rain was called sleet, would be

The probability of rain is p,
The probability of sleet is q,

with the added condition, expressed as a logical proposition, that sleet always implies rain. And this as a statement of the data would, it is affirmed, be equivalent to the former statement. If these were the data of an actual problem, the event of which the probability is sought would require similar translation.

I desire to guard here against a possible misapprehension. I have said that the choice of simple terms, if considered with respect to our power of choice, is arbitrary. I do not mean by this to affirm that the actual growth of language is arbitrary. We know that it is far otherwise. Unity of sensuous impression in the early stages of its growth, unity of thought in the latter, seems to govern the invention and introduction of simple terms. It has indeed been said that there is a λόγος in the constitution of things of which language in its varied forms is the human reflexion, but never without the inseparable human element of choice and voluntary power.

It is then affirmed that whatever the grounds of fitness or propriety (and the existence of such grounds is fully conceded) may be, which have governed the actual choice of the simple terms of language, those grounds have nothing whatever to do with the calculation of probability. This depends upon the *information* contained in the data, information supposed to be derived from actual experience, or at least to be of such a nature that experience might have furnished it.

The different forms in which a problem is capable of being expressed, though differing in consequence of the different arbitrary elections which are possible with respect to its simple terms, are not independent of each other. They are connected together by the Laws of Thought, and pass one into the other by the processes of the Calculus of Logic, which is an organized expression of those Laws.

Among these forms there is one which presents exclusive advantages. It is that in which those events, however originally expressed, the probabilities of which constitute the data, are assumed as the simple events of the problem, and expressed by logical symbols corresponding to the simple terms of ordinary language; the event of which the probability is sought being also expressed logically by means of the same symbols. The Calculus of Logic enables us to do this, determining at the same time in an *explicit* form, *i. e.* in a form capable of expression in ordinary language by definite logical pro-

positions, the connexion which exists among all the events in question—a connexion which in the original form of the data was only *implied*.

This leads us to the statement of the second Principle of the Method.

Principle II.—When the data have been translated into probabilities of events connected by conditions logical in form and explicitly known, the problem may be constructed from a scheme of corresponding ideal events which are free, and of which the probabilities are such that when they (the ideal events) are restricted by the same conditions as the events in the data, their calculated probabilities will become the same as the given probabilities of the events in the data.

To take a material illustration: the problem, in the form to which it is reduced by the Calculus of Logic in accordance with Principle I., might be represented by the supposition of an urn containing balls distinguished by certain properties, *e. g.* by colour, as white or not white, by form, as round or not round, by material, as ivory or not ivory, and by the supposition that, while these properties enter into every conceivable combination, all the balls in which certain combinations are found are attached by strings to the sides of the urn, so that only the balls in which the remaining combinations are realized can be drawn. Suppose, further, that the probabilities of drawing under the actual conditions a white ball, a round ball, an ivory ball, &c. are given, and the probability of drawing a free ball fully defined with respect to the above elements of distinction is required. The principle affirmed is that we must proceed as if the balls were all free, and with probabilities such that the calculated probability of drawing any one of the balls which under the previous supposition are free, would be the same as under that supposition it is given to be.

Confining ourselves to the above material case, I remark, that the supposed mode of solution represents, 1st, a *possible* order of things; 2ndly, an order of things in which no preference is given to any one combination over any other which falls under the same category, or mode of thought. All the procedure of the theory of probabilities is founded upon the mental construction of the problem from some hypothesis, either, 1st, of events known to be independent; or, 2ndly, of events of the connexion of which we are totally ignorant; so that, upon the ground of this ignorance, we can again construct a scheme of alternatives all equally probable, and distinguished merely as favouring or not favouring the event of which the probability is sought. In doing this we are not at liberty to proceed arbitrarily. We are subject, first, to the formal Laws of Thought, which determine the possible conceivable combinations; secondly, to that principle, more easily conceived than explained, which has been differently expressed as the "principle of sufficient reason," the "principle of the equal distribution of knowledge or ignorance*," and the "principle of order." We do not know that the distribution of

* Knowledge and ignorance being in the theory of probabilities supplementary to each other, the equal distribution of the one implies that of the other.

I take this opportunity of explaining a passage in the 'Laws of Thought,' p. 370, relating to certain applications of the principle. Valid objection lies not against the principle itself, but against its applica-

properties in the actual urn is the same as it is conceived to be in the ideal urn of free balls, but the hypothesis that it is so, involves an equal distribution of our actual knowledge, and enables us to construct the problem from ultimate hypotheses which reduce it to a calculation of combinations.

I pass from the particular and material to the general problem. In the form to which this is brought by the Calculus of Logic, the probabilities are those of events of which certain combinations are, as a logical consequence of the original definitions of those events, impossible. It might, at first sight, appear that this establishes a fundamental difference between this problem and that of the urn, in which certain combinations are prevented from issuing by a material hindrance. In the one case the restriction appears as logically necessary, in the other as only actual.

Upon this I remark, that the data of the problem in its ultimate reduced form *might* result from the same kind of dependence as in the actual data; that they, in fact, *would* thus result if the mind of the observer were capable of contemplating, and were in a position to contemplate, each of the events in this ultimate translated form simply as a whole, and of recording, through an approximately infinite series of observations, what combinations of those wholes come into being, and what do not, in the actual universe. What appears as necessary in the translated data would now appear as actual—as a result of observation; what is impossible would be received as non-existent. The question is, then, whether the difference between the conception of what is impossible from involving a logical contradiction, and the conception of what in the actual constitution of things never exists, is of a kind to affect expectation. I do not hesitate to say that it is not. We are concerned with events in so far as they are capable of happening or not happening, of combining or not combining; but we are not concerned with the reasons in virtue of which they happen or do not happen, combine or do not combine. If we went beyond this, we should enter upon a metaphysical question to which I presume that no answer can, upon rational grounds, be given, viz. upon the question whether, when two things or events are in the actual constitution of things incapable of happening together, it would, if our knowledge were sufficiently extended, be found that the resulting conceptions of them were logically inconsistent.

I have but one further observation on Principle II. to make. It is that in the general problem we are not called upon to interpret the ideal events. The whole procedure is, like every other procedure of abstract thought, formal. We do not say that the ideal events exist, but that the events in the translated form of the actual problem are to be considered to have such relations with respect to happening or not happening as a certain system of ideal events would have if conceived first as free, and then subjected, without their freedom being otherwise affected, to relations formally agreeing with those to which the events in the translated problem are subject.

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tion through arbitrary hypotheses, coupled with the assumption that any result thus obtained is necessarily *the* true one. The application of the principle employed in the text, and founded upon the general theorem of development in Logic, I hold to be *not* arbitrary.

We are now able to explain more clearly the nature of the analytical investigation which will follow. Let $p_1, p_2, \ldots p_n$ represent the probabilities given in the data. As these will in general not be the probabilities of unconnected events, they will be subject to other conditions than that of being positive proper fractions, viz. to other conditions beside

$$\left.\begin{array}{l} p_1 \geqq 0,\ p_2 \geqq 0 \ldots p_n \geqq 0 \\ p_1 \leqq 1,\ p_2 \leqq 1 \ldots p_n \leqq 1. \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (1.)$$

Those other conditions will, as will hereafter be shown, be capable of expression by equations or inequations reducible to the general form

$$a_1 p_1 + a_2 p_2 \ldots + a_n p_n + a \geqq 0,$$

$a_1, a_2, \ldots a_n, a$ being numerical constants which differ for the different conditions in question. These, together with the former, may be termed the conditions of possible experience. When satisfied they indicate that the data *may* have, when not satisfied they indicate that the data *cannot* have, resulted from actual observation. On the other hand, the ideal events are regarded as independent, and their probabilities, which enter as auxiliary quantities into the process of solution, are subject to no other condition than that of being positive proper fractions. It is the general object of the analytical investigation to establish the two following conclusions, viz.,—

1st. The probabilities of the ideal independent events, as involved in the method under consideration, will in the process be determinable, without ambiguity, as positive proper fractions whenever the data satisfy the conditions of possible experience, and not otherwise.

And, as a consequence of the above,

2ndly. The probability determined by the method will have such a value as it consistently might have had if, instead of being calculated from the data, it had been determined by observation under the same experience as the data.

These conclusions rest upon the ground of certain analytical theorems relating to functional determinants, and to the possible solutions of simultaneous algebraic equations, which will be demonstrated in this paper. But, in order to explain the application of those theorems, it will be necessary to show, first, how the "conditions of possible experience" in problems in the Theory of Probabilities may be determined; secondly, what the analytical method in question for the solution of such problems is.

Determination of the Conditions of possible Experience.

The method for determining the conditions of possible experience given in the 'Laws of Thought,' chap. xix., may be advantageously replaced by the following one, which is taken from the 'Memoir on the Combination of Testimonies and of Judgments,' already referred to.

Let the events in the data be resolved into the ultimate possible alternatives which they involve, and let the unknown probabilities of these alternatives be represented by λ, μ, ν, &c.; then, as the probability of each event in the data is equal to the sum of the

probabilities of the alternatives which it involves, we shall have a system of equations connecting λ, μ, ν, &c. with p_1, p_2, ... p_n, the probabilities supposed given. Again, λ, μ, ν ..., as probabilities, are subject to the conditions

$$\lambda \geqq 0,\ \mu \geqq 0,\ \nu \geqq 0, \ldots \text{ \&c.},$$

and, as alternatives mutually excluding each other, to the condition

$$\lambda + \mu + \nu + \ldots = 1,$$

or the condition

$$\lambda + \mu + \nu + \ldots \leqq 1,$$

according as the alternatives in question together make up certainty or not.

Thus we have a system consisting of equations and inequations from which λ, μ, ν, &c. must be eliminated. To effect this elimination we must determine as many of the quantities λ, μ, ν... as possible from the equations, substitute their values in the inequations, and then eliminate the remainder of the quantities λ, μ, ν ... by means of the theorem that if we have simultaneously

$$\lambda \leqq a_1,\ \lambda \leqq a_2, \ldots \lambda \leqq a_m,$$
$$\lambda \geqq b_1,\ \lambda \geqq b_2, \ldots \lambda \geqq b_n,$$

then we have the system of conditions of which the type is

$$a_i \geqq b_j,$$

a_i representing any one of the set $a_1, a_2, \ldots a_m$, and b_j any one of the set $b_1, b_2, \ldots b_n$. Thus there are mn conditions in all.

This method is illustrated in the following problem, in the expression and solution of which it is to be noticed, that when in the Calculus of Logic an event is represented by x, the event which consists in its not happening is denoted by $1-x$, or for brevity by $\bar{x}$; that when two events are represented by x and y, their concurrence is denoted by xy, the happening of the first without the second by $x\bar{y}$, and so on.

Problem. Given that the probability of the concurrence of the events x and y is p, of the events y and z, q, and of the events z and x, r. Required the conditions to which p, q, and r must be subject in order that the above data may be consistent with a possible experience.

Resolving the events xy, yz, xz into the possible alternations out of which they are formed, let us write

$$\text{Prob. } xyz = \lambda,\ \text{Prob. } xy\bar{z} = \mu,\ \text{Prob. } x\bar{y}z = \nu,\ \text{Prob. } \bar{x}yz = \varrho.$$

Then we have the equations

$$\lambda + \mu = p,\ \lambda + \varrho = q,\ \lambda + \nu = r,$$

together with the inequations

$$\lambda \geqq 0,\ \mu \geqq 0,\ \nu \geqq 0,\ \varrho \geqq 0,$$
$$\lambda + \mu + \nu + \varrho \leqq 1.$$

From the equations we find

$$\mu = p - \lambda,\ \varrho = q - \lambda,\ \nu = r - \lambda,$$

which, substituted in the inequations, give

$$\lambda \geqq 0,\ p - \lambda \geqq 0,\ q - \lambda \geqq 0,\ r - \lambda \geqq 0,$$
$$p + q + r - 2\lambda \leqq 1;$$

and it only remains to eliminate λ. Now from the above,

$$\lambda \leqq p,\ \lambda \leqq q,\ \lambda \leqq r,\ \lambda \geqq 0,\ \lambda \geqq \frac{p+q+r-1}{2},$$

therefore

$$p \geqq 0,\ q \geqq 0,\ r \geqq 0,$$

$$p \geqq \frac{p+q+r-1}{2},\ q \geqq \frac{p+q+r-1}{2},\ r \geqq \frac{p+q+r-1}{2}.$$

The last three conditions are reducible to the simpler form,

$$p \geqq q+r-1,\ q \geqq r+p-1,\ r \geqq p+q-1.$$

Such are the conditions of possible experience in the data.

Suppose, for instance, it was affirmed as a result of medical statistics that, in two-fifths of a number of cases of disease of a certain character, two symptoms x and y were observed; in two-thirds of the cases the symptoms y and z were observed; and in four-fifths of the cases the symptoms z and x were observed; so that, the number of cases observed being large, we might on a future outbreak of the disease consider the fractions $\frac{2}{5}$, $\frac{2}{3}$, and $\frac{4}{5}$ as the probabilities of recurrence of the particular combinations of the symptoms x, y, and z observed. The above formulæ would show that the evidence was contradictory. For, representing the respective fractions by p, q, and r, the condition $p \geqq q+r-1$ is not satisfied. (*Edinburgh Memoir.*)

In applying the above method to the *à priori* limitation of questions in the theory of probabilities, it will be necessary to represent the probability sought by a single letter u, and treat this as if it were one of the numerical data. The resolution of the event of which the probability is sought into alternatives belonging to the same scheme as those of the events in the data gives us a new equation, which must be combined with the equations involving p, q, r, &c. The elimination of λ, μ, ν, &c. then determines not only the conditions of possible experience limiting p, q, r, but also the conditions which u must satisfy *à priori*, whatever method for its actual determination may be employed.

Thus, if from the foregoing data it were required to determine the *à priori* limits of Prob. xyz, i. e. of the probability of the conjunction of the events x, y, z, we should have as the additional equation

$$u = \lambda,$$

and therefore, after elimination of λ, μ, ν,

$$u \leqq p,\ u \leqq q,\ u \leqq r,$$

$$u \geqq 0,\ u \geqq \frac{p+q+r-1}{2},$$

the conditions required.

It will, however, in most of the following investigations suffice to consider the conditions of possible experience in the data alone, because it will be shown that when these are satisfied the corresponding conditions for the probability sought, when its value is determined by the method of the following section, will also be satisfied.

Statement of the Method for the Solution of Questions in the Theory of Probabilities.

For the general demonstration of this method the reader is referred to the 'Laws of Thought,' chap. xvii. For the purpose of the analytical investigation the statement of the method will suffice.

Let s, t, v, &c. represent the events of which the probabilities are given, p, q, r, &c. those probabilities, and w the event of which the probability is sought; then, whatever the definitions of $s, t \ldots$ and w may be, and whatever connecting relations may exist, it is always possible by the Calculus of Logic to determine the logical dependence of w upon s, t, &c. in the following most general form, viz.

$$w = A + 0B + \frac{0}{0}C + \frac{1}{0}D.$$

Here A, B, C, D are logical combinations of the events s, t, &c., and the connexion in which these stand to the event w and to each other is the following: A expresses those combinations of s, t, &c. which are entirely included in w, i. e. which cannot happen without our being permitted to say that w happens. B represents those combinations which may happen but are not included under w; so that when they happen we may say that w does not happen. C represents those combinations the happening of which leaves us in doubt whether w happens or not. D those combinations the happening of which would involve logical contradiction.

It follows from the above that the *translated* form of the problem is

Given Prob. $s = p$, Prob. $t = q$, Prob. $v = r$, &c., s, t, $v \ldots$ being regarded as events subject to the explicit logical condition

$$A + B + C = 1.$$

Required the probability u of the event of which the logical expression is

$$w = A + \frac{0}{0}C;$$

and it is shown (Laws of Thought, p. 265), upon grounds essentially the same as those expressed in Principles I. and II. of this paper, that the solution of the problem is involved in the following *algebraic* equations, viz.

$$\frac{V_s}{p} = \frac{V_t}{q} \ldots = \frac{A + cC}{u} = V, \quad \ldots\ldots\ldots\ldots \quad (I.)$$

in which the functions V, V_s, $V_t \ldots$ are formed in the following manner, viz.,—

1st. V is derived from A+B+C without change of form by interpreting s, t, &c. no longer as logical symbols, but as symbols of quantity. They represent the probabilities of the ideal events of Principle II.

2ndly. V_s is the sum of those terms in V which contain s as a factor, V_t the sum of those which contain t as a factor, &c.

The quantity c is an arbitrary constant, admitting of any value between 0 and 1.

To effect the solution, the quantities s, t, &c. are to be eliminated from the system (I.), and u then determined as a function of p, q, $r \ldots$ and c. The arbitrary constant c may

not appear in the final result, because the developed form of w may not contain any terms affected with the symbol $\frac{0}{0}$. When such terms do appear, the constant c admits of an interpretation indicating what new data are required to make the solution definite*.

It is proper here to observe that the conditions of possible experience can be determined as well from the 'translated' as from the original form of the problem. That the results will agree is evident *à priori*, but it may be desirable to point out the analytical connexion of the two processes. I will take the example just considered, and then offer some general remarks on the subject.

Representing the events xy, yz, zx by s, t, v, the translated data would be found to be

$$\text{Prob. } s=p, \quad \text{Prob. } t=q, \quad \text{Prob. } v=r,$$

s, t, and v being connected by the explicit logical condition

$$stv+s\bar{t}\,\bar{v}+\bar{s}t\bar{v}+\bar{s}\,\bar{t}v+\bar{s}\bar{t}\,\bar{v}=1.$$

It is easily shown that the first member of this equation represents the sum of those combinations of the events s, t, v, with respect to happening or failing, which involve no logical contradiction.

If, then, we represent under the above condition

$$\text{Prob. } stv=\lambda', \quad \text{Prob. } s\bar{t}\,\bar{v}=\mu', \quad \text{Prob. } \bar{s}t\bar{v}=\nu', \quad \text{Prob. } s\bar{t}v=\varrho',$$

we shall have

$$\lambda'+\mu'=p,\ \lambda'+\nu'=q,\ \lambda'+\varrho'=r,$$
$$\lambda'\geqslant 0,\ \mu'\geqslant 0,\ \nu'\geqslant 0,\ \varrho'\geqslant 0,$$
$$\lambda'+\mu'+\nu'+\varrho'\leqslant 1.$$

This system of equations and inequations agrees with that employed in the previous solution, if we only make

$$\lambda'=\lambda,\ \mu'=\mu,\ \nu'=\varrho,\ \varrho'=\nu,$$

so that the elimination of λ', μ', ν', ϱ' will lead to the same results as before.

In general it may be observed that each combination of s, t, v which is possible without logical contradiction, gives, on substituting for s, t, v... their expressions in the simple terms of the original problem, either a single combination of those simple terms, or a sum of such combinations; but the same combination of those simple terms will not arise from two different combinations of s, t... It is clear from this that the systems of united equations and inequations arising in the two forms of the problem will be related in the following manner, viz.—

For each positive quantity λ' in the one set, there will exist either a single positive quantity λ, or a sum of such quantities $\lambda_1+\lambda_2+$&c. in the other; but each such sum is inseparable, and the elements it is composed of are distinct from those of any other sum arising from any other of the quantities λ'... It is evident, then, that the final results of elimination will be the same. The same formal processes which eliminate single quan-

* Laws of Thought, p. 267.

tities in the one case, will eliminate the corresponding single quantities, or sums of single quantities, in the other.

Simplification of the General Equations for the Solution of Questions in the Theory of Probabilities.

Let us express the system (L.) in the form

$$\frac{V_s}{V}=p,\quad \frac{V_t}{V}=q,\ \&c.,$$

$$u=\frac{A+cC}{V},$$

and let us suppose the quantities $p, q\ldots$ (and therefore $s, t\ldots$) to be n in number. Then all the terms in V will be composed of products of $s, t\ldots\ \bar{s}, \bar{t}\ldots$, each term involving either s or $\bar{s}$, either t or $\bar{t}$, &c., but not the combinations $s\bar{s}$, $t\bar{t}$, &c. Each term is therefore homogeneous and of the nth degree.

It follows, therefore, that if we divide the numerator and denominator of each of the first members of the above system by $\bar{s}\,\bar{t}\,\bar{v}\ldots$, and then make

$$\frac{s}{\bar{s}}=x_1,\quad \frac{t}{\bar{t}}=x_2,\quad \frac{v}{\bar{v}}=x_3,\ \&c.,$$

and if at the same time we, for symmetry, change $p, q, r\ldots$ into $p_1, p_2 \ldots p_n$, the system will assume the following form,—

$$\frac{V_1}{V}=p_1,\quad \frac{V_2}{V}=p_2\ldots\ \frac{V_n}{V}=p_n,$$

$$u=\frac{A+cC}{V},$$

in which V, A, C are formed from their former values by suppressing $\bar{s}$, $\bar{t}$, $\bar{v}$, &c., or, which is the same thing, changing each of them into unity, and then changing $s, t, v\ldots$ into $x_1, x_2, x_3, \ldots$, while V_1 consists of those terms of V which contain x_1, V_2 of those which contain x_2, and so on.

In its new form V is a rational and entire function of $x_1, x_2, \ldots x_n$ not involving powers of those quantities, and with all its coefficients equal to unity. Again, as s, t, &c. are from the theory of their origin required to be positive proper fractions, $x_1, x_2, \ldots x_n$ are, from the nature of their connexion with $s, t\ldots$, required to be positive quantities. And it is sufficient that $x_1, x_2, \ldots x_n$ be determinable as positive quantities in order that $s, t\ldots$ may be determinable as positive fractions.

Now we shall proceed to show that $x_1, x_2, \ldots x_n$ are determinable as positive quantities precisely when $p_1, p_2 \ldots p_n$ satisfy the conditions of possible experience. We shall further show, as a consequence of this, that the value of the probability sought, when determined by the General Rule, will, under the same conditions, lie within such limits as if it were itself given by the same experience. In the order of this proof, we shall first demonstrate the theorems of pure Analysis upon which the conclusions depend, then in a distinct section make the particular application.

Analytical Theorems relating to Functional Determinants and Systems of Algebraic Equations.

A symmetrical determinant may be conveniently expressed in the form

$$\begin{vmatrix} A_1 & A_{12} & \dots & A_{1n} \\ A_{21} & A_2 & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_n \end{vmatrix} \quad \dots\dots\dots\dots \quad (I.)$$

the conditions of symmetry being

$$A_{ij}=A_{ji},\ A_{ii}=A_i.$$

It is desirable to employ fixed language in referring to this. We shall therefore call the quantities $A_1, A_2 \dots A_n$ the 'principal elements,' and the diagonal series of terms which they form the 'principal diagonal.' The elements A_{ij}, when i and j differ, we shall call 'subordinate elements.' The element A_i, together with all the subordinate elements which occur upon the same horizontal or vertical line of the determinant, we shall designate the 'i-system of elements.' Lastly, in comparing two rows or two columns of elements together, those elements will be said to correspond which occupy the same numerical place in their respective rows or columns.

The following Lemma will next be established.

Lemma.—A symmetrical determinant expressed in the form (I.) will be unaltered in value, if from each subordinate element of its i-system we subtract the corresponding element of its j-system multiplied by a quantity λ, which is invariable for the same system,—and for the principal element A_i substitute $A_i-2\lambda A_{ij}+\lambda^2 A_j$.

It is known that a determinant vanishes if two of its lines or columns are identical, and it is known as a consequence of this that if from a particular line or column of a determinant the corresponding elements of another line or column, multiplied each by the same constant, are subtracted, the determinant is unaltered in value. From the ith line of the above symmetrical determinant subtract, term by term, λ times the jth line, and then from the ith column of the resulting determinant subtract λ times the jth column. As respects any subordinate element, the result will obviously accord with the statement in the Lemma. But the element A_i will be successively converted into

$$A_i-\lambda A_{ji}$$

$$(A_i-\lambda A_{ji})-\lambda(A_{ij}-\lambda A_j).$$

The last expression, since $A_{ji}=A_{ij}$, is reducible to

$$A_i-2\lambda A_{ij}+\lambda^2 A_j.$$

Upon this property the demonstration of the following general proposition will be founded.

PROPOSITION I.

Let the symmetrical determinant (I.) *possess the following properties, viz.*:—

1st. *That all its elements are linear homogeneous rational functions of certain quantities* a, b, c, *&c., unlimited in number.*

2ndly. *That if the coefficients of any one of these quantities* a *in the elements of any particular line or column taken in order are* $\alpha_1, \alpha_2, \ldots \alpha_n$, *and in any other line or column* $\beta_1, \beta_2, \ldots \beta_n$, *then these two series of quantities are respectively proportional.*

3rdly. *That the principal terms* $A_1, A_2, \ldots A_n$ *are positive*, i. e. *that the coefficients of all the quantities* a, b, c, *&c. which appear in these terms are positive.*

Then the developed determinant will be itself positive, and will consist of products of the quantities a, b, c, *&c. without powers, each product affected by a positive sign.*

First, it may be observed that any letter a of the set a, b, $c\ldots$ which appears in the subordinate term A_{ij} will appear in both the principal terms A_i, A_j.

For let m be the coefficient of a in A_{ij}, and therefore also in A_{ji}; let l be the coefficient of a in A_i, and n its coefficient in A_j. Thus to the elements A_i, A_{ji} in the i-column correspond A_{ji}, A_j in the j-column. Hence, by the definition of the determinant,

$$l : m :: m : n,$$
$$\therefore \quad m^2 = ln,$$

which implies that neither l nor n vanishes, so that a appears in A_i and A_j.

Secondly, we shall show that the determinant can, without alteration of its final developed value, be reduced to a form in which any letter a of the system a, b, $c\ldots$ shall appear in only one system of elements, and therefore only in the principal term of that system, since every subordinate term is common to *two* systems.

Let us suppose a to be contained in two at least of the systems of elements, and for convenience of expression, let these be the 1-system and the n-system. Let, then, $\alpha_1, \alpha_2, \ldots \alpha_n$ be the successive coefficients of a in $A_1, A_{21}, \ldots A_{n1}$, and therefore, by definition of the determinant, $\lambda\alpha_1, \lambda\alpha_2, \ldots \lambda\alpha_n$, its coefficients in $A_{n1}, A_{n2}, \ldots A_n$. Any of the quantities $\alpha_1, \alpha_2, \ldots \alpha_n$ may be 0. But by the Lemma above demonstrated the determinant may, without alteration of value, be reduced to the following form, viz.:—

$$\left|\begin{array}{l} A_1\ ,\ A_{12}, \ldots A_{1n} - \lambda A_1 \\ A_{21},\ A_2\ , \ldots A_{2n} - \lambda A_{21} \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ A_{n1} - \lambda A_{11}, A_{n2} - \lambda A_{12} \ldots A_n - 2\lambda A_{n1} + \lambda^2 A_1 \end{array}\right| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B.)$$

Now in the determinant thus transformed the quantity a will no longer occur in the n-system.

This is obvious with respect to the subordinate elements of that system. With respect to the principal element, we observe that the coefficient of a is

in A_1, equal to α_1,
in A_{n1}, equal to $\lambda\alpha_1$,
. . . . . .
in A_n, equal to $\lambda \times \lambda\alpha_1$ or $\lambda^2\alpha_1$,

whence the coefficient of a in $A_n - 2\lambda_{n1} + \lambda^2 A_1$ is equal to 0.

Thus a has been eliminated from the n-system, and as the process has not affected any elements but those which belong to the n-system, it will not affect the relations under which a enters into the other systems.

Consider then any other quantity b in the set a, b, c, then by hypothesis the coefficients of b in any line or column of elements

$$A_{i1}, A_{i2}, \ldots A_{in}, \text{ or } A_{1i}, A_{2i}, \ldots A_{ni}$$

may be represented by

$$\mu_i\beta_1, \mu_i\beta_2, \ldots \mu_i\beta_n,$$

$\beta_1, \beta_2, \ldots \beta_n$ being an arbitrary set of quantities which are the same for all lines or columns, while μ_i differs for different lines or columns, and vanishes for those in which b does not enter.

It is to be noted that as $A_{ij}=A_{ji}$, we have in general

$$\mu_i\beta_j=\mu_j\beta_i,$$

while as the principal elements of the determinant (I.) are positive, we have always $\mu_i\beta_i$=a positive quantity.

Now reverting to the derived determinant (B.), we see that its ith line or column of elements will be

$$A_{i1}, A_{i2}, \ldots A_{in}-\lambda A_{1i},$$

and its jth line or column

$$A_{j1}, A_{j2}, \ldots A_{jn}-\lambda A_{ji},$$

supposing i and j to be both less than n.

In these lines or columns the successive coefficients of b will therefore be

$$\mu_i\beta_1, \mu_i\beta_2, \ldots \mu_i\beta_n-\lambda\mu_i\beta_1,$$
$$\mu_j\beta_1, \mu_j\beta_2, \ldots \mu_j\beta_n-\lambda\mu_j\beta_1,$$

which stand to each other in the constant ratio $\mu_i : \mu_j$.

Now let $j=n$. The coefficients of b in the nth line or column of (B.) are obviously

$$\mu_n\beta_1-\lambda\mu_1\beta_1, \mu_n\beta_2-\lambda\mu_1\beta_2, \ldots \mu_n\beta_n-2\lambda\mu_1\beta_n+\lambda^2\mu_1\beta_1,$$

of which the last term may be reduced as follows,

$$\mu_n\beta_n-2\lambda\mu_1\beta_n+\lambda^2\mu_1\beta_1=\mu_n\beta_n-\lambda\mu_1\beta_n-\lambda\mu_n\beta_1+\lambda^2\mu_1\beta_1=(\mu_n-\lambda\mu_1)(\beta_n-\lambda\beta_1);$$

so that the series of coefficients of b becomes

$$(\mu_n-\lambda\mu_1)\beta_1, (\mu_n-\lambda\mu_1)\beta_2 \ldots (\mu_n-\lambda\mu_1)(\beta_n-\lambda\beta_1),$$

and they are now seen to stand to those of b in the i-line and column in the constant ratio $\mu_n-\lambda\mu_1 : \mu_i$.

We have, lastly, to prove that the new principal element $A_n-2\lambda A_{1n}+\lambda^2 A$, is positive.

Let N be the coefficient of any one of the quantities $a, b, c \ldots$ in the above element, L its coefficient in the principal element A_i, and M its coefficient in each of the subordinate elements common to the two systems of which the above are the respective

principal elements, viz. in $A_{1n}-\lambda A_{11}$ and $A_{n1}-\lambda A_{11}$. Then, by what has already been proved,

$$L : M :: M : N,$$

$$\therefore \quad M^2 = LN;$$

but L is positive; therefore N is so, and the principal element in question consists wholly of positive terms.

The above demonstration shows that the elimination of a from the n-system produces a new determinant equivalent to the original one, and in which the characters noted in the original one still remain. Should a occur in any other system or systems of elements of the new determinant beside the 1-system, it can, by repetitions of the same process, be eliminated thence. Ultimately, then, it will only remain in the 1-system, and therefore only in the principal term of that system. Again, as it enters that term in the first degree, it follows that the developed determinant will involve only the first power of a. Hence, as a may represent any of the quantities a, b, c, ..., it is seen that no powers, but only products of these quantities, will appear in the developed determinant.

Let us represent the determinant, after the elimination of a from all the elements but A_{11}, in the form

$$\begin{vmatrix} A_1 & B_{12} \dots & B_{1n} \\ B_{21} & C_2 \dots & C_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ B_{n1} & C_{n2} \dots & C_n \end{vmatrix}$$

Now let ah_1 represent that term in A_1 which involves a. Then the portion of the determinant which involves a will be

$$ah_1 \begin{vmatrix} C_2 \dots & C_{2n} \\ \cdot \quad \cdot \quad \cdot & \cdot \\ C_{n2} \dots & C_n \end{vmatrix}$$

And here it is to be observed that ah_1 is positive, while the new determinant to which it is attached as a coefficient possesses all the characters of the old one. This determinant we can therefore transform in the same way, so as to eliminate any other letter b from all but a single principal element, which we shall suppose to contain it in a term bh_2. That portion of the original determinant which involves ab will therefore assume the form

$$abh_1h_2 \begin{vmatrix} D_3 \dots & D_{3n} \\ \cdot \quad \cdot \quad \cdot & \cdot \\ D_{n3} \dots & D_n \end{vmatrix}$$

Ultimately, then, as the result of such processes continued, the portion of the original determinant which involves any particular combination of n letters selected from a, b, c ... will consist of the product of a series of positive terms, each of which has appeared in some residual principal element. Every such combination being positive, it follows that the determinant itself consists solely of positive terms.

PROPOSITION II.

If V *be any rational entire function of the n variables* $x_1, x_2, \ldots x_n$, *but involving no powers of those variables above the first, and if, further, all the different terms of* V *have positive signs, then the determinant*

$$\begin{vmatrix} V & V_1 & V_2 & \ldots & V_n \\ V_1 & V_1 & V_{12} & \ldots & V_{1n} \\ V_2 & V_{21} & V_2 & \ldots & V_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ V_n & V_{n1} & V_{n2} & \ldots & V_n \end{vmatrix}$$

in which V_i *denotes the sum of the terms in* V *which contain* x_i, *and* V_{ij} *the sum of the terms in* V *which contain* x_i, x_j, *will, when developed as a rational and entire function of* $x_1, x_2, \ldots x_n$, *consist wholly of terms with positive coefficients.*

From the definition it is plain that in general

$$V_{ij}=V_{ji}, \quad V_{ii}=V_i,$$

whence the above determinant is symmetrical.

Again, all its elements are homogeneous linear functions of the terms in V.

Again, if $\alpha, \alpha_1, \alpha_2, \ldots \alpha_n$ represent the successive coefficients of any one of the terms of V in any row or column of the determinant, and $\beta, \beta_1, \beta_2 \ldots \beta_n$ the successive corresponding coefficients of the same term in any other row or column of the determinant, the one series of coefficients shall be proportional to the other.

Let us compare the first column and the i-column headed with the element V_i. Selecting any term in V, suppose it to contain x_i, then in whatever element of the first column that term is found, it will be found in a corresponding element of the i-column, and in each case with unity for its coefficient, since all the elements are mere collections of terms from V. But when it is not found in a particular element of the first column, it will not be found in the corresponding element of the i-column. The entire series of coefficients in the one being then the same as that in the other, the common ratio of the corresponding terms is unity.

Suppose, secondly, that the proposed term is found in V and not in V_i; then in all the elements of the i-column its coefficient is 0, so that the series of coefficients in the i-column might be formed from those in the first column by multiplying the latter successively by 0. This again represents a common ratio.

The same reasoning may be applied to the comparison of any two columns of the determinant. Thus in comparing the i-column and the j-column:—terms of V which contain both x_i and x_j will be found in corresponding elements of both columns—terms which contain x_i but not x_j will be wholly absent from the j-column. Thus in all cases if $\alpha, \alpha_1, \alpha_2, \ldots \alpha_n$ represent the coefficients of a term of V in one column, its coefficients in any other column, taken in the same order, will be of the form $\lambda\alpha, \lambda\alpha_1, \lambda\alpha_2 \ldots \lambda\alpha_n$, the coefficient λ being either 1 or 0.

Lastly, the principal elements consist, as do all the elements, of positive terms.

Therefore by the last proposition the developed determinant will consist of products (without powers higher than the first) of different terms of V, and the coefficients of all such products will be positive.

Therefore the determinant will be expressible as a rational entire function of $x_1, x_2, \ldots x_n$ with positive coefficients.

The rapidity with which the complexity of the determinant increases as the number of variables increases is remarkable. For example, if $n=2$ and $V=axy+bx+cy+d$, the determinant is

$$\begin{vmatrix} axy+bx+cy+d & axy+bx & axy+cy \\ axy+bx & axy+bx & axy \\ axy+cy & axy & axy+cy \end{vmatrix};$$

and its calculated value will be found to be

$$abcx^2y^2+abdx^2y+acdxy^2+bcdxy,$$

consisting of four positive terms.

But if $n=3$ and

$$V=axyz+byz+cxz+dxy+ex+fy+gz+h,$$

the developed determinant will consist of fifty-eight positive terms. Its calculated value will be found in the Memoir on Testimonies and Judgments.

Proposition III.

The functions $V, V_1, V_2, \ldots V_n$ *being defined as above, if* V *be complete in form*, i. e. *if it consist of all the terms which according to definition it can contain, each with a positive coefficient, then the system of equations*

$$\frac{V_1}{V}=p_1, \quad \frac{V_2}{V}=p_2 \ldots \frac{V_n}{V}=p_n \quad \ldots\ldots\ldots \quad (1.)$$

will, when $p_1, p_2, \ldots p_n$ *are proper fractions, admit of one solution, and only one, in positive values of* $x_1, x_2, \ldots x_n$.

We shall show, first, that the above proposition is true when $n=1$, secondly, that on the hypothesis that it is true for $n-1$ variables, it is true for n variables. Hence it will follow that it is true generally.

Suppose $n=1$. Then $V=ax_1+b$, whence the system (1.) reduces to the single equation

$$\frac{ax_1}{ax_1+b}=p_1,$$

$$\therefore \quad x_1=\frac{bp_1}{a(1-p_1)},$$

whence, since a and b are positive, and p is a positive fraction, x_1 is positive.

Thus the proposition is true when $n=1$.

Now, let $x_1=0$, and let $x_2, x_3 \ldots x_n$ be determined to satisfy the last $n-1$ equations of the system (1.). These $n-1$ equations will, when $x_1=0$, form a system of the same

nature with respect to the $n-1$ variables $x_2, x_3 \ldots x_n$, as (1.) is with respect to the n variables $x_1, x_2, \ldots x_n$. This will be at once seen by taking any particular example. Hence by hypothesis $x_2, x_3 \ldots x_n$ will be determinable as positive quantities, and their values substituted in the first member of the first equation of (1.) will reduce it to the form

$$\frac{Ax_1}{Ax_1+B},$$

A and B being finite and positive. Hence the function $\frac{V_1}{V}$ will become 0.

Secondly, let any finite positive value be assigned to x_1. The last $n-1$ equations of the system (1.) will again form a system of the same nature as before, and will by hypothesis determine a set of finite positive values for $x_2, x_3, \ldots x_n$. These values again substituted in $\frac{V_1}{V}$, will give to it again the form

$$\frac{Ax_1}{Ax_1+B},$$

A and B being finite and positive. Hence as x_1 is finite and positive, $\frac{V_1}{V}$ will be a positive fraction.

Lastly, let x_1 be infinite. Still the last $n-1$ equations of the system (1.) will assume the same form as before. Determining thence $x_2, x_3 \ldots x_n$, and substituting in $\frac{V_1}{V}$, we have

$$\frac{V_1}{V}=\frac{Ax_1}{Ax_1+B},$$

in which A and B are finite and positive and x_1 is infinite. Hence $\frac{V_1}{V}=1$. It is seen then that as x_1 varies from 0 to infinity, $x_2, x_3, \ldots x_n$ being at the same time always by hypothesis determined to satisfy the last $n-1$ equations of the system (1.), the function $\frac{V_1}{V}$ will vary from 0 through positive fractional values to unity. It is manifest, too, that it varies continuously. If then it vary by continuous *increase*, it will once, and only once in its change, become equal to p_1, and the whole system of equations thus be satisfied together. I shall show that it does vary by continuous *increase*.

If it vary continuously from 0 to 1 and not by continuous increase, it must in the course of its variation assume at least once a maximum or minimum value. Let us then seek the condition of possibility of

$$\frac{V_1}{V} = \text{a maximum or minimum},$$

the variables being subject to the relations

$$\frac{V_2}{V}=p_2, \quad \frac{V_3}{V}=p_3 \ldots \frac{V_n}{V}=p_n.$$

Here, proceeding in the usual way by differentiation, we have

$$\frac{V dV_1 - V_1 dV}{V^2}=0, \quad \frac{V dV_2 - V_2 dV}{V^2}=0, \ldots \frac{V dV_n - V_n dV}{V^2}=0,$$

or

$$\frac{dV}{V}=\frac{dV_1}{V_1}=\frac{dV_2}{V_2}\ldots=\frac{dV_n}{V_n}.$$

Let the common value of these fractions be represented by $-dt$, then we have a system of $n+1$ equations of which the first is

$$Vdt+dV=0,$$

while the n others are of the type

$$V_idt+dV_i=0.$$

The complete system, therefore, on effecting the total differentiations, becomes

$$Vdt+\frac{dV}{dx_1}dx_1..+\frac{dV}{dx_n}dx_n=0,$$

$$V_1dt+\frac{dV_1}{dx_1}dx_1..+\frac{dV_1}{dx_n}dx_n=0,$$

. . . . . .

$$V_ndt+\frac{dV_n}{dx_1}dx_1..+\frac{dV_n}{dx_n}dx_n=0.$$

Now from the nature of the function V we have

$$\frac{dV}{dx_i}=\frac{V_i}{x_i},\ \frac{dV_i}{dx_i}=\frac{V_i}{x_i},\ \frac{dV_i}{dx_j}=\frac{V_{ij}}{x_j},$$

so that the above equations become

$$Vdt+V_1\frac{dx_1}{x_1}+V_2\frac{dx_2}{x_2}..+V_n\frac{dx_n}{x_n}=0,$$

$$V_1dt+V_1\frac{dx_1}{x_1}+V_{12}\frac{dx_2}{x_2}..+V_{1n}\frac{dx_n}{x_n}=0,$$

$$V_2dt+V_{21}\frac{dx_1}{x_1}+V_2\frac{dx_2}{x_2}..+V_{2n}\frac{dx_n}{x_n}=0,$$

. . . . . .

$$V_ndt+V_{n1}\frac{dx_1}{x_1}+V_{n2}\frac{dx_2}{x_2}..+V_n\frac{dx_n}{x_n}=0,$$

and the elimination of $dt, \frac{dx_1}{x_1}, \frac{dx_2}{x_2}\ldots\frac{dx_n}{x_n}$ from these equations gives the sought condition of possibility of a maximum value of $\frac{V_1}{V}$, consistently with the satisfaction of the last $n-1$ equations of the system (1.).

This condition is therefore expressed by the equation

$$\begin{vmatrix} V & V_1 & V_2 & .. & V_n \\ V_1 & V_1 & V_{12} & .. & V_{1n} \\ V_2 & V_{21} & V_2 & .. & V_{2n} \\ .. & .. & .. & .. & \\ V_n & V_{n1} & V_{n2} & .. & V_n \end{vmatrix}=0.$$

But we have already seen (Prop. II.) that the first member of this equation is essentially positive for positive values of $x_1, x_2 \ldots x_n$. Hence the function $\frac{V_1}{V}$ varies by continuous increase, and on the hypothesis that the proposition to be proved is true for $n-1$ variables, it is true for n variables.

Therefore, connecting this with the former result, the proposition is true universally.

PROPOSITION IV.

If V *be an incomplete function, some of the terms belonging to the complete form being wanting, but the terms present having their coefficients positive, it will in general be necessary not only that the quantities* $p_1, p_2, \ldots p_n$ *should be positive fractions, but also that they should satisfy certain inequations of the form*

$$a_1p_1+a_2p_2\ldots+a_np_n+b \geqq 0,$$

in order that the system

$$\frac{V_1}{V}=p_1, \quad \frac{V_2}{V}=p_2 \ldots \frac{V_n}{V}=p_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

may admit of a solution in positive values of $x_1, x_2 \ldots x_n$.

For let $Ax_r x_s x_t \ldots$ be any term in V, A being a constant which is positive in all the terms, but which may be different in the different terms. Suppose that in V_i there exist e terms like the above, and let the several ratios of these terms to V be denoted by $\lambda_1, \lambda_2 \ldots \lambda_e$. Then the ith equation of the system (1.) will become

$$\lambda_1+\lambda_2\ldots+\lambda_e=p_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

and the system (1.) will be converted into a system of n equations of this nature. We will suppose that there exist m distinct quantities of the nature of $\lambda_1, \lambda_2 \ldots \lambda_e$ in the first members of this transformed system, and we will represent these by $\lambda_1, \lambda_2 \ldots \lambda_m$. Then, if these constitute all the ratios of the separate terms of V to V itself, we have a new equation,

$$\lambda_1+\lambda_2\ldots+\lambda_m=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

If they do not constitute all those separate ratios, we have, on the contrary, an inequation,

$$\lambda_1+\lambda_2\ldots+\lambda_m \leqq 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Lastly, the condition that $\lambda_1, \lambda_2 \ldots \lambda_m$ are positive fractions, gives the inequations

$$\lambda_1 \geqq 0, \ \lambda_2 \geqq 0 \ldots \lambda_m \geqq 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

The conditions $\lambda_1 \leqq 1$, &c. are already implied in (3.) or (4.).

The λ quantities are thus subject to a system of united *equations* and *inequations*, from which they must be eliminated by the method already explained.

The result of such elimination will be a final system of inequations connecting $p_1, p_2, \ldots p_n$. Equations connecting these quantities can only present themselves when the equations of the original system are not independent, or, which really falls under

the same hypothesis, when one or more of the variables $x_1, x_2 \ldots x_n$ is wholly absent from that system. Thus if x_1 were a common factor of all the terms of V, it would divide out from the numerators and denominators of the system, which would thus become a system of n simultaneous equations connecting the $n-1$ variables $x_2, x_3 \ldots x_n$. Considered with reference to these variables, therefore, the equations of the system would not be independent.

All resulting inequations will be capable of expression under the one general form,

$$a_1p_1+a_2p_2 \ldots +a_np_n+b \gtreqless 0,$$

the coefficients $a_1, a_2, \ldots a_n$ and b being positive, negative, or vanishing, numerical constants. For any inequation which presents itself in the form

$$a_1'p_1+a_2'p_2 \ldots +a_n'p_n+b' \lesseqgtr 1$$

may be transformed into

$$-a_1'p_1 \ldots -a_n'p_n+1-b' \gtreqless 0.$$

Again, the general inequation

$$a_1p_1+a_2p_2 \ldots +a_np_n+b \gtreqless 0$$

determines an inferior limit of p_1 when a_1 is positive, and a superior limit of p_1 when a_1 is negative.

For in the former case we have

$$p_1 \gtreqless -\left(\frac{a_2}{a_1}p_1 \ldots +\frac{a_n}{a_1}p_n+\frac{b}{a_1}\right),$$

the second member of which is an inferior limit of p_1; and it will be observed that the calculated value of this member may be positive, as there is no general restriction on the signs of $a_2, \ldots a_n, b$.

In the latter case, changing a_1 into $-a_1'$, and observing that a_1 is positive, we have

$$p_1 \lesseqgtr \frac{a_2}{a_1'}p_2+\frac{a_3}{a_1'}p_3 \ldots +\frac{a_n}{a_1'}p_n+\frac{b}{a_1'},$$

the second member of which is a superior limit of p_1.

Lastly, the final system of inequations is totally independent of the numerical value of the coefficients of V. The only restriction is that these coefficients are positive.

Proposition V.

Let V *be incomplete in form; then, provided that the equations*

$$\frac{V_1}{V}=p_1, \quad \frac{V_2}{V}=p_2 \ldots \frac{V_n}{V}=p_n \quad \ldots\ldots\ldots \quad (1.)$$

are independent with respect to the quantities $x_1, x_2, \ldots x_n$, *and that the inequations of condition deducible by the last proposition are satisfied, the equations will admit of one solution, and only one, in positive finite values of* $x_1, x_2, \ldots x_n$.

The proof of this proposition will, in its general character, resemble the proof of

Proposition III. It will be shown that when we assign to x any value *between* the limits 0 and infinity, the quantities $x_2, x_3 \ldots x_n$ will admit of determination from the last $n-1$ equations of the system as positive finite quantities, and the function $\frac{V_1}{V}$ will receive a value falling within the limits assigned by Proposition IV. to the quantity p_1; that when x_1 is equal to 0 or infinity, $x_2, x_3 \ldots x_n$ will admit of determination either as positive finite quantities, or as limits (0 and ∞) of such quantities; and that these values together will give to $\frac{V_1}{V}$ a value coinciding with the highest of the inferior, or lowest of the superior limits of p_1, as determined by Proposition IV.; that when x_1 varies from 0 to ∞, $x_2, x_3 \ldots x_n$ being determined as above, $\frac{V_1}{V}$ will vary by continuous increase from the highest of the inferior to the lowest of the superior limits of p_1, and once in its variation become equal to p_1. Thus the truth of the proposition for n variables will flow necessarily from its assumed truth for $n-1$ variables. And on this ground it will be shown that it may ultimately be reduced to a direct dependence upon Proposition III.

In the system (1.) let x_1 receive any finite positive value, and let V by the substitution of this value become U; the last $n-1$ equations of (1.) will thus assume the form

$$\frac{U_2}{U}=p_2, \quad \frac{U_3}{U}=p_3 \cdots \frac{U_n}{U}=p_n, \quad \ldots\ldots\ldots \quad (2.)$$

in which the quantities $p_2, p_3 \ldots p_n$ satisfy the conditions to which the direct application of Proposition IV. to this reduced system of equations would lead.

For what is important to notice in the change from V to U is, that any two terms in V which differ only in that one contains x_1 and the other does not, reduce to a single term in U. The effect of the change upon the primary system of equations and inequations formed in the application of Proposition IV. to the system (1.) is the following:—

1st. The equation between $\lambda_1, \lambda_2 \ldots$ derived from the first equation of (1.) will be annulled.

2ndly. In the remaining equations connecting $\lambda_1, \lambda_2 \ldots$ some *pairs* of those quantities may be replaced by single quantities, with corresponding changes in the inequations. Thus if $\lambda_1+\lambda_2$ be replaced by μ, the inequations

$$\lambda_1 \gtreqless 0, \quad \lambda_2 \gtreqless 0$$

will be replaced by what they before *implied*, viz.

$$\mu \gtreqless 0.$$

But these changes do not affect the truth of the relations, or introduce any new relations. They cannot, therefore, lead to any new final conditions. The conditions connecting $p_2, p_3 \ldots p_n$, in accordance with Proposition IV. in the system (2.), must have been already involved in the equations connecting $p_1, p_2 \ldots p_n$ in the system (1.).

Hence by hypothesis the system (2.) gives one set of positive finite values of

$x_2, x_3 \ldots x_n$ corresponding to the assumed positive finite value of x_1. And these values together make $\frac{V_1}{V}$ a positive proper fraction. We may notice that, representing $\frac{V_1}{V}$ under the form

$$\frac{Ax_1}{Ax_1+B},$$

it cannot be that either A or B is wanting so as to reduce $\frac{V_1}{V}$ to the value 0 or 1. For if A were wanting, V would not contain x_1 at all, as by hypothesis it does; and if B were wanting, V would contain x_1 in every term. Thus x_1 would divide out from the system (1.), which would thus become a system of n equations between $n-1$ variables, and would cease to be independent, as by hypothesis it is.

But when $x_1=0$, or $x_1=$ infinity, the form of V, considered as a function of $x_2, x_3 \ldots x_n$, will not generally be the same as in the case last considered; and the conditions connecting $p_2, p_3, \ldots p_n$ will no longer be such that we can affirm the possibility of deducing from the last $n-1$ equations of the system (1.), as transformed, positive finite values of $x_2, x_3, \ldots x_n$.

The theory of this case depends upon a remarkable transformation.

The most general form of the inequations of condition connecting $p_1, p_2 \ldots p_n$, as determined by Proposition IV., is

$$a_1p_1+a_2p_2 \ldots +a_np_n+b \geqq 0. \quad \ldots \quad (3.)$$

Hence, from the nature of the system (1.), it follows that the function

$$a_1V_1+a_2V_2 \ldots +a_nV_n+bV \quad \ldots \quad (4.)$$

must consist wholly of positive terms. Therefore V must consist of terms which would either appear in the development of the above function with positive signs, or not appear in it at all. Let $Ax_rx_sx_t \ldots$ be any term of V. Then, as the coefficient of this term in (4.) would be

$$a_rA+a_sA+a_tA \ldots +bA,$$

and as A is positive, we have

$$a_r+a_s+a_t \ldots +b \geqq 0,$$

a general condition which determines not what terms have actually entered, but what could alone possibly have entered into the constitution of V.

From the system (1.) we have

$$\frac{a_1V_1+a_2V_2 \ldots +a_nV_n+bV}{V}=a_1p_1+a_2p_2 \ldots +a_np_n+b.$$

Hence if we write

$$a_1V_1+a_2V_2 \ldots +a_nV_n+bV=H,$$

we have

$$\frac{H}{V}=a_1p_1+a_2p_2 \ldots +a_np_n+b, \quad \ldots \quad (5.)$$

an equation by which we may replace any one of the equations of the system (1.), and

which has the peculiarity that for every term $Ax_r x_s x_t \ldots$ which appears in the numerator H the particular condition

$$a_r + a_s + a_t \ldots + b > 0$$

is satisfied.

Let K be the aggregate of those terms in V for which the remaining particular condition

$$a_r + a_s + a_t \ldots + b = 0$$

is satisfied. Then $V = H + K$. If we now substitute (5.) in place of the first equation of the system (1.) and then write $H + K$ for V, $H_1 + K_1$ for V_1, &c., the system becomes converted into the following one, viz.

$$\frac{H}{H+K} = a_1 p_1 + a_2 p_2 \ldots + a_n p_n + b, \quad \frac{H_2 + K_2}{H+K} = p_2, \quad \frac{H_3 + K_3}{H+K} = p_3, \ldots \frac{H_n + K_n}{H+K} = p_n \quad . \quad . \quad (6.)$$

Now let us transform the above equations by assuming

$$x_2 = x_1^{\frac{a_2}{a_1}} y_2, \quad x_3 = x_1^{\frac{a_3}{a_1}} y_3 \ldots x_n = x_1^{\frac{a_n}{a_1}} y_n.$$

The general type of these equations is

$$x_i = x_1^{\frac{a_i}{a_1}} y_i,$$

and it includes the particular case of $i = 1$, provided that we suppose, as we shall do, $y_1 = 1$.

Then representing, as before, any term of V by $Ax_r x_s x_t \ldots$, we have

$$Ax_r x_s x_t \ldots = Ax_1^{\frac{a_r + a_s + a_t \ldots}{a_1}} y_r y_s y_t \ldots$$

Let this substitution be made in the different terms both of the numerators and denominators of the fractions which form the first members of the above system, and then let each numerator and denominator be multiplied by $x_1^{\frac{b}{a_1}}$. The result will be the same as if for each term $Ax_r x_s x_t \ldots$ in numerator or denominator we substituted the term

$$Ax_1^{\frac{a_r + a_s + a_t \ldots + b}{a_1}} y_r y_s y_t \ldots$$

In considering the effect of this transformation we will first suppose a_1 positive, and afterwards suppose it negative.

Case 1; the coefficient a_1 positive. Here, since for all the terms in H and in $H_2, H_3 \ldots H_n$ we have

$$\frac{a_r + a + a_t \ldots + b}{a_1} > 0,$$

all such terms in the transformed equations will be affected with positive powers of x_1.

And since for all terms in K, $K_2, \ldots K_n$ we have

$$\frac{a_r + a_s + a_t \ldots + b}{a_1} = 0,$$

all such terms in the transformed equations will be free from x_1.

Now let $$a_1p_1+a_2p_2\ldots+a_np_n+b=0.$$

This, as a_1 is positive, is to suppose that p_1 coincides with one of its own inferior limits. We must suppose this to be the highest of those limits, as otherwise some of the other limiting conditions would be violated. Now, since all the terms in H are affected with positive powers of x_1, while those in K do not contain x_1, the first equation of the system (6.) will be satisfied by $x_1=0$, provided that the remaining $n-1$ equations give finite positive values for $y_2\ldots y_n$. But the vanishing of x_1 reduces these equations to the form

$$\frac{K_2}{K}=p_2,\ \frac{K_3}{K}=p_3,\ldots\ \frac{K_n}{K}=p_n. \quad \ldots\ldots\ldots \quad (7.)$$

It is therefore necessary to show that $p_2, p_3\ldots p_n$ in this system are actually subject to the conditions to which the application of the method of Proposition IV. to the system itself would lead.

The n quantities $p_1, p_2\ldots p_n$ are by hypothesis subject to the conditions furnished by the application of the method of Proposition IV. to the original system (1.). In applying this method each of the original equations yields an equation of the form

$$\lambda_1+\lambda_2\ldots+\lambda_s=p_i; \quad \ldots\ldots\ldots\ldots \quad (8.)$$

and to the equations thus formed are added the inequations

$$\lambda_1+\lambda_2\ldots+\lambda_m\leqq 1,$$
$$\lambda_1\geqq 0,\ \lambda_2\geqq 0,\ \ldots\lambda_m\geqq 0;$$

$\lambda_1, \lambda_2\ldots\lambda_m$ having reference to the whole system of original equations.

Now the satisfaction of the equation

$$\frac{H}{H+K}=0$$

by the value $x_1=0$, involves the vanishing of all those quantities of the system $\lambda_1, \lambda_2\ldots\lambda_m$, which are derived from terms in V that are also found in H. Hence the λ quantities that do not vanish are those derived from terms in V which appear in K.

Again, the condition

$$a_1p_1+a_2p_2\ldots+a_np_n+b=0$$

shows that the system of equations of which (8.) is the type are not independent. They must, under the particular circumstances of the case, be such that the above equation shall be derivable from them. Hence one of these equations may be rejected. If we reject the first, viz. the one which contains p_1, and then reduce the others by making the λ quantities which are not derived from K to vanish, the system typified by (8.) evidently reduces to the system which we should have to employ if we applied the method of Proposition IV. directly to the system of $n-1$ equations (7.). Hence the quantities $p_2, p_3, \ldots p_n$ satisfy the final conditions to which that application would lead, and therefore by hypothesis the equations (7.) admit of solution by a single system of finite positive values of $y_2, y_3\ldots y_n$.

Now in general

$$x_i=x_1^{\frac{a_i}{a_1}}y_i.$$

Hence since $x_1=0$ and y_i is finite and positive for all values of i from 2 to n, we see that x_i will be 0 for all values of i for which a_i is positive, finite and positive for all values of i for which a_i is 0, and infinite for all values of i for which a_i is negative.

Case 2; the coefficient a_i negative. Here the inequation of condition (3.) must be supposed to determine the lowest of the superior limits of p_1, and therefore when p_1 coincides with that limit we have

$$a_1p_1+a_2p_2\ldots+a_np_n+b=0.$$

The transformations remaining formally the same as before, the following results will present themselves.

The terms in H and in H_2, $H_3\ldots H_n$ will be affected with negative instead of positive powers of x_1. Hence the same determination of y_2, $y_3\ldots y_n$ from the last $n-1$ equations of (6.), which before followed from the assumption $x_1=0$, will now follow from the assumption $x_1=\infty$, which at the same time satisfies the first equation of (6.).

The equation

$$x_i=x_1^{\frac{a_i}{a_1}}y_i$$

shows, since a_1 is here negative and x_1 infinite, that x_i will be infinite for those values of i for which a_i is negative, finite for those values of i for which a_i is 0, nothing for those values of i for which a_i is positive.

In all these cases the values 0 and ∞ appear as limits of finite positive values. This results from the connexion of the second member of the first equation of the system (6.) with the condition (3.).

Lastly, as the incompleteness of form of V only causes certain terms of the developed determinant of Proposition II. to vanish, but leaves the signs of the terms which remain positive, it follows that as x_1 varies from 0 to infinity, x_2, x_3, $\ldots x_n$ being always determined by the last $n-1$ equations of (1.), the function $\frac{V_1}{V}$ will vary by continuous increase between the limits above investigated, viz. from the highest inferior to the lowest superior limit of p_1. Once, therefore, in its progress it becomes equal to p_1, and all the equations are satisfied together.

The above reasoning establishes rigorously that if the proposition is true for $n-1$ variables, it is true for n variables. It remains then to consider the limiting case of $n=1$.

Here, however, only the complete form of V, viz. $V=ax+b$, leads to a definite value of x, and this, as has been seen, is finite and positive. If we give to V the particular form ax, the equation $\frac{V_x}{V}=p$ becomes

$$\frac{ax}{ax}=p, \text{ or } p=1,$$

which determines p, but leaves x indefinite. If we employ the other particular form $V=b$, we obtain no equation whatever, and here again x is indefinite. But as the

reducing transformations are all definite, the above indefinite forms cannot present themselves in the last stage of the problem when the original equations are independent and admit of definite solution.

The proposition is therefore established.

Application.

The general system of algebraic equations upon which the solution of questions in the theory of probabilities depends, is a particular case of that discussed in Proposition V. Its peculiarity is, that all the coefficients which appear in the function V are equal to unity.

The conditions of possible experience, as determined by the method, agree with the conditions shown in Proposition IV. to be necessary, and in Proposition V. to be sufficient, in order that $x_1, x_2 \ldots x_n$ may be determinable as positive finite quantities. For in both cases the quantities λ_1, λ_2, &c. correspond to the different terms in V, and in both cases the equations among those quantities depend simply on the forms of the functions $V_1, V_2 \ldots V_n$, and therefore ultimately on the form of V, irrespectively of the values of the positive coefficients of V. In both cases the systems of inequations are the same.

It follows, therefore, that precisely when the data represent a possible experience, the probabilities of the ideal events from which in the process of solution the problem is mentally constructed admit of determination as positive proper fractions.

Again, as the process for determining the *à priori* limits of the probability sought rests ultimately upon the assumption that the ratio of any term or partial aggregate of terms in V to V itself is a positive fraction, and as this assumption is satisfied when $x_1, x_2 \ldots x_n$ are positive quantities, it follows that the calculated value of the probability sought will always lie within the limits which it would have had if determined by observation from the same experience as the data.

But though the test last mentioned is one which must necessarily be satisfied by a true method, it is of infinitely less theoretical importance than that from which it is derived, viz. the test which consists in the absolute connexion between possibility in the data and formal consistency in the method.

As the conclusions of Propositions IV. and V. depend upon the form of the function V and the fact that its coefficients are positive, it follows that if in the application of the method to questions of probability we substituted any other positive values for unity in the coefficients of V, leaving the rest of the process as before, we should still be able to determine $x_1, x_2, \ldots x_n$ as positive quantities, or as limits of such, and the altered value of the probability sought would still be consistent with the experience from which the data are supposed to be derived. It would, however, properly speaking, be a value of interpolation, not a probability.

I will close with a few remarks upon the general nature of the method, and of the solutions to which it leads.

1st. The probability determined is not precisely of the same nature as the probabilities given.

For the data are supposed to be derived from experience; and therefore, on the supposition that the future will resemble the past, the events of which the probabilities are given will in the long run recur with a frequency proportioned to their probability.

But the probability determined is always an intellectual rather than a material probability. We cannot affirm that in the long run an event will occur with a frequency proportional to its calculated probability; but we can affirm that it is more likely to occur with this than with any other precise degree of frequency; that if it do not occur with this degree of frequency, the data are in some measure *one-sided*.

At the same time the limits of possible deviation are determined.

2ndly. General solutions obtained by the method do sometimes, but not always, admit* of being verified by other methods. I believe that this is solely because it is not often possible to solve the problem by other methods without introducing hypotheses which are of the nature of additional data, and, in effect, limit the problem. Every general solution, however, admits of a number of particular verifications by necessary consequence from the theorems established in this paper.

3rdly. It has been seen that a calculated probability is not necessarily a definite numerical value. It may be of the form $A+cC$, in which c is an arbitrary positive fraction. Here it is implied that the probability admits of any value between A and $A+C$. If, further, $A=0$ and $C=1$, it is implied that the probability may have any value between 0 and 1,—is therefore quite indefinite. This would really arise if we applied the method to a case in which the event of which the probability is sought had absolutely no connexion with those of which the probabilities are given.

Hence in the present theory the numerical expression for the probability of an event about which we are totally ignorant is not $\frac{1}{2}$, but c†. Hence, also, when all the probabilities given are measured by $\frac{1}{2}$, it is not to be concluded (upon the ground of *e nihilo nihil*) that the probability sought will also be $\frac{1}{2}$.

4thly. While extending the real power of the theory of probabilities, the method tends in some cases to diminish the apparent value of its results. For all problems in which the data admit of logical expression can be solved by it; but the resulting solutions, founded upon the bare data, may be of an indeterminate character, in place of the determinate results to which ordinary methods, aided by hypotheses not really involved in the data, lead.

This is the case with the problem of the combination of different grounds of belief or opinion. The general solution is indefinite. In two limiting cases, however, it assumes a definite form; one of these, which agrees with the formula generally accepted, representing the extreme cumulative force of testimonies, the other the mean weight of

* Professor Donkin has verified a general solution (Laws of Thought, p. 362).

† See on this subject a paper by Bishop Terrot, Edinburgh Transactions, vol. xxi. part 3.

judgments. Both these, however, occur as limiting cases, and they can only be applied with confidence under extreme circumstances, such as probably never occur in human affairs. (*Edinburgh Memoir*, pp. 630–645.)

5thly. I have, in effect, remarked that there is reason to suppose that all questions in the theory of probabilities can ultimately be reduced to questions in which the immediate subjects of probability are *logical*, i. e. involve no other essential relations than those of genus and species, whole and part. This is a question of theoretical rather than of practical interest. For instance, whether the formula of the arithmetical mean, which is the basis of the theory of astronomical observations, is self-evident, or whether it rests upon an ultimate logical basis, or whether, as I am inclined to believe, it may lawfully be regarded in either of these distinct but not conflicting lights, the superstructure remains the same.

XIII. *On the Calculus of Symbols.*—Second Memoir. *By* W. H. L. Russell, *Esq.*, *A.B.* *Communicated by* Arthur Cayley, *Esq.*, *F.R.S.*

Made up of two Memoirs: one received January 7,—Read January 30, 1862; the other received June 18,—Read June 19, 1862.

This Memoir is a continuation of one on the Calculus of Symbols which I had the honour to lay before the Society in December 1860, and which has since been published in the Philosophical Transactions. I commence this paper with some extensions of the method given in the former memoir for resolving functions of non-commutative symbols into binomial factors. I then explain a method, analogous to the process for extracting the square root in ordinary algebra, for resolving such functions into equal factors. I next investigate a process for finding the highest common internal divisor of two functions of non-commutative symbols, or, in other words, of finding if two linear differential equations admit of a common solution. After this, I give a rule for multiplying linear factors of non-commutative symbols, analogous to the ordinary algebraical rule for linear algebraical factors. I then resume the consideration of the binomial theorem explained in the former memoir. Two new forms of this binomial theorem are here given; and the method by which these forms are proved identical will, I hope, be considered an interesting portion of symbolical algebra, and as exhibiting in a remarkable manner its peculiar nature.

In the next place, I proceed to calculate the general values of the coefficients which occur in the form of the binomial theorem given in the first memoir; I then obtain an expression for the symbolical coefficient of the general term of the multinomial theorem as previously explained; and also a theorem for the multiplication of symbolical factors emanating from each other after a given law; lastly, I investigate a binomial theorem reciprocal to the binomial theorem already considered.

In the former memoir I explained a process by which the symbolical function

$$\varrho^n\varphi^n(\pi)+\varrho^{n-1}\varphi_{n-1}(\pi)+\varrho^{n-2}\varphi_{n-2}(\pi)+\ldots+\varphi_0(\pi)$$

could be resolved in all possible cases into factors of the form

$$(\varrho\psi_1^{(n)}\pi+\psi_0^{(n)}\pi)(\varrho\psi_1^{(n-1)}\pi+\psi_0^{(n-1)}\pi)\ldots(\varrho\psi_1\pi+\psi_0\pi).$$

I shall now give a method by which the same symbolical function may be resolved into factors of the form

$$(\varrho^n\psi_n^{(n)}\pi+\psi_0^{(n)}\pi)(\varrho^\beta\psi_\beta^{(n-1)}\pi+\psi_0^{(n-1)}\pi)\ldots(\varrho^n\psi_n(\pi)+\psi_0\pi).$$

By pursuing methods similar to those employed in the preceding paper, we find the following equations as the condition that $\varrho^s\psi_s(\pi)+\psi_0(\pi)$ may be an internal factor of

the given symbolical function,

$$\varphi_0(\pi)-\frac{\psi_0(\pi)}{\psi_2(\pi-2)}\varphi_2(\pi-2)+\frac{\psi_0\pi\psi_0(\pi-2)}{\psi_2(\pi-2)\psi_2(\pi-4)}\varphi_4(\pi-4)-\ldots=0,$$

$$\varphi_1\pi-\frac{\psi_0(\pi)}{\psi_1(\pi-2)}\varphi_3(\pi+2)+\frac{\psi_0\pi\psi_0(\pi-2)}{\psi_2(\pi-2)\psi_2(\pi-4)}\varphi_5(\pi-4)-\ldots=0.$$

Again, the conditions that $\varrho^3\psi_3(\pi)+\psi_0(\pi)$ may be an external factor of the same symbolical function will be given by the equations

$$\varphi_0(\pi)-\frac{\psi_0\pi}{\psi_3\pi}\varphi_3(\pi)+\frac{\psi_0\pi\psi_0(\pi+2)}{\psi_3\pi\psi_3(\pi+2)}\varphi_6(\pi)-\ldots=0,$$

$$\varphi_1(\pi)-\frac{\psi_0(\pi+1)}{\psi_3(\pi+2)}\varphi_3(\pi)+\frac{\psi_0(\pi+1)\psi_0(\pi+3)}{\psi_3(\pi+1)\psi_3(\pi+3)}\varphi_6(\pi)-\ldots=0.$$

We may also find in like manner the conditions that $\varrho^3\psi_3(\pi)+\psi_1(\pi)$, $\varrho^4\psi_4(\pi)+\psi_0(\pi)$ may be internal or external factors of the given symbolical function: in every case the number of equations of condition will be equal to the degree of (ϱ) in the given factor.

By applying the method of divisors, as explained in the former paper, we may ascertain the forms of $\psi_3(\pi)$, $\psi_0(\pi)$ in order that $\varrho^3\psi_3(\pi)+\psi_0(\pi)$ may be an internal factor of the given symbolical function. In the present case, however, $\psi_3(\pi)$ must be a divisor both of $\varphi_n(\pi)$ and $\varphi_{n-1}(\pi)$, $\psi_0(\pi)$ a divisor both of $\varphi_1(\pi)$ and $\varphi_0(\pi)$,—a consideration which will greatly simplify the process. We proceed, in like manner, should there be no internal factors of the form $\varrho^3\psi_3\pi+\psi_0\pi$, $\varrho^3\psi_3(\pi)+\psi_0(\pi)$, to ascertain if there are any of the form $\varrho^3\psi_3(\pi)+\psi_1(\pi)$, &c.; and continuing the investigation as before, we are able, in all possible cases, to resolve the given symbolical function

$$\varrho^n\varphi_n(\pi)+\varrho^{n-1}\varphi_{n-1}(\pi)+\ldots+\varphi_0(\pi)$$

into binomial factors.

Hence, in any linear differential equation,

$$X_r\frac{d^ru}{dx^r}+X_{r-1}\frac{d^{r-1}u}{dx^{r-1}}+X_{r-2}\frac{d^{r-2}u}{dx^{r-2}}+\ldots=X,$$

if we put

$$\varrho=x,\ \pi=x\frac{d}{dx},$$

we shall be able in all possible cases to reduce it to a series of equations:—

$$\begin{aligned}
\varrho^\alpha\psi_\alpha^{(n-1)}(\pi)u_{n-1}+\psi_0^{(n-1)}(\pi)u_{n-1}&=X,\\
\varrho^\beta\psi_\beta^{(n-2)}(\pi)u_{n-2}+\psi_0^{(n-2)}(\pi)u_{n-2}&=u_{n-1},\\
\&c.\qquad\qquad\qquad\qquad&=\&c.,\\
\varrho^\mu\psi_\mu\ (\pi)u\ +\psi_0\ (\pi)u\ &=u_1,
\end{aligned}$$

a series of binomial equations, each of which may be treated by the methods due to Professor Boole. I shall now explain a process analogous to that denominated 'evolution' in ordinary algebra. To resolve the symbolical function

$$\varrho^{2m}\varphi_{2m}(\pi)+\varrho^{2m-1}\varphi_{2m-1}(\pi)+\ldots+\varphi_0(\pi)$$

into two equal factors.

For this purpose, let us assume

$$\varrho^{2n}\varphi_{2n}(\pi)+\varrho^{2n-1}\varphi_{2n-1}(\pi)+\varrho^{2n-2}\varphi_{2n-2}(\pi)+\ldots+\varphi_0(\pi)$$
$$=(\varrho^n\theta_n(\pi)+\varrho^{n-1}\theta_{n-1}(\pi)+\varrho^{n-2}\theta_{n-2}(\pi)+\ldots+\theta_0(\pi))^2.$$

From this we find, equating the coefficients of ϱ^{2n},

$$\theta_n(\pi)\theta_n(\pi+n)=\varphi_{2n}(\pi),$$

from whence

$$\theta_n(\pi+n)\theta_n(\pi+2n)=\varphi_{2n}(\pi+n);$$

or

$$\theta_n(\pi+2n)=\frac{\varphi_{2n}(\pi+n)}{\varphi_{2n}(\pi)}\,\theta_n(\pi),$$

whence

$$\theta_n(\pi)=\frac{\varphi_{2n}(\pi-n)\ \varphi_{2n}(\pi-3n)\varphi_{2n}(\pi-5n)\ldots}{\varphi_{2n}(\pi-2n)\varphi_{2n}(\pi-4n)\varphi_{2n}(\pi-6n)\ldots}.$$

Again, equating the coefficients of ϱ^{2n-1}, we shall have

$$\theta_{n-1}(\pi+n)\theta_n\pi+\theta_n(\pi+n-1)\theta_{n-1}(\pi)=\varphi_{2n-1}(\pi)\ldots; \quad . \quad . \quad . \quad . \quad . \quad (A.)$$

$\theta_n(\pi)$, $\varphi_{2n-1}(\pi)$ are known rational functions of (π), wherefore assume

$$\theta_n(\pi)=a+b\pi+c\pi^2+\ldots+k\pi^s,$$

$$\varphi_{2n-1}(\pi)=\alpha+\beta\pi+\gamma\pi^2+\ldots+\varkappa\pi^r,$$

where $a, b, c, \ldots \alpha, \beta, \gamma, \ldots$ are known, and

$$\theta_{n-1}(\pi)=\mathrm{A}+\mathrm{B}\pi+\mathrm{C}\pi^2+\ldots+\mathrm{K}\pi^{r-s},$$

where A, B, C are known. They may be easily found by equating the coefficients of (π) in equation (A.) and $\theta_{n-1}(\pi)$ thus determined.

If we equate the coefficients of ϱ^{2n-2}, we shall have

$$\theta_{n-2}(\pi+n)\theta_n\pi+\theta_{n-2}(\pi)\theta_n(\pi+n-2)+\theta_{n-1}\pi\theta_{n-1}(\pi+n-1)=\varphi_{2n-2}(\pi),$$

from which $\theta_{n-2}(\pi)$ may be determined in like manner. By this method we may in all possible cases reduce a proposed differential equation,

$$\mathrm{X}_r\frac{d^ru}{dx^r}+\mathrm{X}_{r-1}\frac{d^{r-1}u}{dx^{r-1}}+\ldots+\mathrm{X}_1\frac{du}{dx}+\mathrm{X}_0u=\mathrm{X},$$

to the form

$$\left\{\Xi_r\frac{d^r}{dx^r}+\Xi_{r-1}\frac{d^{r-1}}{dx^{r-1}}+\ldots+\Xi_0\right\}^2u=\mathrm{X},$$

when Ξ_r, Ξ_{r-1}, &c. are rational functions of (x).

The methods for finding the highest common divisor in ordinary algebra apply equally to the present Calculus, as will be seen by the following examples:—

To find the highest common internal divisor of the symbolical functions $\varrho^2+\varrho-\pi^2$ and $\varrho^2+\varrho(\pi^2+\pi+1)-\pi^2$.

Take $\varrho^3+\varrho-\pi^3$ for a divisor, and proceed as follows:—

$$\begin{array}{r|l}
\varrho^3+\varrho-\pi^3 & \varrho^3+\varrho(\pi^3+\pi+1)-\pi^3 \\
\varrho^3+\varrho \qquad -\pi^3 & \\
\hline
\varrho(\pi^3+\pi)-\pi^3+\pi^3 &
\end{array}$$

It is easily seen that the remainder may be written $\pi(\pi-1)(\varrho-\pi)$.

Hence, taking $\varrho-\pi$ as divisor,

$$\begin{array}{r|l}
\varrho-\pi & \varrho^3+\varrho-\pi^3(\varrho+\pi \\
\varrho^2-\varrho\pi & \\
\hline
\varrho(\pi+1)-\pi^3 & \\
\varrho(\pi+1)-\pi^3 & \\
\hline
\end{array}$$

Hence $\varrho-\pi$ is the highest common divisor of $\varrho^3+\varrho-\pi^3$ and $\varrho^3+\varrho(\pi^3+\pi+1)-\pi^3$.

Again, to find the highest common internal divisor of $\varrho^3\pi(\pi+1)+\varrho(\pi^3+\pi^3)+\pi^3(\pi-1)$ and $\varrho^3\pi-2\varrho^3\pi+\varrho(\pi^3+\pi)+\pi^4$.

The first of these quantities is equivalent to $(\pi-1)(\varrho^3\pi+\varrho(\pi^3+\pi)+\pi^3)$.

We take $\varrho^3\pi+\varrho(\pi^3+\pi)+\pi^3$ for divisor, and proceed as follows:—

$$\begin{array}{r|l}
\varrho^3\pi+\varrho(\pi^3+\pi)+\pi^3 & \varrho^3\pi-2\varrho^3\pi+\varrho(\pi^3+\pi)+\pi^4(\varrho-(\pi+1) \\
\varrho^3\pi+\varrho^3(\pi^3+\pi)+\varrho\pi^3 & \\
\hline
-\varrho^3(\pi^3+3\pi)+\varrho\pi+\pi^4 & \\
-\varrho^3(\pi^3+3\pi)-\varrho(\pi^3+3\pi^3+2\pi)-\pi^3(\pi+1) & \\
\hline
\varrho(\pi^3+3\pi^3+3\pi)+\pi^4+\pi^3+\pi^3 &
\end{array}$$

The remainder is equivalent to $(\pi^3+\pi+1)(\varrho\pi+\pi^3)$.

Hence, taking $\varrho\pi+\pi^3$ for a new divisor,

$$\begin{array}{r|l}
\varrho\pi+\pi^3 & \varrho^3\pi+\varrho(\pi^3+\pi)+\pi^3(\varrho+1 \\
\varrho^3\pi+\varrho\pi^3 & \\
\hline
\varrho\pi+\pi^3 & \\
\varrho\pi+\pi^3 & \\
\hline
\end{array}$$

Hence $\varrho\pi+\pi^3$ is the highest common internal divisor of

$$\varrho^3\pi(\pi+1)+\varrho(\pi^3+\pi^3)+\pi^3(\pi-1)$$

and

$$\varrho^3\pi-2\varrho^3\pi+\varrho(\pi^3+\pi)+\pi^4.$$

It is evident, as was mentioned in the introduction to this memoir, that this process is equivalent to finding the conditions that two linear differential equations may have a common solution.

I shall next proceed to find the general term of the continued product

$$(\varrho+\theta_1\pi)(\varrho+\theta_2\pi)(\varrho+\theta_3\pi)\ldots(\varrho+\theta_n\pi).$$

This product, when developed, will be of the form

$$\varrho^n\varphi_n(\pi)+\varrho^{n-1}\varphi_{n-1}(\pi)+\ldots+\varrho^m\varphi_m(\pi)+\ldots+\varrho\varphi_1\pi+\varphi_0(\pi).$$

Then $\varphi_m(\pi)$ is given by the following rule:—
Write down the following symbolical product:

$$\theta_1(\pi+m)\theta_2(\pi+m)\theta_3(\pi+m)\ldots\ldots\theta_{n-m}(\pi+m);$$

take every possible combination of the quantities 1, 2, 3,...n taken $(n-m)$ at a time, and substitute them as the weight* of θ in this continued product, diminishing (m) in each factor by the increment of the weight of the factor; add all the results together, and we obtain the value of $\varphi_m(\pi)$. The truth of this rule will be manifest to every one who will consider the following result obtained by actual multiplication:—

$$\begin{aligned}
&(\varrho+\theta_1\pi)(\varrho+\theta_2\pi)(\varrho+\theta_3\pi)(\varrho+\theta_4\pi)(\varrho+\theta_5\pi)=\varrho^5\\
&+\varrho^4\{\theta_1(\pi+4)+\theta_2(\pi+3)+\theta_3(\pi+2)+\theta_4(\pi+1)+\theta_5(\pi)\}\\
&+\varrho^3\{\theta_1(\pi+3)\theta_2(\pi+3)+\theta_1(\pi+3)\theta_3(\pi+2)+\theta_2(\pi+2)\theta_3(\pi+2)+\theta_1(\pi+3)\theta_4(\pi+1)\\
&+\theta_2(\pi+2)\theta_4(\pi+1)+\theta_3(\pi+1)\theta_4(\pi+1)+\theta_1(\pi+3)\theta_5\pi+\theta_2(\pi+2)\theta_5(\pi)+\theta_3(\pi+1)\theta_5\pi+\theta_4\pi\theta_5\pi\}\\
&+\varrho^2\{\theta_1(\pi+2)\theta_2(\pi+2)\theta_3(\pi+2)+\theta_1(\pi+2)\theta_2(\pi+2)\theta_4(\pi+1)+\theta_1(\pi+2)\theta_3(\pi+1)\theta_4(\pi+1)\\
&+\theta_2(\pi+1)\theta_3(\pi+1)\theta_4(\pi+1)+\theta_1(\pi+2)\theta_3(\pi+2)\theta_5\pi+\theta_1(\pi+2)\theta_3(\pi+1)\theta_5\pi\\
&+\theta_2(\pi+1)\theta_3(\pi+1)\theta_5\pi+\theta_1(\pi+2)\theta_4\pi\theta_5\pi+\theta_2(\pi+1)\theta_4\pi\theta_5\pi+\theta_3\pi\theta_4\pi\theta_5\pi\}\\
&+\varrho\{\theta_1(\pi+1)\theta_2(\pi+1)\theta_3(\pi+1)\theta_4(\pi+1)+\theta_1(\pi+1)\theta_2(\pi+1)\theta_3(\pi+1)\theta_5\pi\\
&+\theta_1(\pi+1)\theta_2(\pi+1)\theta_4\pi\theta_5\pi+\theta_1(\pi+1)\theta_3\pi\theta_4\pi\theta_5\pi+\theta_2\pi\theta_3\pi\theta_4\pi\theta_5\pi\}\\
&+\theta_1\pi\theta_2\pi\theta_3\pi\theta_4\pi\theta_5\pi.
\end{aligned}$$

I now come to the investigation of the two new forms of the binomial theorem as explained in the former memoir.

It is evident, in the first place, that in multiplying any binomial $(\varrho^3+\varrho\theta(\pi))^n$ by $\varrho^3+\varrho\theta(\pi)$, the result in this case will be the same whether we employ internal or external multiplication.

Let

$$(\varrho^3+\varrho\theta(\pi))^n=\varrho^{3n}+\varrho^{3n-1}\varphi_{2n-1}(\pi)+\varrho^{3n-2}\varphi_{2n-2}\pi+\ldots,$$

where $\varphi_{2n-1}(\pi)$, $\varphi_{2n-2}(\pi)$, $\varphi_{2n-3}(\pi)$ are unknown functions of (π) which we seek to determine.

Then multiplying externally and internally by $(\varrho^3+\varrho\theta(\pi))$

$$\begin{aligned}
(\varrho^3+\varrho\theta(\pi))^{n+1}&=\varrho^{3n+3}+\varrho^{3n+1}\varphi_{2n-1}(\pi)\quad+\varrho^{3n}\varphi_{2n-2}(\pi)\\
&\quad+\varrho^{3n+1}\theta(\pi+2n)\quad+\varrho^{3n}\theta(\pi+2n-1)\varphi_{2n-1}(\pi)+\ldots\\
&=\varrho^{3n+3}+\varrho^{3n+1}\varphi_{2n-1}(\pi+2)+\varrho^{3n}\varphi_{2n-2}(\pi+2)+\ldots\\
&\quad+\varrho^{3n+1}\theta(\pi)\quad+\varrho^{3n}\theta\pi\varphi_{2n-1}(\pi+1)+\ldots
\end{aligned}$$

* The use I have here made of the term 'weight' will be familiar to every one who is conversant with the modern Higher Algebra.

From whence, by equating the coefficients of (ϱ),

$$\varphi_{2n-1}(\pi)+\theta(\pi+2n)=\varphi_{2n-1}(\pi+2)+\theta(\pi),$$

$$\therefore \quad \left(\varepsilon^{2\frac{d}{d\pi}}-1\right)\varphi_{2n-1}(\pi)=\left(\varepsilon^{2n\frac{d}{d\pi}}-1\right)\theta(\pi),$$

whence

$$\varphi_{2n-1}(\pi)=\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi).$$

Again,

$$\varphi_{2n-2}(\pi)+\theta(\pi+2n-1)\varphi_{2n-1}(\pi)=\varphi_{2n-2}(\pi+2)+\varphi_{2n-1}(\pi+1)\theta\pi,$$

$$\therefore \quad \varphi_{2n-2}(\pi)=\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left\{\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right\}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta\pi\right\}-\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left\{\frac{\varepsilon^{(2n+1)\frac{d}{d\pi}}-\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta\pi\right\}\theta\pi.$$

We shall now investigate another form of this expansion, by which we shall be able to obtain a remarkable expression for the general term. We shall express the unknown functions by a notation slightly differing from that which we have just employed. The reason for so doing will be easily seen by the reader.

Let

$$(\varrho^2+\varrho\theta\pi)^n=\varrho^{2n}+\varrho^{2n-1}\varphi_1^{(n)}\pi+\varrho^{2n-2}\varphi_2^{(n)}(\pi)+\dots$$

Then

$$\begin{aligned}(\varrho^2+\varrho\theta(\pi))^{n+1}&=\varrho^{2n+2}+\varrho^{2n+1}\varphi_1^{(n)}(\pi+2)+\varrho^{2n}\varphi_2^{(n)}(\pi+2)+\dots\\&\quad+\varrho^{2n+1}\theta(\pi)+\varrho^{2n}\theta(\pi)\varphi_1^{(n)}(\pi+1)+\dots\\&=\varrho^{2n+2}+\varrho^{2n+1}\varphi_1^{(n+1)}(\pi)+\varrho^{2n}\varphi_2^{(n+1)}(\pi)+\dots;\end{aligned}$$

$$\therefore \quad \varphi_1^{(n+1)}(\pi)=\varphi_1^{(n)}(\pi+2)+\theta(\pi),$$

or

$$\varphi_1^{(n+1)}(\pi)-\varepsilon^{2\frac{d}{d\pi}}\varphi_1^{(n)}\pi=\theta(\pi).$$

Wherefore, solving this equation in finite differences, we have

$$\varphi_1^{(n)}(\pi)=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi).$$

Again,

$$\varphi_2^{(n+1)}\pi=\varphi_2^{(n)}(\pi+2)+\varphi_1^{(n)}(\pi+1)\theta\pi,$$

whence

$$\varphi_2^{(n+1)}\pi-\varepsilon^{2\frac{d}{d\pi}}\varphi_2^{(n)}\pi=\theta\pi\varepsilon^{\frac{d}{d\pi}}\varphi_1^{(n)}\pi.$$

Hence

$$\begin{aligned}\varphi_2^{(n)}(\pi)&=\left\{\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\right\}\left(\theta(\pi)\varepsilon^{\frac{d}{d\pi}}\right)\left\{\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\right\}\theta(\pi)\\&=\varepsilon^{-\frac{d}{d\pi}}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\right\}\theta(\pi)\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\right\}\theta(\pi)\\&=\varepsilon^{-\frac{d}{d\pi}}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right\}^2;\end{aligned}$$

and similarly,

$$\varphi_r^{(n)}\pi=\varepsilon^{-\frac{d}{d\pi}}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right\}^r,$$

where, however, a proper correction must be added after each performance of the symbol Σ.

As there are some peculiarities connected with this form, I shall calculate its value at length for the values $r=1$ and $r=2$.

First, let $r=1$. Then

$$\varphi_1^{(n)}(\pi)=\varepsilon^{\pi(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)$$

$$=\varepsilon^{\pi(n-1)\frac{d}{d\pi}}\Pi_1-\frac{\theta(\pi)}{\varepsilon^{2\frac{d}{d\pi}}-1}.$$

Let $n=1$. Then

$$\varphi_1^{(1)}\pi=\theta\pi, \quad \text{and} \quad \Pi_1=\frac{\varepsilon^{2\frac{d}{d\pi}}\theta(\pi)}{\varepsilon^{2\frac{d}{d\pi}}-1}.$$

Consequently

$$\varphi_1^{(n)}(\pi)=\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi),$$

which coincides with the value of the coefficient of ξ^{2n-1} obtained by the former process.

Again,

$$\varphi_2^{(n)}(\pi)=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\varepsilon^{\frac{d}{d\pi}}\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\theta(\pi-2n)\varepsilon^{\frac{d}{d\pi}}.\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi-2n)$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\left(\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right)\varepsilon^{\frac{d}{d\pi}}\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left(\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right)$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\left\{\left(\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right)\left(\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)\right\}$$

$$-\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\left\{\theta(\pi).\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right\}$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Pi_2+\varepsilon^{2(n-1)\frac{d}{d\pi}}\left(\frac{\varepsilon^{-2n\frac{d}{d\pi}}}{\varepsilon^{-2\frac{d}{d\pi}}-1}\theta(\pi)\right)\left(\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)$$

$$-\varepsilon^{2(n-1)\frac{d}{d\pi}}\frac{\varepsilon^{-2n\frac{d}{d\pi}}}{\varepsilon^{-2\frac{d}{d\pi}}-1}.\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Pi_2-\left(\frac{\theta(\pi)}{\varepsilon^{2\frac{d}{d\pi}}-1}\right)\left(\frac{\varepsilon^{(2n-1)\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)$$

$$+\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right),$$

where Π_2 is a function of (π) to be determined.

For this purpose put $n=1$, then $\varphi_2^{(n)}(\pi)=0$, and

$$\Pi_2=\left(\frac{\theta(\pi)}{\varepsilon^{2\frac{d}{dx}}-1}\right)\left(\frac{\varepsilon^{\frac{d}{dx}}\theta(\pi)}{\varepsilon^{2\frac{d}{dx}}-1}\right)-\frac{1}{\varepsilon^{2\frac{d}{dx}}-1}\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{dx}}}{\varepsilon^{2\frac{d}{dx}}-1}\theta(\pi)\right);$$

$$\therefore\ \varphi_1^{(n)}(\pi)=\left(\frac{\varepsilon^{2(n-1)\frac{d}{dx}}-1}{\varepsilon^{2\frac{d}{dx}}-1}\theta(\pi)\right)\left(\frac{\varepsilon^{(2n-1)\frac{d}{dx}}}{\varepsilon^{2\frac{d}{dx}}-1}\theta(\pi)\right)-\frac{\varepsilon^{2(n-1)\frac{d}{dx}}-1}{\varepsilon^{2\frac{d}{dx}}-1}\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{dx}}}{\varepsilon^{2\frac{d}{dx}}-1}\theta(\pi)\right).$$

We next proceed to show the identity of this value of the coefficient of ε^{2n-2} with that formerly obtained.

The truth of the following theorem is easily seen:—

$$\left(\varepsilon^{2\frac{d}{dx}}-1\right)f_1(\pi)f_2(\pi)=\left(\varepsilon^{2\frac{d}{dx}}-1\right)f_1(\pi)\left(\varepsilon^{2\frac{d}{dx}}-1\right)f_2(\pi)+f_2(\pi)\left(\varepsilon^{2\frac{d}{dx}}-1\right)f_1(\pi)+f_1(\pi)\left(\varepsilon^{2\frac{d}{dx}}-1\right)f_2(\pi).$$

Hence

$$\left(\varepsilon^{2\frac{d}{dx}}-1\right)\varphi_1^n\pi=\left\{\left(\varepsilon^{2(n-1)\frac{d}{dx}}-1\right)\theta(\pi)\varepsilon^{(2n-1)\frac{d}{dx}}\theta(\pi)\right\}+\left\{\left[\varepsilon^{2(n-1)\frac{d}{dx}}-1\right)\theta(\pi)\right\}\left\{\frac{\varepsilon^{(2n-1)\frac{d}{dx}}}{\varepsilon^{2\frac{d}{dx}}-1}\theta(\pi)\right\}$$

$$+\left\{\left(\varepsilon^{(2n-1)\frac{d}{dx}}\right)\theta\pi\right\}\left\{\left(\frac{\varepsilon^{2(n-1)\frac{d}{dx}}-1}{\varepsilon^{2\frac{d}{dx}}-1}\right)\theta(\pi)\right\}-\left(\varepsilon^{2(n-1)\frac{d}{dx}}-1\right)\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{dx}}}{\varepsilon^{2\frac{d}{dx}}-1}\theta(\pi)\right).$$

But

$$\left(\varepsilon^{2(n-1)\frac{d}{dx}}-1\right)\left(\theta\pi.\frac{\varepsilon^{\frac{d}{dx}}}{\varepsilon^{2\frac{d}{dx}}-1}\theta(\pi)\right)=\left(\varepsilon^{2(n-1)\frac{d}{dx}}\theta(\pi)\right)\left(\frac{\varepsilon^{(2n-1)\frac{d}{dx}}}{\varepsilon^{2\frac{d}{dx}}-1}\theta(\pi)\right)-\theta\pi\frac{\varepsilon^{\frac{d}{dx}}}{\varepsilon^{2\frac{d}{dx}}-1}\theta(\pi);$$

$$\therefore\ \left(\varepsilon^{2\frac{d}{dx}}-1\right)\varphi_1^{(n)}\pi=\left(\varepsilon^{2(n-1)\frac{d}{dx}}\theta(\pi)\right)\left(\varepsilon^{(2n-1)\frac{d}{dx}}\theta(\pi)\right)+\left(\frac{\varepsilon^{2(n-1)\frac{d}{dx}}-1}{\varepsilon^{2\frac{d}{dx}}-1}\theta(\pi)\right)\left(\varepsilon^{(2n-1)\frac{d}{dx}}\theta(\pi)\right)$$

$$-\theta(\pi)\varepsilon^{(2n-1)\frac{d}{dx}}\theta\pi-\theta(\pi).\frac{\varepsilon^{(2n-1)\frac{d}{dx}}}{\varepsilon^{2\frac{d}{dx}}-1}\theta\pi+\theta(\pi).\frac{\varepsilon^{\frac{d}{dx}}}{\varepsilon^{2\frac{d}{dx}}-1}\theta\pi$$

$$=\left\{\frac{\varepsilon^{2n\frac{d}{dx}}-1}{\varepsilon^{2\frac{d}{dx}}-1}\theta\pi\right\}\left\{\varepsilon^{(2n-1)\frac{d}{dx}}\theta(\pi)\right\}-\left\{\frac{\varepsilon^{(2n+1)\frac{d}{dx}}-\varepsilon^{\frac{d}{dx}}}{\varepsilon^{2\frac{d}{dx}}-1}\theta(\pi)\right\}\theta(\pi),$$

or

$$\varphi_1^{(n)}\pi=\frac{1}{\varepsilon^{2\frac{d}{dx}}-1}\left\{\frac{\varepsilon^{2n\frac{d}{dx}}-1}{\varepsilon^{2\frac{d}{dx}}-1}\theta\pi\right\}\left\{\varepsilon^{(2n-1)\frac{d}{dx}}\theta\pi\right\}-\frac{1}{\varepsilon^{2\frac{d}{dx}}-1}\left\{\frac{\varepsilon^{(2n+1)\frac{d}{dx}}-\varepsilon^{\frac{d}{dx}}}{\varepsilon^{2\frac{d}{dx}}-1}\theta(\pi)\right\}\theta\pi,$$

which agrees with the value of the symbolical coefficient of ε^{2n-2} as before obtained.

It is proper to add that the same method of investigation applies to all binomials of the form $(\varepsilon^\alpha+\varepsilon^\beta\theta(\pi))^n$, of which I have, for the sake of simplicity, selected the case $(\varepsilon^2+\varepsilon\theta\pi)^n$.

I now come to the calculation of the coefficients of the general term of the form of the binomial theorem as given in the first memoir.

Let us assume

$$\begin{aligned}(\xi^3+\xi\theta(\pi))^n = \quad & \xi^{3n} \quad +A_1^{(0)}\xi^{3n-1}+A_2^{(0)}\xi^{3n-2}+A_3^{(0)}\xi^{3n-3}+\&c.\\ &+(A_1^{(1)}\xi^{3n-1}+A_2^{(1)}\xi^{3n-2}+A_3^{(1)}\xi^{3n-3}+\&c.)\pi\\ &+(A_1^{(2)}\xi^{3n-1}+A_2^{(2)}\xi^{3n-2}+A_3^{(2)}\xi^{3n-3}+\&c.)\pi^2+\&c.\\ &+(A_1^{(r)}\xi^{3n-1}+A_2^{(r)}\xi^{3n-2}+A_3^{(r)}\xi^{3n-3}+\&c.)\pi^r+\&c.\end{aligned}$$

Then we have, writing $\theta_m(\pi)$ for $\frac{\theta^{(m)}(\pi)}{1.2.3....m}$,

$$\begin{aligned}A_1^{(0)}&=\Sigma\theta(2n)\\ A_2^{(0)}&=\Sigma\theta(2n-1)\Sigma\theta(2n)\\ A_3^{(0)}&=\Sigma\theta(2n-2)\Sigma\theta(2n-1)\Sigma\theta 2n\\ \&c.&=\&c.\\ A_s^{(0)}&=\Sigma\theta(2n-s+1)\Sigma\theta(2n-s+2)....\Sigma\theta 2n\\ A_{(1)}^{(1)}&=\Sigma\theta_1(2n)\\ A_2^{(1)}&=\Sigma\theta(2n-1)\Sigma\theta_1(2n)+\Sigma\theta_1(2n-1)\Sigma\theta(2n)\\ A_3^{(1)}&=\Sigma\theta_1(2n-2)\Sigma\theta(2n-1)\Sigma\theta(2n)\\ &\quad+\Sigma\theta(2n-2)\Sigma\theta_1(2n-1)\Sigma\theta(2n)\\ &\quad+\Sigma\theta(2n-2)\Sigma\theta(2n-1)\Sigma\theta_1(2n)\\ \&c.&=\&c.\\ A_s^{(1)}&=\Sigma\theta_1(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta(2n)\\ &\quad+\Sigma\theta(2n-s+1)\Sigma\theta_1(2n-s+2)\ldots\Sigma\theta(2n)\\ &\quad+\ldots\\ &\quad+\Sigma\theta(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta_1(2n)\\ A_1^{(2)}&=\Sigma\theta_2(2n)\\ A_2^{(2)}&=\Sigma\theta_2(2n-1)\Sigma\theta 2n+\Sigma\theta_1(2n-1)\Sigma\theta_1(2n)\\ &\quad+\Sigma\theta(2n-1)\Sigma\theta_2(2n)\\ A_s^{(2)}&=\Sigma\theta_2(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta(2n)\\ &\quad+\Sigma\theta(2n-s+1)\Sigma\theta_2(2n-s+2)\ldots\Sigma\theta(2n)\\ &\quad+\ldots\\ &\quad+\Sigma\theta(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta_2(2n)\\ &\quad+\Sigma\theta_1(2n-s+1)\Sigma\theta_1(2n-s+2)\ldots\Sigma\theta(2n)\\ &\quad+\Sigma\theta_1(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta_1(2n)\\ &\quad+\ldots\end{aligned}$$

Where there are (s) terms in the first part of this expression, and $s\cdot\frac{s-1}{2}$ in the second:

$$\begin{aligned}A_s^{(r)}=&\Sigma\theta_{\alpha_1}(2n-s+1)\Sigma\theta_{\beta_1}(2n-s+2)\Sigma\theta_{\gamma_1}(2n-s+3)\ldots\Sigma\theta_{\nu_1}(2n)\\ &+\Sigma\theta_{\alpha_2}(2n-s+1)\Sigma\theta_{\beta_2}(2n-s+2)\Sigma\theta_{\gamma_2}(2n-s+3)\ldots\Sigma\theta_{\nu_2}(2n)\\ &+\Sigma\theta_{\alpha_3}(2n-s+1)\Sigma\theta_{\beta_3}(2n-s+2)\Sigma\theta_{\gamma_3}(2n-s+3)\ldots\Sigma\theta_{\nu_3}(2n)\\ &+\ldots\end{aligned}$$

where $\alpha_1, \beta_1, \gamma_1, \dots \nu, \alpha_2, \beta_2, \gamma_2, \dots \nu_2$, &c. are all the whole numbers which satisfy the equation

$$\alpha+\beta+\gamma+\dots+\nu=r.$$

In the preceding investigation we have used $\theta_m(\pi)$ as an abbreviation for $\frac{\theta^{(m)}\pi}{1.2.3\dots m}$, where $\theta^{(m)}\pi$ is the mth function derived from $\theta(\pi)$. In the following investigation, it is proper to remark that $\theta_1(\pi)$, $\theta_2(\pi)$, $\theta_3(\pi)\dots$ are any rational and entire functions of (π) whatever.

To find an expression for the general term of the multinomial theorem, by which

$$(\varrho^s+\varrho^{s-1}\theta_1(\pi)+\varrho^{s-2}\theta_2(\pi)+\varrho^{s-3}\theta_3(\pi)+\dots)^n$$

is expanded in powers of (ϱ).

Let us assume

$$(\varrho^s+\varrho^{s-1}\theta_1(\pi)+\varrho^{s-2}\theta_2(\pi)+\dots)^n=\varrho^{sn}+\varrho^{sn-1}\varphi_1^{(n)}(\pi)+\varrho^{sn-2}\varphi_2^{(n)}(\pi)+\varrho^{sn-3}\varphi_3^{(n)}(\pi)+\dots$$

Then multiplying internally by the factor

$$\varrho^s+\varrho^{s-1}\theta_1(\pi)+\varrho^{s-2}\theta_2(\pi)+\varrho^{s-3}\theta_3\pi+\dots,$$

and equating coefficients of like powers of (ϱ), we have the following series of equations:—

$$\varphi_1^{(n+1)}(\pi)-\varphi_1^{(n)}(\pi+\alpha)=\theta_1(\pi),$$

$$\varphi_2^{(n+1)}(\pi)-\varphi_2^{(n)}(\pi+\alpha)=\theta_1(\pi)\varphi_1^{(n)}(\pi+\alpha-1)+\theta_2(\pi),$$

$$\varphi_3^{(n+1)}(\pi)-\varphi_3^{(n)}(\pi+\alpha)=\theta_1(\pi)\varphi_2^{(n)}(\pi+\alpha-1)+\theta_2(\pi)\varphi_1^{(n)}(\pi+\alpha-2)+\theta_3(\pi),$$

and thus we proceed: hence we have, putting the symbol $\varepsilon^{(n+1)n\frac{d}{d\pi}}\Sigma\varepsilon^{-(n+1)n\frac{d}{d\pi}}=\Pi$,

$$\varphi_1^{(n)}(\pi)=\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_1\pi,$$

$$\varphi_2^{(n)}\pi=\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)+\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_2(\pi)$$

$$\varphi_3^{(n)}\pi=\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1\pi+\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_2(\pi)+\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_2(\pi)\varepsilon^{-2\frac{d}{d\pi}}\Pi\theta_1\pi+\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_3\pi$$

$$\varphi_4^{(n)}(\pi)=\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)$$

$$+\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_2\pi+\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_1\pi\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_2(\pi)\varepsilon^{-2\frac{d}{d\pi}}\Pi\theta_1\pi+\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_2(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1\pi\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1\pi$$

$$+\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_2(\pi)+\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_2(\pi)\varepsilon^{-2\frac{d}{d\pi}}\Pi\theta_1(\pi)+\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_2\pi\varepsilon^{-2\frac{d}{d\pi}}\Pi\theta_2\pi+\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_4(\pi).$$

We easily see that the general term may be expressed thus: construct the formula

$$\varepsilon^{-n\frac{d}{d\pi}}\Pi\theta_a(\pi)\varepsilon^{-a\frac{d}{d\pi}}\Pi\theta_b(\pi)\varepsilon^{-b\frac{d}{d\pi}}\Pi\theta_c(\pi)\varepsilon^{-c\frac{d}{d\pi}}\dots\Pi\theta_l(\pi),$$

and give to $\alpha, \beta, \gamma, \dots \nu$ all the values which satisfy the equation

$$a+b+c+\dots+e=r,$$

where $\alpha n-r$ is the index of (ϱ). Then the sum of all the terms so formed will be the required result.

To determine the product of the factors

$$(\varrho+\chi_\alpha(\pi))(\varrho+\chi_{\alpha-1}(\pi))\dots(\varrho+\chi(\pi))\dots(\varrho+\chi_{\alpha-1}(\pi))(\varrho+\chi_\alpha\pi),$$

whence $\chi_1(\pi)$, $\chi_2(\pi)$, $\chi_3(\pi)$ emanate from each other after a given law.

Let

$$(\varrho+\chi_n(\pi))\dots(\varrho+\chi\pi)\dots\varrho+\chi_n(\pi)=\varrho^{2n+2}+\varrho^{2n}\varphi_1^{(n)}\pi+\varrho^{2n-1}\varphi_2^{(n)}\pi+\dots$$

Then multiplying internally and externally by $\varrho+\chi_{n+1}(\pi)$,

$$\begin{aligned}&\varrho^{2n+3}+\varrho^{2n+2}\varphi_1^{(n+1)}\pi+\varrho^{2n+1}\varphi_2^{(n+1)}\pi+\&c.\\&=\varrho^{2n+3}+\varrho^{2n+2}\{\varphi_1^{(n)}(\pi+1)+\chi_{n+1}(\pi+2n+2)+\chi_{n+1}(\pi)\}\\&\quad+\varrho^{2n+1}\{\varphi_2^{(n)}(\pi+1)+\varphi_1^{(n)}(\pi+1)\chi_{n+1}(\pi+2n+1)\\&\quad+\varphi_1^{(n)}(\pi)\chi_{n+1}(\pi)+\chi_{n+1}(\pi+2n+1)\chi_{n+1}(\pi)\}.\end{aligned}$$

Hence

$$\varphi_1^{(n+1)}\pi-\varphi_1^{(n)}(\pi+1)=\chi_{n+1}(\pi+2n+2)+\chi_{n+1}\pi,$$

whence

$$\varphi_1^{(n)}(\pi)=\varepsilon^{\pi\frac{d}{d\varrho}}\Sigma\left(\varepsilon^{(2n+1)\frac{d}{d\varrho}}+\varepsilon^{-(n+1)\frac{d}{d\varrho}}\right)\chi_{n+1}(\pi),$$

and also

$$\begin{aligned}\varphi_2^{(n+1)}(\pi)-\varphi_2^{(n)}(\pi+1)&=\varphi_1^{(n)}(\pi+1)\chi_{n+1}(\pi+2n+1)\\&\quad+\varphi_1^{(n)}(\pi)\chi_{n+1}(\pi)+\chi_{n+1}(\pi+2n+1)\chi_{n+1}(\pi);\end{aligned}$$

$$\begin{aligned}\therefore\quad\varphi_2^{(n)}(\pi)&=\varepsilon^{\pi\frac{d}{d\varrho}}\Sigma\varepsilon^{-(n+1)\frac{d}{d\varrho}}\left\{\varepsilon^{(2n+1)\frac{d}{d\varrho}}\chi_{n+1}(\pi)\right\}\left\{\varepsilon^{\pi\frac{d}{d\varrho}}\Sigma\left(\varepsilon^{(n+1)\frac{d}{d\varrho}}-\varepsilon^{-(n+1)\frac{d}{d\varrho}}\right)\varepsilon^{\pi\frac{d}{d\varrho}}\chi_{n+1}(\pi)\right\}\\&\quad+\varepsilon^{\pi\frac{d}{d\varrho}}\Sigma\varepsilon^{-(n+1)\frac{d}{d\varrho}}\chi_{n+1}(\pi)\varepsilon^{\pi\frac{d}{d\varrho}}\Sigma\left(\varepsilon^{(n+1)\frac{d}{d\varrho}}-\varepsilon^{-(n+1)\frac{d}{d\varrho}}\right)\chi_{n+1}(\pi)\\&\quad+\varepsilon^{\pi\frac{d}{d\varrho}}\Sigma\varepsilon^{-(n+1)\frac{d}{d\varrho}}\chi_{n+1}(\pi)\left\{\varepsilon^{(2n+1)\frac{d}{d\varrho}}\chi_{n+1}(\pi)\right\};\end{aligned}$$

and in like manner we find the values of the succeeding symbolical coefficients.

I now come to the form of the binomial theorem which is reciprocal to that previously investigated.

To expand $(\pi^2+\theta(\varrho).\pi)^n$ in powers of (π).

Let us assume

$$(\pi^2+\theta(\varrho).\pi)^n=\pi^{2n}+\varphi_{2n-1}(\varrho).\pi^{2n-1}+\varphi_{2n-2}(\varrho).\pi^{2n-2}+\dots,$$

we know that

$$\pi^r\theta(\varrho)=\theta(\varrho).\pi^r+r\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho).\pi^{r-1}+r.\frac{r-1}{2}\left(\varrho\frac{d}{d\varrho}\right)^2\theta(\varrho)\pi^{r-2}+\dots$$

Hence, multiplying internally and externally by $\pi^2+\theta(\varrho)\pi$, we shall have

$$\begin{aligned}&\pi^{2n+2}+\varphi_{2n-1}(\varrho)\pi^{2n+1}+\varphi_{2n-2}(\varrho).\pi^{2n}+\varphi_{2n-3}(\varrho).\pi^{2n-1}+\dots\\&\quad+\left\{\theta(\varrho)\pi^{2n}+2n\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho)\pi^{2n-1}+2n.\frac{2n-1}{2}\left(\varrho\frac{d}{d\varrho}\right)^2\theta(\varrho)\pi^{2n-2}+\dots\right\}\pi\\&\quad+\varphi_{2n-1}(\varrho)\left\{\theta(\varrho).\pi^{2n-1}+(2n-1)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho)\pi^{2n-2}+\dots\right\}\pi\\&\quad+\varphi_{2n-2}(\varrho)\left\{\theta(\varrho).\pi^{2n-2}+(2n-2)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho).\pi^{2n-3}+\dots\right\}\pi\\&\quad+\&c.\\&=\pi^{2n+2}+\left\{\varphi_{2n-1}(\varrho).\pi^2+2\left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-1}(\varrho).\pi+\left(\varrho\frac{d}{d\varrho}\right)^2\varphi_{2n-1}(\varrho)\right\}\pi^{2n-1}\\&\quad+\left\{\varphi_{2n-2}(\varrho).\pi^2+2\left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-2}(\varrho).\pi+\left(\varrho\frac{d}{d\varrho}\right)^2\varphi_{2n-2}(\varrho)\right\}\pi^{2n-2}\end{aligned}$$

$$+ \&c. + \phi_\varrho \pi^{2n+1} + \theta(\varrho).\left(\varphi_{2n-1}(\varrho).\pi + \left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-1}(\varrho)\right)\pi^{2n-1}$$

$$+ \theta(\varrho)\left(\varphi_{2n-2}(\varrho).\pi + \left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-2}(\varrho)\right)\pi^{2n-2} + \&c.$$

Hence equating coefficients of the powers of (ϱ),

$$\varphi_{2n-2}(\varrho) + 2n\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho). + \varphi_{2n-1}(\varrho).\theta(\varrho) = 2\left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-1}(\varrho) + \varphi_{2n-2}(\varrho) + \theta(\varrho)\varphi_{2n-1}(\varrho),$$

whence

$$\varphi_{2n-1}(\varrho) = n\theta(\varrho),$$

$$\varphi_{2n-2}(\varrho) + 2n.\frac{2n-1}{2}\left(\varrho\frac{d}{d\varrho}\right)^2\theta(\varrho) + (2n-1)\varphi_{2n-1}(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho) + \varphi_{2n-2}(\varrho)\phi_\varrho$$

$$= \left(\varrho\frac{d}{d\varrho}\right)^2\varphi_{2n-1}(\varrho) + 2\left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-2}(\varrho) + \theta(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-1}(\varrho) + \theta(\varrho)\varphi_{2n-2}(\varrho) + \varphi_{2n-2}(\varrho),$$

whence

$$\varphi_{2n-2}(\varrho) = n(n-1)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho) + \frac{n(n-1)}{2}(\theta(\varrho))^2;$$

$$\therefore \quad (\pi^2 + \theta(\varrho).\pi)^n = \pi^{2n} + n\phi_\varrho.\pi^{2n-1} + \left\{n(n-1)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho) + \frac{n(n-1)}{2}(\theta(\varrho))^2\right\}\pi^{2n-2} + \ldots,$$

the required expansion.

XIV. *On the Calculus of Functions.* *By* W. H. L. Russell, *Esq.*, *A.B.* *Communicated by* Arthur Cayley, *Esq.*, *F.R.S.*

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One of the first efforts towards the formation of the Calculus of Functions is due to Laplace, whose solution of the functional equation of the first order, by means of two equations in finite differences, is well known. Functional equations were afterwards treated systematically by Mr. Babbage; his memoirs were published in the Transactions of this Society, and there is some account of them in Professor Boole's Treatise on the Calculus of Finite Differences. A very important functional equation was solved by Poisson in his Memoirs on Electricity; which suggested to me the investigations I have now the honour to lay before the Society.

I have commenced by discussing the linear functional equation of the first order with constant coefficients, when the subjects of the unknown functions are rational functions of the independent variable, and have shown how the solution of such equations may in a variety of cases be effected by series, and by definite integrals. I have then considered functional equations with constant coefficients of the higher orders, and have proved that they may be solved by methods similar to those used for equations of the first order. I have next proceeded with the solution of functional equations with variable coefficients.

In connexion with functional equations, I have considered equations involving definite integrals, and containing an unknown function under the integral sign; the methods employed for their resolution depend chiefly upon the solution of functional equations, as effected in this paper. The Calculus of Functions has now for a long time engaged the attention of analysts; and I hope that the following investigations will be found to have extended its power and resources.

Let the functional equation be

$$\varphi\left\{\frac{n^2+2n-8}{4r}-nx+rx^2\right\}-a\varphi(x)=\mathrm{F}(x),$$

where φ is an unknown, and F a known function.

Let

$$x=\frac{n}{2r}+\frac{2}{r}\cos z,$$

and the equation becomes

$$\varphi\left\{\frac{n}{2r}+\frac{2}{r}\cos 2z\right\}-a\varphi\left\{\frac{n}{2r}+\frac{2}{r}\cos z\right\}=\mathrm{F}\left\{\frac{n}{2r}+\frac{2}{r}\cos z\right\};$$

or if

$$\varphi\left\{\frac{n}{2r}+\frac{2}{r}\cos z\right\}=\varphi_1(z),$$

we shall have

$$\varphi_1(2z)-\alpha\varphi_1(z)=\mathrm{F}\left\{\frac{n}{2r}+\frac{2}{r}\cos z\right\}.$$

Let

$$z=\frac{1}{2^{v+1}},$$

then

$$\varphi_1\left(\frac{1}{2^v}\right)-\alpha\varphi_1\left(\frac{1}{2^{v+1}}\right)=\mathrm{F}\left\{\frac{n}{2r}+\frac{2}{r}\cos\frac{1}{2^{v+1}}\right\}.$$

Let

$$\mathrm{F}\left\{\frac{n}{2r}+\frac{2}{r}\cos\frac{1}{2^{v+1}}\right\}=\chi\left(\frac{1}{2^{v+1}}\right),$$

then

$$\varphi_1\left(\frac{1}{2^v}\right)-\alpha\varphi_1\left(\frac{1}{2^{v+1}}\right)=\chi\left(\frac{1}{2^{v+1}}\right),$$

and

$$\varphi_1\left(\frac{1}{2^v}\right)=\chi\left(\frac{1}{2^{v+1}}\right)+\alpha\chi\left(\frac{1}{2^{v+2}}\right)+\alpha^2\chi\left(\frac{1}{2^{v+3}}\right)+\ldots+\frac{\mathrm{C}}{\alpha^{v+1}}.$$

Now

$$\int_0^\infty\frac{(\chi(\epsilon^{i\theta})+\chi(\epsilon^{-i\theta}))d\theta}{1-2\alpha\cos\theta+\alpha^2}=\frac{2\pi}{1-\alpha^2}\chi(\alpha),$$

whence (α) is less than unity: then if

$$\epsilon^w=2^{v+n+1},$$

and

$$w=(v+1)\log_\epsilon 2+n\log_\epsilon 2,$$

also

$$f(\theta)=\chi(\epsilon^{i\theta})+\chi(\epsilon^{-i\theta}),$$

we shall have

$$\chi\left(\frac{1}{2^{v+n+1}}\right)=\frac{1}{2\pi}\int_0^\infty\frac{\epsilon^w-\epsilon^{-w}}{\epsilon^w-2\cos\theta+\epsilon^{-w}}f(\theta)d\theta$$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\infty d\theta d\varrho f(\theta).\frac{\epsilon^{(w-i\theta)\varrho}+\epsilon^{-(w-i\theta)\varrho}}{\epsilon^{w\varrho}-\epsilon^{-w\varrho}}\sin\{\varrho\log_\epsilon 2^{v+1}+n\varrho\log_\epsilon 2\};$$

$$\therefore\quad\chi\left(\frac{1}{2^{v+1}}\right)+\alpha\chi\left(\frac{1}{2^{v+2}}\right)+\alpha^2\chi\left(\frac{1}{2^{v+3}}\right)+\alpha^3\chi\left(\frac{1}{2^{v+4}}\right)+\ldots$$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\infty d\theta d\varrho f(\theta).\frac{\epsilon^{(w-i\theta)\varrho}+\epsilon^{-(w-i\theta)\varrho}}{\epsilon^{w\varrho}-\epsilon^{-w\varrho}}\cdot$$

$$\{\sin(\varrho\log_\epsilon 2^{v+1})+\alpha\sin(\varrho\log_\epsilon 2^{v+1}+\varrho\log_\epsilon 2)+\alpha^2\sin(\varrho\log_\epsilon 2^{v+1}+2\varrho\log_\epsilon 2)+\&c.\}$$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\infty d\theta d\varrho f(\theta).\frac{\epsilon^{(w-i\theta)\varrho}+\epsilon^{-(w-i\theta)\varrho}}{\epsilon^{w\varrho}-\epsilon^{-w\varrho}}.\frac{\sin(\varrho\log_\epsilon 2^{v+1})-\alpha\sin(\varrho\log_\epsilon 2^v)}{1-2\alpha\cos(\varrho\log_\epsilon 2^{v+1})+\alpha^2}.$$

Hence

$$\varphi(x)=\frac{1}{\pi}\int_0^\infty\int_0^\infty d\theta d\varrho f(\theta).\frac{\epsilon^{(w-i\theta)\varrho}+\epsilon^{-(w-i\theta)\varrho}}{\epsilon^{w\varrho}-\epsilon^{-w\varrho}}.\frac{\sin\left\{\varrho\log_\epsilon\dfrac{2}{\cos^{-1}\dfrac{2rx-n}{4}}\right\}-\alpha\sin\left\{\varrho\log_\epsilon\dfrac{1}{\cos^{-1}\dfrac{2rx-n}{4}}\right\}}{1-2\alpha\cos\left\{\varrho\log_\epsilon\dfrac{1}{\cos^{-1}\dfrac{2rx-n}{4}}\right\}+\alpha^2}$$

$$+\frac{\mathrm{C}}{\left(\cos^{-1}\dfrac{2rx-n}{4}\right)^{\frac{\log_\epsilon\alpha}{\log_\epsilon\frac{1}{2}}}},$$

where

$$f(\theta)=\mathrm{F}\left\{\frac{n}{2r}+\frac{1}{r}\cos\cos\theta(\epsilon^{\theta}+\epsilon^{-\theta})+\frac{i}{r}\sin\cos\theta(\epsilon^{\theta}-\epsilon^{-\theta})\right\}$$
$$+\mathrm{F}\left\{\frac{n}{2r}+\frac{1}{r}\cos\cos\theta(\epsilon^{\theta}+\epsilon^{-\theta})-\frac{i}{r}\sin\cos\theta(\epsilon^{\theta}-\epsilon^{-\theta})\right\}.$$

Next let the functional equation be

$$\varphi\left\{\left(\frac{n^3}{27r^3}-\frac{4n}{3r}\right)+\left(\frac{n^2}{3r}-3\right)x+nx^2+rx^3\right\}-\alpha\varphi(x)=\mathrm{F}(x).$$

Let

$$x=-\frac{n}{3r}+\frac{2}{\sqrt{r}}\cos z,$$

and the equation will be transformed into

$$\varphi\left\{-\frac{n}{3r}+\frac{2}{\sqrt{r}}\cos 3z\right\}-\alpha\varphi\left\{-\frac{n}{3r}+\frac{2}{\sqrt{r}}\cos z\right\}=\mathrm{F}\left\{-\frac{n}{3r}+\frac{2}{\sqrt{r}}\cos z\right\};$$

this may be written

$$\varphi_1(3z)-\alpha\varphi_1(z)=\chi(z).$$

Let

$$z=\frac{1}{3^{s+1}}$$

$$\varphi_1\left(\frac{1}{3^s}\right)-\alpha\varphi_1\left(\frac{1}{3^{s+1}}\right)=\chi\left(\frac{1}{3^{s+1}}\right),$$

whence

$$\varphi_1\left(\frac{1}{3_s}\right)=\chi\left(\frac{1}{3^{s+1}}\right)+\alpha\chi\left(\frac{1}{3^{s+2}}\right)+\alpha^2\chi\left(\frac{1}{3^{s+3}}\right)+\dots$$

The same method evidently applies, and we thus obtain the value of $\varphi(x)$.

In the same way we may treat the equation

$$\varphi(a_0+a_1x+a_2x^2+a_3x^3+\dots+a_mx^m)-\alpha\varphi(x)=\mathrm{F}(x),$$

where a_0, a_1, a_2, $\dots$ a_{m-2} must be supposed given in terms of a_{m-1}, a_m.

We will now consider the equation

$$\varphi\left\{\frac{(n-r)r^2+2rn^2+2r(rr'-nr'-2nn')x+((n-r)r'^2+2rn'^2)x^2}{(n'-r')r^2+2r'n^2+2r'(rr'-nr-2nn')x+((n'-r')r'^2+2r'n'^2)x^2}\right\}-\alpha\varphi(x)=\mathrm{F}(x).$$

Let

$$x=\frac{n+r\cos z}{n'+r'\cos z}$$

Then the equation becomes

$$\varphi\left\{\frac{n+r\cos 2z}{n'+r'\cos 2z}\right\}-\alpha\varphi\left\{\frac{n+r\cos z}{n'+r'\cos z}\right\}=\mathrm{F}\left\{\frac{n+r\cos z}{n'+r'\cos z}\right\},$$

which may be treated as before: as we may write it

$$\varphi_1(2z)-\alpha\varphi_1(z)=\chi(z).$$

Similar methods will evidently apply to the equation

$$\varphi\left\{\frac{a_0+a_1x+a_2x^2+\dots+a_mx^m}{b_0+b_1x+b_2x^2+\dots+b_mx^m}\right\}-\alpha\varphi(x)=\mathrm{F}(x).$$

The general linear functional equation of the nth order with constant coefficients is

$$\varphi(\psi^n x)+a\varphi(\psi^{n-1}x)+b\varphi(\psi^{n-2}x)+\ldots+h\varphi\psi(x)+k\varphi(x)=\mathrm{F}'(x),$$

where the subject ψx is supposed to be the same as in the preceding equations.

Let $x=\chi(z)$, and $\psi(x)=\psi\chi(z)=\chi(mz)$ suppose; then $\psi^2(x)=\chi m^2 z$, $\psi^3 x=\chi m^3 z$, &c., and the equation becomes

$$\varphi\chi(m^n z)+a\varphi\chi(m^{n-1}z)+b\varphi\chi(m^{n-2}z)+\ldots+h\varphi\chi(mz)+k\varphi\chi(z)=\mathrm{F}\chi(z).$$

Let $z=\frac{1}{m^{\nu+n}}$, and the equation becomes

$$\varphi\chi\left(\frac{1}{m^\nu}\right)+a\varphi\chi\left(\frac{1}{m^{\nu+1}}\right)+b\varphi\chi\left(\frac{1}{m^{\nu+2}}\right)+h\varphi\chi\left(\frac{1}{m^{\nu+n-1}}\right)+k\varphi\chi\left(\frac{1}{m^{\nu+n}}\right)=\mathrm{F}\chi\left(\frac{1}{m^{\nu+n}}\right).$$

This equation may be written

$$\left\{1+a\delta^{\frac{d}{dx}}+b\delta^{2\frac{d}{dx}}+\ldots h\delta^{(n-1)\frac{d}{dx}}+k\delta^{n\frac{d}{dx}}\right\}\varphi\chi\left(\frac{1}{m^\nu}\right)=\mathrm{F}\chi\left(\frac{1}{m^{\nu+n}}\right),$$

or

$$\left\{\left(1+\alpha_1\delta^{\frac{d}{dx}}\right)\left(1+\alpha_2\delta^{\frac{d}{dx}}\right)\ldots\left(1+\alpha_n\delta^{\frac{d}{dx}}\right)\right\}\varphi\chi\left(\frac{1}{m^\nu}\right)=\mathrm{F}\chi\left(\frac{1}{m^{\nu+n}}\right),$$

$$\therefore\quad \varphi\chi\left(\frac{1}{m^\nu}\right)$$

$$=\mathrm{A}_1\left\{\mathrm{F}\chi\left(\frac{1}{m^{\nu+n}}\right)-\alpha_1\mathrm{F}\chi\left(\frac{1}{m^{\nu+n+1}}\right)+\alpha_1^2\mathrm{F}\chi\left(\frac{1}{m^{\nu+n+2}}\right)+\&c.\right\}$$

$$+\mathrm{A}_2\left\{\mathrm{F}\chi\left(\frac{1}{m^{\nu+n}}\right)-\alpha_2\mathrm{F}\chi\left(\frac{1}{m^{\nu+n+1}}\right)+\alpha_2^2\mathrm{F}\chi\left(\frac{1}{m^{\nu+n+2}}\right)+\&c.\right\}$$

$$+\mathrm{A}_3\left\{\mathrm{F}\chi\left(\frac{1}{m^{\nu+n}}\right)-\alpha_3\mathrm{F}\chi\left(\frac{1}{m^{\nu+n+1}}\right)+\alpha_3^2\mathrm{F}\chi\left(\frac{1}{m^{\nu+n+2}}\right)+\ldots\right\}+\&c.$$

$$+\frac{\mathrm{C}_1}{(-\alpha_1)^\nu}+\frac{\mathrm{C}_2}{(-\alpha_2)^\nu}+\frac{\mathrm{C}_3}{(-\alpha_3)^\nu}+\&c.,$$

where A_1, A_2, A_3, &c. are certain functions of α_1, α_2, α_3, &c., and C_1, C_2, C_3, &c. are arbitrary constants, from whence we at once obtain the value of $\varphi(x)$.

We now proceed to consider functional equations with variable coefficients. And first let the equation be

$$\varphi(x)-\chi(x)\varphi\left\{\frac{a+bx}{c+ex}\right\}=\mathrm{F}(x),$$

where $\chi(x)$ and $\mathrm{F}(x)$ are known rational functions of (a).

Let

$$x=u_x,\quad \frac{a+bu_x}{c+eu_x}=u_{x+1},$$

then

$$u_{x+1}=\frac{a+bu_x}{c+eu_x}.$$

Suppose a solution of this equation of finite differences to be

$$u_x=\frac{\mathrm{A}+\mathrm{B}x}{\mathrm{C}+\mathrm{E}x},$$

the equation becomes

$$\varphi u_x - \chi(u_x)\varphi(u_{x+1}) = \mathrm{F}u_x.$$

Let

$$\chi u_x = \alpha \cdot \frac{(\alpha_1+x)(\alpha_2+x)(\alpha_3+x)\dots}{(\beta_1+x)(\beta_2+x)(\beta_3+x)\dots},$$

and the equation may then be written

$$\alpha^x \frac{\Gamma(\alpha_1+x)\Gamma(\alpha_2+x)\Gamma(\alpha_3+x)\dots}{\Gamma(\beta_1+x)\Gamma(\beta_2+x)\Gamma(\beta_3+x)\dots}\varphi(u_x) - \alpha^{x+1}\frac{\Gamma(\alpha_1+x+1)\Gamma(\alpha_2+x+1)\Gamma(\alpha_3+x+1)\dots}{\Gamma(\beta_1+x+1)\Gamma(\beta_2+x+1)\Gamma(\beta_3+x+1)\dots}\varphi(u_{x+1})$$
$$= \alpha^x \frac{\Gamma(\alpha_1+x)\Gamma(\alpha_2+x)\Gamma(\alpha_3+x)\dots}{\Gamma(\beta_1+x)\Gamma(\beta_2+x)\Gamma(\beta_3+x)\dots}\mathrm{F}u_x.$$

Hence

$$\varphi(u_x) = \mathrm{F}u_x + \alpha \cdot \frac{(\alpha_1+x)(\alpha_2+x)(\alpha_3+x)\dots}{(\beta_1+x)(\beta_2+x)(\beta_3+x)\dots}\mathrm{F}(u_{x+1}) + \alpha^2 \frac{(\alpha_1+x)(\alpha_1+x+1)(\alpha_2+x)(\alpha_2+x+1)\dots}{(\beta_1+x)(\beta_1+x+1)(\beta_2+x)(\beta_2+x+1)\dots}\mathrm{F}(u_{x+2})$$
$$+ \dots + \frac{\mathrm{C}}{\alpha^x} \cdot \frac{\Gamma(\beta_1+x)\Gamma(\beta_2+x)\dots}{\Gamma(\alpha_1+x)\Gamma(\alpha_2+x)\dots}.$$

$\mathrm{F}u_x$ is a rational function of (x), and may therefore be decomposed into a series of terms of the form

$$\mathrm{F}u_x = \frac{1}{h_1+k_1 x} + \frac{1}{h_2+k_2 x} + \frac{1}{h_3+k_3 x} + \dots$$

Hence

$$\varphi u_x = \frac{1}{h_1+k_1 x} + \alpha \frac{(\alpha_1+x)(\alpha_2+x)(\alpha_3+x)\dots}{(\beta_1+x)(\beta_2+x)(\beta_3+x)\dots} \cdot \frac{1}{h_1+k_1(x+1)} + \&c.$$
$$+ \frac{1}{h_2+k_2 x} + \alpha \frac{(\alpha_1+x)(\alpha_2+x)(\alpha_3+x)\dots}{(\beta_1+x)(\beta_2+x)(\beta_3+x)\dots} \cdot \frac{1}{h_2+k_2(x+1)} + \&c.$$
$$+ \frac{1}{h_3+k_3 x} + \alpha \frac{(\alpha_1+x)(\alpha_2+x)(\alpha_3+x)\dots}{(\beta_1+x)(\beta_2+x)(\beta_3+x)\dots} \cdot \frac{1}{h_3+k_3(x+1)} + \&c.$$
$$+ \frac{\mathrm{C}}{\alpha^x} \cdot \frac{\Gamma(\beta_1+x)\Gamma(\beta_2+x)\Gamma(\beta_3+x)\dots}{\Gamma(\alpha_1+x)\Gamma(\alpha_2+x)\Gamma(\alpha_3+x)\dots}.$$

We may obtain a multiple integral which shall be equivalent to any of the above series, by remembering that

$$\frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\dots(\beta+n-1)} = \frac{\Gamma\beta}{\Gamma\alpha\Gamma(\beta-\alpha)}\int_0^1 v^{\alpha+n-1}(1-v)^{\beta-\alpha-1}dv$$

$$\frac{1}{\beta(\beta+1)(\beta+2)\dots(\beta+n-1)} = \frac{\Gamma\beta \cdot s}{2\pi}\int_{-\infty}^{\infty}\frac{e^{iv}dv}{(1+iv)^{\beta+n}},$$

also

$$\frac{1}{h+kn} = \int_0^\infty e^{-v(h+kn)}dv,$$

and summing accordingly. We may hence immediately deduce the value of $\varphi(x)$.

It is evident that the functional equation

$$\varphi(a+bx+cx^2) - \alpha\frac{\psi(x)}{\psi(a+cx+cx^2)}\varphi(x) = \mathrm{F}(x),$$

when ψ_1 and F_1 are known functions of (x), may be reduced to the form

$$\varphi_1(a+bx+cx^2)-\alpha\varphi_1(x)=F(x)\psi(a+bx+cx^2)$$

by making

$$\varphi_1(x)=\psi(x)\varphi(x).$$

The second equation has already been considered; it becomes, therefore, interesting to ascertain what equations are included under the form of the first equation, in other words, to consider what forms the algebraical expression $\frac{\psi(x)}{\psi(a+bx+cx^2)}$ can possibly take.

The following are a few of them:—

$$\frac{\sqrt{(x+2\sqrt{ac}-b)}}{\sqrt{a}+\sqrt{c}.x},\quad \frac{x-a}{x(b+cx)},\quad \frac{x^2-a^2}{(2a+bx+cx^2)x(b+cx)},\quad \frac{\sqrt{\{x^2-2bx+b^2-4ac\}}}{a-cx^2}.$$

Many others may, in like manner, be imagined; and the same methods, *mutatis mutandis*, apply to functional equations of the higher orders with variable coefficients.

I now come to the consideration of equations involving definite integrals, when the equation contains an unknown function under the sign of definite integration.

Let us take the equation

$$\int_0^1 \frac{F_1(\alpha)+F_2(\alpha).x^2}{F_3(\alpha)+F_4(\alpha)x^2+F_5(\alpha).x^4}\,dx\varphi(x)=F(\alpha),$$

where $\varphi(x)$ is an unknown function of (x) not containing (α), $F_1(\alpha)$, $F_2(\alpha)\ldots$, $F_5(\alpha)$ rational functions of (α) which is supposed to vary independently of (x), to determine $\varphi(x)$.

Suppose the equation can be written in the form

$$\int_0^1 \frac{\lambda(x)dx}{\sqrt{1-x^2}}\left\{\frac{1-(\chi(\alpha))^2}{1-2\chi\alpha(1-2x^2)+(\chi\alpha)^2}+\frac{\mu(1-\alpha^2)}{1-2\alpha(1-2x^2)+\alpha^2}\right\}=F(\alpha),$$

where

$$\lambda(x)=\varphi(x)\sqrt{1-x^2}.$$

Let $x=\sin\frac{\theta}{2}$, and the equation reduces to

$$\int_0^\pi d\theta\lambda\left(\sin\frac{\theta}{2}\right)\left\{\frac{1-(\chi\alpha)^2}{1-2\chi\alpha\cos\theta+(\chi\alpha)^2}+\frac{\mu(1-\alpha^2)}{1-2\alpha\cos\theta+\alpha^2}\right\}=2F(\alpha);$$

or if

$$\psi(\alpha)=\int_0^\pi d\theta.\frac{(1-\alpha^2)\lambda\left(\sin\frac{\theta}{2}\right)}{1-2\alpha\cos\theta+\alpha^2},$$

we find

$$\psi\chi(\alpha)+\mu\psi(\alpha)=2F(\alpha).$$

Suppose the solution of this equation be determined by the former investigations to be

$$\psi(\alpha) \;=\; f(\alpha),$$

then

$$\int_0^\pi \frac{(1-\alpha^2)d\theta\lambda\left(\sin\frac{\theta}{2}\right)}{1-2\alpha\cos\theta+\alpha^2}=f(\alpha).$$

Assume

$$\lambda\sin\frac{\theta}{2}=a_0+a_1\cos\theta+a_2\cos 2\theta+a_3\cos 3\theta+\ldots,$$

then since

$$\frac{1-\alpha^2}{1-2\alpha\cos\theta+\alpha^2}=1+2\alpha\cos\theta+2\alpha^2\cos 2\theta+\ldots,$$

we have

$$\pi(a_0+a_1\alpha+a_2\alpha^2+\ldots)=f(\alpha);$$

$$\therefore \quad a_0+a_1\epsilon^{i\theta}+a_2\epsilon^{2i\theta}+\ldots=\frac{1}{\pi}f(\epsilon^{i\theta})$$

$$a_0+a_1\epsilon^{-i\theta}+a_2\epsilon^{-2i\theta}+\ldots=\frac{1}{\pi}f(\epsilon^{-i\theta}).$$

Hence

$$\lambda\left(\sin\frac{\theta}{2}\right)=\frac{1}{2\pi}\{f\epsilon^{i\theta}+f\epsilon^{-i\theta}\},$$

whence $\varphi(x)$ can be determined.

Similar treatment will of course apply to the equation

$$\int_0^1\frac{F_1\alpha+F_2\alpha x^2+F_3\alpha\,.\,x^4+\ldots}{F_n(\alpha)+F_{n+1}(\alpha)x^2+\ldots}\varphi(x)=F(\alpha),$$

but the functional equation employed for its solution (when possible by this method) will be of a higher order.

Let us, lastly, consider the equation

$$\int_0^1 F_1(x)\varphi(x\psi(\alpha))=F(\alpha),$$

to find φ, where F_1x is a known function of (x) not containing (α), and (α) varies independently of (x).

Let $\psi(\alpha)=\beta$, then $\alpha=\psi^{-1}\beta$, and the equation becomes

$$\int_0^1 F_1(x)\varphi(x\beta)=F\psi^{-1}(\beta).$$

Let

$$\varphi(x\beta)=A_0+A_1x\beta+A_2x^2\beta^2+\ldots,$$

then we shall have

$$A_0\int_0^1 dx F_1x+A_1\int_0^1 dx\,.\,xF(x)\,.\,\beta+A_2\int_0^1 dx x^2F(x)\,.\,\beta^2+\ldots$$

$$=F\psi^{-1}0+F'\psi^{-1}0\,.\,\beta+F''\psi^{-1}0\,.\,\frac{\beta^2}{1.2}+\ldots;$$

then

$$A_0=\frac{F\psi^{-1}(0)}{\int_0^1 dx F_1x},\quad A_1=\frac{F'\psi^{-1}0}{\int_0^1 dx\,.\,xF(x)}\ldots$$

Hence

$$\varphi(\xi)=\frac{F\psi^{-1}0}{\int_0^1 dx F_1x}+\frac{F'\psi^{-1}0}{\int_0^1 dx\,.\,xF_1x}\,.\,\xi+\frac{F''\psi^{-1}0}{\int_0^1 dx\,.\,x^2F_1x}\,.\,\frac{\xi^2}{1.2}+\frac{F'''\psi^{-1}0}{\int_0^1 dx\,.\,x^3\,.\,F_1x}\,.\,\frac{\xi^3}{1.2.3}+\ldots,$$

ξ being any variable.

Now suppose

$$\frac{1}{\int_0^1 dx\,.\,x^n F_1x}=\chi(n),$$

and that we are able to express $\chi(n)$ by a definite integral, so that

$$\chi(n)=\int f(v)(\lambda(v))^n dv,$$

the integral being supposed to have certain finite limits, we shall have

$$\varphi(\xi)=\int f(v)\mathrm{F}\psi^{-1}(\xi\lambda(v))\,.\,dv.$$

Thus if

$$\int_0^1(x+1)\varphi(x\psi(\alpha))=\mathrm{F}(\alpha),$$

we have

$$\varphi(\xi)=\tfrac{1}{2}\xi\frac{d}{d\xi}\mathrm{F}\psi^{-1}(\xi)+\tfrac{3}{4}\mathrm{F}\psi^{-1}\xi-\frac{1}{8}\int_0^\infty s^{-\frac{3v}{4}}dv\mathrm{F}\psi^{-1}(\xi s^{-v}).$$

XV. *On the Difference in the Magnetic Properties of Hot-Rolled and Cold-Rolled Malleable Iron, as regards the power of receiving and retaining Induced Magnetism of Subpermanent Character.* By George Biddell Airy, *Astronomer Royal.*

Received April 22,—Read May 15, 1862.

In reflecting on the differences exhibited by different iron-built ships in the change of their subpermanent magnetism, it has often occurred to me as a subject worthy of experimental investigation, whether a portion of this difference might not depend on the temperature at which the plates of iron are passed through the rollers in the last stage of their manufacture. No favourable opportunity of making these experiments presented itself until, in the course of the last winter, I became aware that Mr. Fairbairn had been engaged in experiments on the difference of the strength of plates of malleable iron, according as they had been rolled at a high or at a low temperature. I immediately requested Mr. Fairbairn's kind offices for procuring for me bars adapted to magnetic experiment, divided into the four classes of—1. Hot-Rolled, with the length of the bars parallel to the direction in which the rolling had lengthened the iron, or parallel to the direction of fibre; 2. Hot-Rolled, with the length of the bars transverse to the direction of fibre; 3. Cold-Rolled, with the length of the bars parallel to the direction of fibre; 4. Cold-Rolled, with the length of the bars transverse to the direction of fibre (which classes will hereafter be described by the words, 1. Hot-Rolled Longitudinal; 2. Hot-Rolled Transversal; 3. Cold-Rolled Longitudinal; 4. Cold-Rolled Transversal). Upon Mr. Fairbairn's application, the bars which I requested were promptly and gratuitously furnished by Richard Smith, Esq., Superintendent of Lord Dudley's Iron Works at the Round Oak Works near Dudley.

The number of bars was 24, namely, 6 in each of the four classes above described. Each bar was 16 inches long, 4 inches broad, and about $\frac{1}{4}$ inch thick: the aggregate weight of the bars in each class was,—1st, 28 lbs. 8 oz.; 2nd, 28 lbs. 10 oz.; 3rd, 27 lbs. 10 oz.; 4th, 27 lbs. 8 oz. The manufacture of the bars is described to me in substance as follows:—The hot-rolled and cold-rolled bars were all manufactured in the same way up to the stage of producing sheets of iron of the desired thickness; the last rollings having commenced with large bars at a welding heat, and having terminated with the bars (now converted into sheets) at a dull red heat. Then the sheets to be cold-rolled were allowed to cool to a perfectly cold state, and in that state were rolled afresh between other rollers. After this, the experimental bars were cut out of the sheets. Each set of six bars was packed in one box, with the maker's inscription on every bar reading forward in the same direction in all.

The bars when received by me, after resting some days in a room, were all placed

upright in the same direction relatively to the direction of the maker's inscription, and the distinctive number of the bar was painted on the upper end of each; the end on which it was painted being that which is called "the Lettered End." The bars rested thus for several days, with the lettered end upwards.

The following apparatus was prepared for the experiments:—A wooden frame was constructed about 11½ feet in length; and this length, in the use of the frame, was placed very approximately in the direction of magnetic E. and W. Its ridge was 21 feet N. of the old front, or 13 feet N. of the new front, of the anteroom of the Magnetic Observatory. The upper and essential part of the frame consisted of two planes, each about 2 feet broad; of which one was very approximately in the position transverse to the direction of dip at Greenwich; and the other, at right angles to the former, included in its plane the direction of dip. These are called "Equatorial Plane" and "Dip Plane" respectively. Ledges of wood were attached to the planes, for the support of flagstones resting on the planes and lying parallel to them. And whether the wooden surface or the stone surface was employed, a frame of laths was placed upon it, which retained the length of each bar in the position nearest to the vertical, and prevented one bar from touching another. In this position the bars were struck with a hammer. If they were upon the equatorial plane, any induced magnetism was instantly struck out of them; if upon the dip plane, they became powerfully magnetized, the lower end having the same properties as the marked end of a compass-needle, or being charged with austral magnetism.

For testing the magnetism, a vertical wooden rod was provided, carrying two horizontal planes or stages, also of wood. The upper stage supported a prismatic Kater's compass, with which a well-defined mark (the ball-mast) was viewed, and its apparent azimuth was read. The lower stage supported the bar under experiment, which was placed in a horizontal position below the compass, at the distance of 5 inches, as nearly as could be conveniently measured, between the centre of the compass-needle and the centre of the bar. The reading of the compass-card under view increases as the card turns in the direction opposite to that of the sun's diurnal motion. From this it will appear that when the end of a bar, which has been downwards on the dip surface, is placed eastward on the lower plane, the austral magnetism of that end will attract the needle's south end towards the east, and will cause the card-reading to increase.

The bars were numerated as follows:—

(Hot-Rolled Longitudinal) Nos. 1, 2, 3, 4, 5, 6 (painted in white);
(Hot-Rolled Transversal) Nos. 7, 8, 9, 10, 11, 12 (painted in black);
(Cold-Rolled Longitudinal) Nos. 13, 14, 15, 16, 17, 18 (painted in red);
(Cold-Rolled Transversal) Nos. 19, 20, 21, 22, 23, 24 (painted in blue).

In the conduct of the experiments, the following rules were uniformly followed:—

The bars of the different classes were systematically intermixed, the same order being preserved in the whole series of experiments. Thus the order in which they were placed upon either plane (equatorial or dip), and the order in which their effects in disturbing the compass were observed, was always the following:—

Order of position, or experiment.	No. of bar.	Order of position, or experiment.	No. of bar.	Order of position, or experiment.	No. of bar.
1	1	9	3	17	5
2	7	10	9	18	11
3	13	11	15	19	17
4	19	12	21	20	23
5	2	13	4	21	6
6	8	14	10	22	12
7	14	15	16	23	18
8	20	16	22	24	24

The bars, when on the equatorial plane, were struck in the order 1 to 24 of the "Order of Position, or Experiment;" when on the dip plane, they were struck in the opposite order. They were always placed on the experimental stage under the compass in the order 1 to 24 of the "Order of Position, or Experiment." A printed skeleton form had been prepared, and the entries in it of the compass-readings were made without the slightest risk of confusion. The lettered end of the bar was always in the first instance placed towards the east; and, as soon as the compass-card was read, the bar was reversed in length, and its lettered end placed towards the west, and the compass-card was again read. The difference between these two readings (which has been carefully verified by examination, and by collateral formation and difference of sums of readings) is the number recorded in this paper. The zero, or card-reading when no bar was near, was usually taken at the beginning and at the end of each experiment, merely to enable me to detect erroneous readings, but no further use was made of it.

In two instances in Experiment 13, the presumption of error of 5° (undetected at the proper moment) was so strong, and the uncertainty as to the side on which the error was made was so great, that I judged it best to omit them, and to multiply the result for four bars by $\frac{3}{2}$, in order to produce the result for six bars.

The "Diff. readings" in the following Tables denotes the excess, with its proper algebraical sign, of the compass-reading when the lettered end of the bar is East above the compass-reading when the lettered end is West, and is taken as measure of the intensity of magnetism of the bar. When the bars have been struck with the lettered end downwards, the "Diff. readings" is positive.

Experiment 1, 1862, February 6, 0^h.

The bars were brought from the painting-room, and were immediately tried.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+18° 5′	7	+ 5° 15′	13	+ 1° 0′	19	+25° 20′
2	+13 30	8	+ 7 40	14	+ 8 0	20	+14 5
3	+14 30	9	+ 7 45	15	+ 7 20	21	+18 20
4	+ 7 0	10	+ 7 30	16	+ 1 10	22	+12 50
5	+11 20	11	+ 6 50	17	+20 30	23	+ 6 20
6	+12 10	12	+ 3 40	18	+ 7 0	24	+17 30
Sum...	+76 35	Sum...	+38 40	Sum...	+45 0	Sum...	+94 25

Total sum . . . +254° 40′.

The sign of the magnetism in this experiment is opposite to that which would have been given by terrestrial induction upon the bars in the position in which they had been standing for many days. It appears therefore that the magnetism had been produced by some circumstance in the manufacture, and that the terrestrial action upon the bars in a quiescent state had not reversed or destroyed it.

The bars were deposited on the flagstones of the equatorial plane, lettered end upwards.

Experiment 2, February 7, 0^h.

The bars, without being struck or subjected to any other violence, were tried.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+18° 0′	7	+ 3° 0′	13	− 2° 10′	19	+25° 0′
2	+13 45	8	+ 6 45	14	+ 6 30	20	+13 20
3	+12 20	9	+ 6 0	15	+ 5 25	21	+18 10
4	+ 5 0	10	+ 4 10	16	+ 0 20	22	+11 50
5	+11 35	11	+ 4 25	17	+19 40	23	+ 5 30
6	+10 30	12	+ 3 10	18	+ 7 15	24	+17 40
Sum...	+71 10	Sum...	+27 30	Sum...	+37 0	Sum...	+91 30

Total sum . . . +227° 10′.

The magnetism, it appears, had very slightly diminished. The bars were returned to the equatorial plane.

Experiment 3, February 7, 1^h.

Each bar as it lay on the flagstones of the equatorial plane, lettered end upwards, was struck with three rather heavy blows, from the unlettered to the lettered end. The hammer used was a geological hammer, weighing (with handle) about $2\frac{3}{4}$ lbs., and it fell in the blows about 2 feet. The bars were immediately examined.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	−2° 10′	7	− 2° 20′	13	−4° 35′	19	+2° 40′
2	+0 55	8	− 2 40	14	+0 40	20	−1 30
3	+0 50	9	− 2 0	15	−7 45	21	+3 50
4	−4 40	10	− 1 20	16	−2 15	22	+0 50
5	−1 20	11	− 7 20	17	+0 10	23	−3 50
6	−2 10	12	− 6 0	18	+6 25	24	+7 0
Sum...	−8 35	Sum...	−21 40	Sum...	−7 20	Sum...	+9 0

Total sum . . . −28° 35′.

Upon the whole, the antecedent magnetism is destroyed, and a small amount of mag-

netism of the opposite kind is left, but it is distributed very capriciously among the bars. It seems to have no relation to the former magnetism.

The bars were immediately placed on the flagstones of the dip plane, with lettered end upwards, and were immediately struck with the same hammer and in the same manner as above, but were not examined on this day.

Experiment 4, February 7, 22^h.

The bars had been resting on the flagstones of the dip plane since Feb. 7, 1^h, with lettered end upwards, and without further disturbance were taken from it for examination.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	− 22° 10′	7	− 21° 0′	13	− 27° 0′	19	− 18° 40′
2	− 19 50	8	− 19 10	14	− 23 40	20	− 25 40
3	− 18 0	9	− 17 30	15	− 30 30	21	− 14 30
4	− 21 20	10	− 23 20	16	− 29 40	22	− 20 50
5	− 21 20	11	− 20 40	17	− 25 15	23	− 26 10
6	− 19 40	12	− 20 30	18	− 25 20	24	− 30 0
Sum...	−122 20	Sum...	−122 10	Sum...	−161 25	Sum...	−135 50

Total sum . . . −541° 45′.

The bars were immediately returned to the flagstones of the dip plane, in the same position as before, with lettered end upwards, and were immediately struck for the next experiment.

Experiment 5, February 7, 23^h.

The bars on the flagstones of the dip plane, with lettered end upwards, were struck each with three blows, with the same hammer and with the same force as in Experiment 5.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	− 21° 20′	7	− 19° 10′	13	− 25° 5′	19	− 20° 0′
2	− 20 30	8	− 18 40	14	− 23 40	20	− 25 10
3	− 16 40	9	− 18 10	15	− 28 40	21	− 17 25
4	− 19 50	10	− 17 10	16	− 29 0	22	− 20 27
5	− 16 40	11	− 15 10	17	− 23 10	23	− 27 40
6	− 20 20	12	− 20 40	18	− 25 20	24	− 31 0
Sum...	−115 20	Sum...	−109 0	Sum...	−154 55	Sum...	−141 42

Total sum . . . −520° 57′.

The magnetism of the bars is, upon the whole, slightly diminished. The bars were

placed on the flagstones of the dip plane, with lettered end downwards, and were left undisturbed in that state.

Experiment 6, February 10, 23^h.

The bars, which had been resting undisturbed on the dip plane with lettered end downwards for three days, were taken out for examination without striking or disturbance.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	−16° 10′	7	−12° 35′	13	− 17° 50′	19	− 17° 40′
2	−14 20	8	−14 10	14	− 21 10	20	− 23 10
3	−15 0	9	−14 40	15	− 24 20	21	− 14 30
4	− 8 40	10	−13 40	16	− 23 10	22	− 17 30
5	−15 30	11	−12 30	17	− 19 20	23	− 23 35
6	−14 20	12	−16 0	18	− 20 30	24	− 23 0
Sum...	−84 0	Sum...	−83 35	Sum...	−126 20	Sum...	−119 25

Total sum . . . −413° 20′.

About $\frac{1}{3}$th part of the magnetism has been destroyed by three days' exposure to antagonistic terrestrial magnetism.

The bars were returned to the dip-plane flagstones in the same position, with lettered end downwards.

Experiment 7, February 11, 0^h.

The bars, immediately after the last examination, were returned to the flagstone dip plane with lettered end downwards, and each bar was struck once lightly at its centre with a joiner's hammer, weighing with its handle about $\frac{1}{2}$ lb., and were immediately examined.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+12° 10′	7	+16° 10′	13	+ 9° 0′	19	+17° 0′
2	+11 40	8	+15 30	14	+15 0	20	+11 20
3	+15 20	9	+12 0	15	+13 10	21	+16 30
4	+ 8 0	10	+11 0	16	+ 8 20	22	+ 9 40
5	+12 10	11	+10 15	17	+16 10	23	+ 8 40
6	+12 40	12	+10 40	18	+15 30	24	+ 5 40
Sum...	+72 0	Sum...	+75 35	Sum...	+77 10	Sum...	+68 50

Total sum . . . +293° 35′.

The light blow has destroyed the former magnetism, and has given an opposite magnetism of nearly three-fourths its amount.

The bars were returned to the flagstones of the dip plane in the same position, with lettered end downwards.

Experiment 8, February 12, 0^h.

The bars had not been disturbed in any way since the last examination. They had been resting on the flagstones of the dip plane, with lettered end downwards.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+14 30	7	+17 10	13	+ 7 55	19	+16 30
2	+11 25	8	+10 20	14	+16 0	20	+ 9 45
3	+11 55	9	+ 8 50	15	+12 10	21	+16 15
4	+ 8 20	10	+ 8 50	16	+ 9 40	22	+10 10
5	+11 0	11	+10 30	17	+13 30	23	+ 8 30
6	+11 10	12	+ 9 5	18	+13 15	24	+ 5 30
Sum...	+68 20	Sum...	+64 45	Sum...	+72 30	Sum...	+66 40

Total sum . . . +272° 15′.

It is remarkable that the magnetism, though favoured by the inductive force of terrestrial magnetism, has in this day's rest somewhat diminished.

The bars were returned to the flagstones of the dip plane, with lettered end downwards.

Experiment 9, February 12, 1^h.

The bars on the flagstones of the dip plane, with lettered end downwards, were struck each three times (from top to bottom) with the light hammer and with light blows, and were immediately examined.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+15 55	7	+16 0	13	+14 10	19	+18 15
2	+11 5	8	+14 10	14	+17 0	20	+14 30
3	+12 55	9	+12 20	15	+14 10	21	+18 50
4	+11 15	10	+13 15	16	+14 15	22	+18 5
5	+13 5	11	+12 55	17	+15 50	23	+10 25
6	+12 30	12	+12 40	18	+16 10	24	+11 0
Sum...	+76 45	Sum...	+81 20	Sum...	+91 35	Sum...	+91 5

Total sum . . . +340° 45′.

The magnetism is not much increased.

The bars were placed on the equatorial plane flagstones.

Experiment 10, February 15, 1^h.

The bars had been lying undisturbed on the equatorial plane, lettered end downwards.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+1° 20′	7	+1° 0′	13	+1° 30′	19	+1° 45′
2	+11 25	8	+ 9 55	14	+13 5	20	+12 50
3	+10 20	9	+ 9 45	15	+ 7 55	21	+17 5
4	+ 6 40	10	+10 30	16	+ 7 5	22	+16 55
5	+ 9 40	11	+10 35	17	+10 25	23	+ 6 55
6	+ 7 30	12	+ 9 50	18	+ 9 20	24	+ 9 5
Sum...	+56 55	Sum...	+62 35	Sum...	+59 20	Sum...	+78 35

Total sum . . . +257° 25′.

No specific cause is known for this decided decrease of magnetism.

The bars were returned to the flagstone equatorial plane.

Experiment 11, February 15, 2^h.

The bars, lying on the flagstones of equatorial plane, with lettered end downwards, were struck each in the middle of its length, with a single blow of the light hammer. The hammer was raised about 18 inches.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+1° 20′	7	+0° 35′	13	− 4 20	19	+3° 25′
2	−0 20	8	−0 25	14	− 1 10	20	−1 40
3	−0 35	9	−0 55	15	− 5 55	21	+3 50
4	−3 0	10	−0 55	16	− 5 5	22	+0 15
5	+0 30	11	−0 20	17	− 2 35	23	−5 30
6	−1 20	12	−1 35	18	− 2 25	24	−5 30
Sum...	−3 25	Sum...	−3 35	Sum...	−21 30	Sum...	−5 10

Total sum . . . −33° 40′.

The bars were returned to the flagstones of equatorial plane, lettered end downwards.

Experiment 12, February 20, 23^h.

The flagstones had been removed, and the bars were lying on the wooden boards of the equatorial plane, with the lettered end upwards. Each bar was slightly rapped three times with a wooden mallet, weighing about $1\frac{7}{8}$ lb.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
	° ′		° ′		° ′		° ′
1	+ 1 0	7	− 1 20	13	− 4 50	19	+ 0 10
2	− 1 50	8	− 2 30	14	− 1 30	20	− 1 55
3	− 1 5	9	− 2 20	15	− 7 10	21	+ 3 5
4	− 4 5	10	− 2 40	16	− 6 20	22	− 0 35
5	− 1 30	11	− 1 50	17	− 3 30	23	− 6 10
6	− 3 20	12	− 2 40	18	− 4 25	24	− 3 25
Sum...	− 10 50	Sum...	− 13 20	Sum...	− 27 45	Sum...	− 8 50

Total sum . . . −60° 45′.

It would appear that, after a certain diminution, the shocks on the equatorial plane have no tendency to reduce the magnetism further.

The bars were placed on the wood of the dip plane, with lettered end upwards.

Experiment 13, February 21, 0^h.

The bars upon the boards of the dip plane, with lettered end upwards, were fairly struck each three times with the mallet.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
	° ′		° ′		° ′		° ′
1	−18 20	7	.....	13	− 19 25	19	− 15 5
2	−15 30	8	−14 20	14	− 19 10	20	− 20 5
3	−17 50	9	−15 40	15	− 23 40	21	− 10 25
4	−17 55	10	.....	16	− 24 20	22	− 14 0
5	−15 50	11	−14 0	17	− 19 55	23	− 20 5
6	−14 25	12	−14 10	18	− 20 0	24	− 21 20
Sum...	−99 50	Sum...	−87 15	Sum...	−126 30	Sum...	−101 0

Total sum . . . −414° 35′.

The readings for bars 7 and 10 appeared suspicious, and have been omitted, the "sum" for the second series being formed by multiplying the sum for bars 8, 9, 11, 12 by $\frac{4}{3}$. It scarcely differs from that which would have been given by the use of the recorded numbers.

The bars were returned to the wood of the dip plane, with lettered end downwards.

Experiment 14, February 21, 1^h.

The bars upon the wood surface of the dip plane, with lettered end downwards, were struck each with one light blow of the wooden mallet.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+ 8 5	7	+ 7 15	13	+ 1 20	19	+ 4 10
2	+ 6 50	8	+ 3 35	14	+ 6 55	20	− 0 10
3	+ 5 20	9	+ 5 10	15	+ 3 45	21	+ 6 55
4	+ 5 20	10	+ 5 20	16	− 0 50	22	− 1 50
5	+ 8 20	11	+ 7 55	17	+ 8 50	23	+ 0 40
6	+ 8 25	12	+ 6 35	18	+ 9 55	24	+ 3 15
Sum...	+42 20	Sum...	+35 50	Sum...	+29 55	Sum...	+13 0

Total sum . . . +121° 5′.

The bars were placed on the wood of the dip plane, with lettered end upwards, and were struck each three times with the wooden mallet, and so left. The mallet was raised about a foot for each blow.

Experiment 15, February 22, 0^h.

The bars were not disturbed before examination.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	− 19 10	7	− 16 40	13	− 23 15	19	− 18 30
2	− 18 50	8	− 18 50	14	− 24 5	20	− 21 10
3	− 17 50	9	− 18 15	15	− 26 55	21	− 13 55
4	− 18 30	10	− 19 35	16	− 24 10	22	− 17 55
5	− 17 35	11	− 15 50	17	− 22 40	23	− 24 45
6	− 15 45	12	− 18 25	18	− 23 35	24	− 26 45
Sum...	−107 40	Sum...	−107 35	um ...	−144 40	Sum...	−123 0

Total sum . . . −482° 55′.

The bars were placed on the wood of the equatorial plane, lettered end downwards.

Experiment 16, March 24, 0^h.

The bars had been lying 30 days undisturbed on the equatorial plane.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	−16 50	7	−14 50	13	− 20 15	19	− 15 5
2	−14 10	8	−14 40	14	− 19 30	20	− 18 55
3	−14 20	9	−12 45	15	− 20 50	21	− 10 45
4	−14 15	10	−13 30	16	− 18 20	22	− 13 55
5	−13 40	11	−11 50	17	− 18 55	23	− 21 10
6	−12 15	12	−13 45	18	− 19 20	24	− 22 35
Sum...	−85 30	Sum...	−81 20	Sum...	−117 10	Sum...	−102 25

Total sum . . . −386° 25′.

Experiment 17, March 24, 1^h.

Immediately after the last examination, the bars were placed on the wood of the equatorial plane, lettered end downwards, and were struck with the wooden mallet very lightly, the mallet being lifted about 2 inches, each bar three times. As soon as the series of blows was finished, the bars were struck again in the same manner. They were then immediately examined.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	−11° 5′	7	−11° 45′	13	−17° 40′	19	−11° 20′
2	−10 15	8	−11 10	14	−15 5	20	−14 15
3	− 7 5	9	−10 15	15	−13 30	21	− 7 35
4	−10 30	10	− 9 45	16	−11 55	22	−11 45
5	−11 5	11	− 9 50	17	−12 50	23	−19 0
6	− 9 10	12	−11 30	18	−15 50	24	−20 50
Sum...	−59 10	Sum...	−64 15	Sum...	−86 50	Sum...	−85 5

Total sum . . . −295° 20′.

The bars were placed on the wood of the equatorial plane, lettered end upwards.

Experiment 18, March 24, 2^h.

The bars on the wood of the equatorial plane, lettered end upwards, were struck with the mallet, each bar three times, the mallet being raised more than 1 foot; and the series of blows was then repeated. The bars were examined immediately.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	− 1° 45′	7	− 3° 30′	13	− 1° 5′	19	+ 1° 30′
2	− 2 0	8	− 1 5	14	+ 0 50	20	− 1 5
3	− 2 5	9	− 0 35	15	− 3 10	21	+ 3 45
4	− 1 25	10	− 0 5	16	− 2 15	22	+ 1 20
5	− 1 10	11	− 0 5	17	− 0 55	23	− 3 35
6	− 0 15	12	− 0 40	18	+ 0 10	24	− 0 15
Sum...	− 8 40	Sum...	− 6 0	Sum...	− 6 25	Sum...	+ 1 40

Total sum . . . −19°.25′.

The bars were returned to the wood of equatorial plane, lettered end upwards.

Experiment 19, March 25, 22^h.

The bars were shifted to the wood of dip plane, with lettered end downwards, and were struck heavily with the geological hammer swung out horizontally to arm's length, each bar three times. They were examined immediately.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+ 21° 45′	7	+ 19° 40′	13	+ 24° 55′	19	+ 28° 10′
2	+ 18 25	8	+ 19 0	14	+ 26 15	20	+ 24 55
3	+ 17 25	9	+ 20 25	15	+ 21 30	21	+ 27 5
4	+ 15 20	10	+ 15 30	16	+ 23 25	22	+ 26 25
5	+ 18 15	11	+ 16 35	17	+ 22 10	23	+ 20 15
6	+ 15 55	12	+ 15 50	18	+ 24 45	24	+ 29 10
Sum...	+107 5	Sum...	+107 0	Sum...	+143 0	Sum...	+156 0

Total sum . . . +513° 5′.

The bars were returned to the wood of dip plane, lettered end upwards.

Experiment 20, March 25, 23^h.

The bars, on wood of dip plane, lettered end upwards, were struck heavily with the geological hammer at arm's length, each three times. They were then immediately placed with lettered end downwards, and struck with blows of similar force, each three times. They were then examined immediately.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+ 21° 20′	7	+ 18° 50′	13	+ 25° 55′	19	+ 27° 25′
2	+ 17 30	8	+ 18 30	14	+ 24 35	20	+ 19 35
3	+ 16 15	9	+ 18 10	15	+ 24 10	21	+ 27 0
4	+ 17 0	10	+ 19 25	16	+ 24 55	22	+ 26 55
5	+ 20 0	11	+ 16 40	17	+ 25 0	23	+ 23 0
6	+ 17 35	12	+ 16 30	18	+ 26 30	24	+ 35 10
Sum...	+109 40	Sum...	+108 5	Sum...	+151 5	Sum...	+159 5

Total sum . . . +527° 55′.

It is worthy of remark that the diff. reading for the bar No. 20 (8th in the order of examination) at first appeared to be 24° 25′. After examining bar No. 5, bar No. 20 was again examined, and the diff. readings was found to be 19° 35′ as is given above. If the first reading was correct (which there is no special reason for doubting), it would appear that the bar lost one-fifth part of its magnetism in a few minutes.

The bars were returned to the dip plane, lettered end upwards. On March 27, 1^h, they were moved to the equatorial plane, lettered end upwards.

Experiment 21, March 31, 1^h.

The bars had been resting undisturbed on the equatorial plane since March 27.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+12° 40′	7	+12° 25′	13	+ 16° 5′	19	+ 23° 35′
2	+13 40	8	+15 10	14	+ 21 30	20	+ 19 30
3	+12 55	9	+16 15	15	+ 17 40	21	+ 21 40
4	+15 5	10	+14 15	16	+ 19 0	22	+ 22 50
5	+15 55	11	+15 35	17	+ 22 30	23	+ 22 10
6	+15 50	12	+15 5	18	+ 24 35	24	+ 32 5
Sum...	+86 5	Sum...	+88 45	Sum...	+121 20	Sum...	+141 50

Total sum . . . +438° 0′.

The bar No. 20 does not appear to have undergone any further change in its magnetism.

After this, the apparatus was dismounted, and the bars were packed up in the same order as at first.

I shall now proceed to exhibit the principal results deducible from the details above.

For abbreviation, I shall sometimes use the expression "tendency 0" to denote that the operation on the bars has been performed while they were lying on the equatorial plane; "tendency +" to denote that the bars were lying on the dip plane, with lettered end downwards; "tendency −" to denote that they were on the dip plane with lettered end upwards.

I. Aggregate of magnetism in the twenty-four bars.

Exp. 1.	After standing several days, tendency −	+254° 40′
2.	After resting one day, tendency 0	+227 10
3.	Struck heavily, on flagstones, large iron hammer, tendency 0 . .	− 28 35
4.	Struck heavily, tendency −; examined after a day's rest, with tendency −	−541 45
5.	(Immediately after Exp. 4), struck heavily, tendency −	−520 57
6.	Not struck, but under + tendency for three days	−413 20
7.	(Immediately), struck lightly with light iron hammer, tendency +	+293 35
8.	After resting quietly one day, under + tendency	+272 15
9.	(Immediately), struck fairly with light iron hammer, tendency +	+340 45
10.	After three days' rest, tendency 0	+257 25
11.	(Immediately), struck fairly with light iron hammer, tendency 0	− 33 40

From this time the bearing is on wood, and the blows are given with a wooden mallet, unless otherwise mentioned.

Exp. 12.	Lightly struck, tendency 0	− 60° 45′
13.	Fairly struck, tendency −	−414 35
14.	Very lightly struck, tendency +	+121 5
15.	Fairly struck, tendency −, rested one day	−482 55
16.	After thirty days' rest, tendency 0	−386 25
17.	Struck very lightly, tendency 0	−295 20
18.	Struck fairly, tendency 0	− 19 25
19.	Struck heavily with geological hammer, tendency +	+513 5
20.	(Immediately), first struck heavily, tendency −; then struck heavily (immediately), tendency +	+527 55
21.	After one day's rest, tendency −, and four days' rest, tendency 0	+438 0

It appears that the greatest amount of magnetism is about $\pm 530°$, and this requires a heavy blow; but it is nearly or quite indifferent whether the bars are supported on stone or on wood, and whether the blow is given by an iron hammer or a wooden mallet. This amount of magnetism, the bars lying with tendency 0, is diminished in one or two days by about one-fifth part, and is scarcely diminished further in thirty days. The loss of magnetism is not greater when the bars, instead of lying with tendency 0, are placed in a tendency opposite to the magnetism with which they are charged. When the charge of magnetism is smaller than the maximum, the diminution in a day or two is nearly in the same proportion.

It appears also that the effect of violence on the bars while lying with tendency 0 is not completely to destroy the magnetism; and that sometimes the magnetism is actually increased by the violence.

II. Proportions of the Aggregate of Magnetism carried by the different classes of Bars; excluding those experiments in which the Bars had been struck while lying with tendency 0.

	Hot-rolled longitudinal.	Hot-rolled transversal.	Cold-rolled longitudinal.	Cold-rolled transversal.
Experiment 4, tendency −	·226	·225	·297	·250
5, −	·222	·209	·298	·272
6, −	·204	·203	·306	·289
7, +	·245	·258	·263	·235
8, +	·251	·238	·266	·245
9, +	·225	·239	·269	·267
10, +	·221	·244	·231	·306
13, −	·241	·211	·306	·244
14, +	·349	·295	·247	·107
15, −	·223	·222	·299	·255
16, −	·221	·211	·303	·266
17, −	·201	·218	·293	·288
19, +	·209	·208	·279	·305
20, +	·207	·205	·287	·302
21, +	·196	·202	·277	·323
Mean	·229	·226	·281	·264

These results appear to leave no doubt that, as a general rule, the cold-rolled iron will, under similar circumstances, take up a heavier charge of magnetism than the hot-rolled iron, in the proportion of 545 : 455, or 1·2 : 1·0, very nearly. It also appears that the cold-rolled longitudinal takes a heavier charge than the cold-rolled transversal.

The departures from this result, in Experiments 8, 9, 10, 14, are confined to small charges of magnetism, in which accident might have greater play. They are also confined to magnetism of + character; of this, perhaps, the following is the explanation.

The bars when received from the manufacturer had + magnetism (see Experiments 1 and 2). It would seem therefore that the hot-rolled bars differ from the cold-rolled bars in this particular, that the hot-rolled bars retain rather more of their primitive magnetism, under all changes, than the cold-rolled bars retain.

III. Proportions of the Aggregate of Magnetism, without regard of sign, carried by the different classes of Bars, when they have been struck while lying with tendency 0.

	Hot-rolled longitudinal.	Hot-rolled transversal.	Cold-rolled longitudinal.	Cold-rolled transversal.
Experiment 3.	·184	·464	·157	·193
11.	·102	·106	·638	·154
12.	·178	·220	·457	·145
18.	·380	·264	·282	·073
Mean	·211	·264	·384	·141

I do not attach great importance to the apparent tendency of the cold-rolled longitudinal bars to retain much magnetism and of the cold-rolled transversal bars to retain little. The observed deviations are small, and a trifling accidental error may materially disturb the results.

IV. Spontaneous Losses of Magnetism in the Bars of different Classes, when they have rested undisturbed.

	Hot-rolled longitudinal.	Hot-rolled transversal.	Cold-rolled longitudinal.	Cold-rolled transversal.	Sum.
Exp. 1 to 2, 1 day, tendency 0	° ′ 5 25	° ′ 11 10	° ′ 8 0	° ′ 2 55	° ′ 27 30
5 to 6, 3 days, tendency +, opposing	31 20	25 25	28 35	22 17	107 37
7 to 8, 1 day, +, favouring	3 40	10 50	4 40	2 10	21 20
9 to 10, 3 days, 0	19 50	18 45	32 15	12 30	83 20
15 to 16, 30 days, 0	22 10	26 15	27 30	20 35	96 30
20 to 21, 1 day, —, opposing; 4 days, 0 ...	23 35	19 20	29 45	17 15	89 55
Sum.....	106 0	111 45	130 45	77 42	426 12

V. Proportions of the Losses of Magnetism in the Bars of different Classes to the Aggregate of Losses.

	Hot-rolled longitudinal.	Hot-rolled transversal.	Cold-rolled longitudinal.	Cold-rolled transversal.
Experiment 1 to 2.	·197	·407	·291	·106
5 to 6.	·293	·236	·266	·207
7 to 8.	·172	·509	·218	·101
9 to 10.	·238	·225	·387	·150
15 to 16.	·230	·272	·285	·214
20 to 21.	·263	·215	·331	·192
Means	·232	·311	·296	·162

It appears, as I think (though the last line of Table IV. would lead to a different conclusion), that there is a real difference in the retentive powers for magnetism among the different classes of bars; and that the cold-rolled bars, under the circumstances of the experiments, lose less magnetism, spontaneously, than the hot-rolled bars lose. If, instead of comparing the absolute losses of magnetism, we had compared the proportions for loss of magnetism in each class to entire magnetism in that class, the difference would have been still more remarkable.

XVI. *On the Relations of the Vomer, Ethmoid, and Intermaxillary Bones.*

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In a paper read before the Royal Physical Society of Edinburgh, in March 1859*, I have drawn attention to the fact, not before noticed, that in Mammalia the lateral masses of the ethmoid always lie edge to edge with the dilated part of the vomer, and are usually united with it to form one bone. I have shown that in the human subject the sphenoidal spongy bones represent, in a disrupted and altered form, the plates by which in other mammals this union is effected; and these plates I have ventured to call the ethmovomerine laminæ.

A further inquiry into the variations of these bones and their development has not only confirmed my former statements, but led to other conclusions of most important morphological bearing.

It will be convenient to consider, in the first place, the typical connexions of the vomer in the mammal. We shall then examine the peculiar modifications which the vomer, ethmoid, and intermaxillaries undergo in the human subject. Afterwards we shall review in detail the varieties of their connexions in a number of mammalian families, and endeavour to deduce their morphological relations. Lastly, we shall glance at the arrangement of these bones in the other classes of Vertebrata, and draw such additional morphological conclusions as our observations shall seem to warrant.

Typical Connexions of the Mammalian Vomer.

In all young mammals the septal cartilage of the nose extends backwards to the presphenoid bone, and the vomer embraces its inferior edge. The vomer is known to be developed from two centres of ossification, one on each side; it always consists therefore fundamentally of two alæ; and between these a groove runs along its whole length, and is occupied by the septal cartilage.

Anteriorly it articulates generally with the mesial-palatine processes of the intermaxillaries. Sometimes this articulation does not take place, to wit, where the intermaxillaries have either no mesial-palatine processes or very short ones; but, when it exists, it is always of such a character that the vomerine groove is continued forwards by a groove between the intermaxillaries, while also the inferior aspect of the mesial processes of the intermaxillaries lies in a continuous line with the inferior edge of the vomer.

* Edinburgh New Philosophical Journal, October 1859.

Since the vomerine groove extends from the presphenoid to the intermaxillaries, it is obvious that the primary laminæ of the vomer (its alæ) can have no contact with the bones forming the back part of the palate. The articulation which the vomer forms usually with the maxillaries, and often with the palatals, is brought about by the production, downwards and backwards from the line of junction of the alæ, of a mesial plate, which is an after-development, and of varying extent in different species.

Between the vomer and the central plate of the ethmoid the only connexion that ever occurs is of a very secondary description. No doubt the appearances in human skulls are often such as would incline one to a different conclusion; but in the skulls of other animals there is, even in the adult condition, in the majority of cases, no contact between the two bones at all; and when there is, it is a contact of contiguity, not of continuity. The central plate of the ethmoid grows from above downwards at the expense of the septal cartilage, and is an ossification of that structure; while, on the other hand, the vomer only embraces the cartilage, never invades it.

But the most intimate and constant connexion of the vomer is with the lateral masses of the ethmoid; although no one who had confined his attention to the arrangement in adult human skulls would think that these bones were at all related. This connexion can be very well seen in the Dog, the Cat, the Sheep, or the Pig. In all these animals, and indeed, as far as I am aware, in all mammalia except Man, the Orang, the Horse, the Elephant, and the Giraffe (keeping the Cetacea out of view at present), the vomer and lateral masses of the ethmoid form one continuous bone. The free superior edge of each ala of the vomer can be traced all the way back in contact with the mesial structures embraced; and, forming a sharp angle with this edge when examined from above, but appearing to be gradually continuous with the ala when looked at from beneath, a very thin but often broad plate of bone passes outwards underneath the ethmoidal convolutions to the external inferior angle of the lateral mass of the ethmoid, and is continuous with the principal lamina which forms the framework of that bone. This plate is what I have called the ethmovomerine lamina. The vomer frequently extends some distance backwards beyond its junction with the ethmovomerine laminæ; but the position of the latter is so far constant that their posterior edges are always in contact with the sphenoidal processes of the palate-bones. The place of junction of the ethmovomerine lamina and ethmoid always corresponds to the upper margin of the internal aspect of the foramen called in human anatomy sphenopalatine, but which has only a seeming relation to the sphenoid bone in the human subject, and none whatever in other mammals, and which it will be better therefore in this paper to call the nasal foramen of the palate-bone.

The Vomer and Sphenoidal Spongy Bones in Man.

If we now proceed to examine the development of the vomer and the bones connected with it in the human subject, we shall see how beautifully the adult appearances are derived from conditions exactly corresponding to those just described. The key to these

appearances is to be found principally in the sphenoidal spongy bones. These curious little bones reach their maximum of development at a very early age. Of four specimens beside me illustrating this condition, No. 1 (Plate IV. fig. 1) is from an infant possibly about a year old; No. 2 (fig. 2) from a child two or three years old; No. 3 (fig. 3) from a child which, judging from the appearances of the rest of the skull, may have been four years old; and No. 4 from an infant a few months old. In all these cases the body of the presphenoid is distinguishable from the postsphenoid by a mark at the line of junction, and projects forwards in continuation of it, in a somewhat rounded form, presenting little of that compression which has taken place to so great an extent in the adult. The presphenoid at this date is quite unconnected with the sphenoidal sinuses, and consists simply of the body just described, and the wings of Ingrassias. The body dips between the sphenoidal spongy bones, but is separated from them by some thickness of tissue. In Nos. 1 and 2, the sphenoidal spongy bones are already adherent to the ethmoid; in 3 and 4 they are unattached either to the sphenoid or ethmoid. They are of the shape of hollow pyramids with their apices directed backwards, and their inner aspects parallel to their fellows. Their cavities (the commencing sphenoidal sinuses) open at their bases in front into the nasal fossæ.

There are, I believe, three distinct ossifications which go to the formation, normally, of each sphenoidal spongy bone: certainly there are three portions which require description. A superior plate, very distinctly marked off in the specimens alluded to, limits the sphenoidal sinus more or less perfectly on the superior and internal aspects; it forms a convexity directed upwards and inwards, and lies in contact with the fibrous tissue which separates it from the body of the sphenoid. An inferior plate forms the whole or the greater part of the floor of the sphenoidal sinus, and may be said to be the only recognizable part in the adult. Internally this plate comes in contact with the lower margin of the superior plate, beneath which it is prolonged downwards so as to lie edge to edge with the corresponding lamina of the vomer, immediately in front of the thick dilated part. Posteriorly it lies above the sphenoidal process of the palate-bone. Besides these two plates, there enters properly into the formation of the sphenoidal spongy bone a third element (an orbital plate) connected with the anterior and external part of the inferior plate, and whose characteristic property is that by an aspect which articulates behind with the sphenoid, in front with the ethmoid, inferiorly with the palatals, and sometimes above with the frontal, it takes a small part in the formation of the orbit. It may be attached to the ethmoid, palatal, or sphenoid bone, instead of to the remainder of the sphenoidal spongy bone. In specimen No. 3 it is certainly unusually large and distinct, but it can be easily distinguished in Nos. 1 and 2 also. It is absent in No. 4; but there it exists incorporated with the orbital process of the palate-bone, at least that process projects upwards between the sphenoid and ethmoid. This enlarged form of the orbital process of the palate-bone is not at all uncommon; and that it arises from union with it of the orbital element of the sphenoidal spongy bone I am satisfied, from its occupying the position of that element. Moreover, in an adult speci-

men, in which I carefully disarticulated a palate-bone with an orbital process of that description, I found that that process had the whole sphenoidal spongy bone in union with it*.

It will be seen from this description that the sphenoidal spongy bone forms an arch of communication between the ethmoid and exactly that part of the vomer from which in other mammals the ethmovomerine lamina springs. Also it is it, and not the sphenoid, which completes with the palate-bone the sphenopalatine foramen; thus it agrees entirely with the ethmovomerine lamina in its relations. If there be yet any link wanting to prove its identity with that lamina, we shall find it supplied when we examine the skull of the young Orang.

In a fœtus of the third or fourth month, the vomer consists merely of two alæ which meet beneath the septal cartilage and form one bone. The inferior edge of the scooped bone so formed presents, in a fœtus of the fifth or sixth month, a broad surface marked by a raphe in the middle line for articulation with the maxillaries strictly so called, *i. e.* the part behind the anterior palatine foramen (fig. 4, B). This surface narrows behind into a mere edge to articulate with the palatals; while in front it ceases abruptly, and only the lamina bounding the groove is prolonged on the intermaxillary part of the palate. Already the intermaxillaries have begun to be elevated in the middle line above the level of the maxillaries, so as to form the crista incisiva, consisting of a process on each bone ("semicrista incisiva" of Henle), and a groove between the two on such a level as to continue the groove of the vomer, which it does throughout life.

In the skulls of young subjects (for example in the specimen referred to as No. 1 (fig. 1, B), the vomer is seen in its most characteristic form. The alæ have reached their maximum of development. At their posterior extremity is seen the thick dilated part which passes back beneath the sphenoid, and lies between the upper extremities of the pterygoids, which are sometimes called *vaginal processes*, but improperly, since they only lie edge to edge with the vomer and never sheath it. In front of the dilatation the margins of the alæ continue for some distance to ascend as they pass forwards, and it is at this part, as far forwards as the points at which they begin to slope downwards, that the sphenoidal spongy bones are in contact with them. Also, by the elongation of the face, the vomerine groove has become sloped, while the extremity which rests on the crista incisiva is, by the increased elevation of that process, raised above the level of the maxillaries; and the space thus left between the groove and the hard palate is spanned by a mesial plate, about one line deep in front and three behind. As the face developes further this mesial plate becomes deeper, and sometimes there is a process continuous with the anterior perpendicular edge of the mesial plate, sent downwards between the incisive foramina, behind the crista incisiva (fig. 4, A). This downward process is occasionally a separate bone, and is so in a specimen of mine taken from a subject

* Henle mentions that the orbital process of the palate-bone "aids in closing the sphenoidal sinus when its wall is imperfect." The condition which he refers to is no doubt that mentioned in the text. See Henle, Handbuch der System. Anat. des Menschen, i. 174.

possibly about ten or twelve years old. But apparently the most frequent arrangement in the adult is that the crista incisiva is prolonged backwards so as to articulate with the maxillaries behind the anterior palatine canal; and in that case, as the maxillary margin of the vomer falls short of the canal, the process in question does not exist.

At an early age the vomer begins to lose its characteristic form and its individuality; and the sphenoidal spongy bones lose theirs even sooner. In consequence of this, the ordinary descriptions, however faithfully they may give an account of the appearances in adult skulls, are imperfect as descriptions of these separate bones. The fact is that the maturity of a skull does not correspond with the period of most perfect development of the individual bones of which it is composed. This is recognized by us when we resort to young skulls for specimens of disarticulated bones, and would be acknowledged still more readily in describing an elephant's skull or a whale's; but, with regard to the bones in question, the period of this most characteristic condition is so early that it has escaped due attention, and the natural result has followed, that the adult appearances have not been properly understood.

The alterations which the sphenoidal spongy bones undergo take place in connexion with the increased hollowing out of the sphenoidal sinuses. As these sinuses dilate, the superior plates of the spongy bones are pressed against the body of the presphenoid, which also becomes narrowed and deepened, until, instead of a thick bone in the middle line and a lamina supporting it on each side, we meet with only one central lamina between two large sinuses, and that deviating often a great distance to one side. The processes which projected downwards to the vomer disappear by the dilatation of the sinuses, until there is nothing left of them but a slight ridge corresponding to the edge of the vomer; inside which, what was once a perpendicular lamina forms part of the convex floor of the sinus, and on reaching the middle line is incorporated with the body of the presphenoid to form the septum sphenoidale. Thus the sphenoidal spongy bones become blended with the sphenoid, while at the same time their appearance of continuity with the vomer is gradually effaced; so much so, that the latter is habitually and correctly described as resting, in the adult, against their under surface. In the Anatomical Museum of this University there is a skull, seemingly from a subject about twelve years of age, in which the descending processes of the sphenoidal spongy bones have at the fore part not yet begun to be rounded away, but articulate accurately edge to edge with the vomer.

The sphenoidal and other sinuses no doubt have the important function of serving as caves for the reverberation of the voice, but they are likewise useful in giving lightness to the bones they occupy; and though in old persons they cease to communicate with the nose, they nevertheless grow larger and larger as age advances and the bones grow denser, until, as in a specimen beside me, the antrum of Highmore invades the malar bones, and the sphenoidal sinuses so hollow-out the sphenoid that the walls of the Vidian canals are laid bare and traverse them like tubes. In these aged subjects the septum sphenoidale is a very thin partition indeed, and is usually very much driven to one side.

The alterations which take place with growth in the form of the vomer are results of the peculiar conformation of the human face, which is developed with a view to speech and expression, and not, as in the lower animals, for the mere prehension of food. The more the maxilla projects the wider is the gape allowed; but in Man, especially in the civilized races, instead of there being any such projection, the palate is short and the incisors perpendicular; and, as adult life approaches, the nasal fossæ and antra of Highmore are enlarged to such an extent by downward growth of the maxillæ as to render the perpendicular height of the posterior nares quite a distinctive characteristic of the human skull. In consequence of this elongation of the face, the distance between the cribriform plate and the palate is rapidly increased. This distance the central plate of the ethmoid and the vomer must span; and as they stretch to accomplish this object, they become attenuated and most frequently lose their symmetry. Often indeed the first degeneration from its typical form which the vomer undergoes is a deviation of its laminæ from the middle line. This is found even in children.

As the central plate of the ethmoid grows downwards, its edge, which is advancing at the expense of the septal cartilage, corresponds with that structure in thickness. But only the edge is so thick; the rest of the plate gets thinner and thinner as it becomes elongated, and may even present irregular perforations. The vomer, on the other hand, embraces the cartilage from below. But when the central plate of the ethmoid, growing down from above, comes in contact with the alæ as they stretch upwards in the perichondrium, the ossifications from above and below unite, and if the middle line has been perfectly preserved, the cartilage between them is surrounded above, below, and on each side by bone. Very rarely this vestige of cartilage disappears entirely, and there is then a thin lamina of bone extending from the cribriform plate to the palate, and in it a diagonal of stronger bone stretching from the crista incisiva to the sphenoid. Generally, however, the deviation from the middle line is considerable, and the central plate of the ethmoid, though anchylosed with both alæ at the back part, is in front united more with one lamina than the other (usually most with that on the concave side of the septum), while the other tends to undergo atrophy. The cartilage between them is thus left more exposed on one side than the other, or sometimes is exposed on one side at one part, and on the other side further back. Occasionally, even in old subjects, a thread of cartilage can be traced back into one of the sphenoidal sinuses.

When the vomer is described as only grooved at its upper and back part, and exhibiting a cul-de-sac for the reception of the rostrum, the description is that of an irregular piece of bone consisting of the vomer and part of the ethmoid. The specimens chosen for study by the describers have been taken from skulls in which the vomer has already become thoroughly united to the central plate; and when it has been sought to separate it, the central plate has given way at its weakest part. In other instances the fracture is effected through the alæ of the vomer. It is utterly impossible to separate a vomer accurately from the central plate after the two bones have come into contact. The ridge which one so frequently meets with on one side of the vomer in specimens

purchased with disarticulated skulls, is formed by an atrophied ala of the vomer, or by the edge of the central plate of the ethmoid projecting to one side, which it may do in an infinite variety of ways.

The same cause accounts for the imperfect description of the inferior margin of the vomer. In particular, the projection which lies on the crista incisiva is very liable to be destroyed by anchylosis; and even when it is not anchylosed it often requires very careful disarticulation to exhibit it. Sometimes the crista itself undergoes changes. It may have the margins of its groove worn away, and present a flat surface having the fore part of the vomer imbedded between its halves; or it may be completely bent over to one side, so that one semicrista shall lie above the other, while the fore part of the vomer, lying between them, is necessarily distorted, but may yet admit of disarticulation. At an early age the united margins of the maxillaries behind the anterior palatine canal become elevated into a sharp ridge, and the margins of the originally flat surface of the vomer in contact with them bend down to grasp it, and become irregular, thin away, and disappear more or less completely. Vestiges of them, as they die away, may sometimes be seen still very distinct on careful disarticulation, even in adult subjects. The edges of the palate-bones often form a more prominent ridge than the maxillaries, rising into a spine a couple of lines or more in height; and the portion of vomer in contact with this is a thin edge.

These remarks, I believe, are sufficient to show that if the human vomer is to be described with accuracy in its character as a separate bone, it must be studied as it shows itself in early life*.

If it be asked why, even in the early condition, when its form and connexions most resemble the typical arrangement, the human vomer should yet differ from the arrangement in other mammals by having the arch of communication between it and the ethmoid a separate bone, some explanation will be arrived at by considering that in Man the ethmoid is very feebly developed, and that the distance between the ethmoid and vomer is increased,—circumstances which depend on the feeble development of the sense of smell, and on the rapid curvature of the arch of the cranium. To the cursory glance the human ethmoid presents a rather well-developed appearance, since it forms a large part of the inner wall of the orbit, and the orbital plate is a peculiarity confined to Man and the Quadrumana. Yet in Man the ethmoid is rudimentary, and in Monkeys still more so; and the existence of the orbital plate is a mere consequence of the great curvature of the cranial arch. In the generality of mammals the ethmoid is shielded by the frontal and palatals; but in Man the palatals are removed from it by the downward development of the face; while by the great size of the anterior lobes of the brain it is pushed from under the shelter of the frontal, and the cribriform plate is depressed into the horizontal plane, instead of occupying, as in the Cat, the Sheep, &c., an almost vertical position. But whereas in most mammalia the lateral masses of the ethmoid

* Of this the first Monro was well aware. See the Works of Alexander Monro, M.D., published by his son in 1781, p. 120.

are composed of large turbinations imbedded in a dense mass of small leaflets, and both leaflets and turbinations lie at right angles to the cribriform plate, in Man there are no such leaflets and only two stunted turbinated processes, which, instead of descending to fill the nasal cavity, immediately curve backwards so as to occupy as little room as possible. And not only is the ethmoid so rudimentary, but by the curvature of the cranial arch, the vomer is moved away from it; its anterior extremity sloping rapidly to a lower level, and its posterior being pushed backwards under the sphenoid. Of this latter fact we may be convinced by comparing the position of the human vomer with that of the same bone in the Cat. The Cat has got sphenoidal sinuses as well as Man, but they are floored-in entirely by projections downwards from the wings of INGRASSIAS; and the part of the vomer which is continued into the ethmovomerine laminæ lies, not underneath the sinuses, but underneath the ethmoid, immediately in front of the sinuses. Moreover, it is very important to keep in mind that the margin of the ethmoid which is placed posteriorly in the human skull is that which is morphologically inferior, and which corresponds to that to which the ethmovomerine lamina is habitually attached, and to which therefore it behoved that the human vomer should be connected. This will be best illustrated by using the vomer and ethmoid of the Sheep, Dog, or Rabbit for comparison. It will then be easily understood, on referring to Plate IV. fig. 5, how there comes to be a space left between the ethmoid and vomer, to bridge over which the large and peculiar development of ethmovomerine laminæ as sphenoidal spongy bones becomes necessary (fig. 5).

Varieties presented by the Vomer and Intermaxillaries in different Classes of Mammalia.

In the *Quadrumana* the vomer comes to a point anteriorly, which is so directed that the vomerine groove is continuous, as in other animals, with the upper surface of the intermaxillaries; but it never comes further forwards than just to touch them, and sometimes falls short of them by a slight interval. This is owing to a want of development of the mesial-palatine processes of the intermaxillaries, these bones having in the Quadrumana begun to suffer that atrophy which they undergo in the human subject; and as their bulk is occupied almost entirely by the sockets of the large incisor teeth, while, on the other hand, the cartilages of the nose are little developed, there is no appearance of the crista incisiva found in Man. The ethmoid and vomer are continuous, according to the normal plan, save only in the Orang; and we may therefore confine our attention to the arrangement in it and in the Chimpanzee. In two skulls of Chimpanzees in the University Museum, the posterior part of the vomer arches out on each side into a lamina which passes up in front of the sphenoid to be continuous with the orbital plate of the ethmoid, as in other Quadrumana; but both are skulls of animals of such an age that they present anchylosis of many sutures. In the Orang, however, there are certainly separate sphenoidal spongy bones. The examination of three young Orang skulls gives the following results. In the largest skull, the sphenoidal spongy bone is quite distinct on the right side, both from the orbital plate of the ethmoid and

from the vomer; but on the left side their suture of connexion with the ethmoid has begun to disappear, and it scarcely comes into contact with the vomer. In the smallest skull, the sphenoidal spongy bones are perfectly free from the ethmoid on both sides, and also free from the vomer, though in contact with it; but while that on the left side is perfectly free from the sphenoid, the right one is anchylosed with the anterior inferior margin of that bone. In the remaining skull, the sphenoidal spongy bones are separate from the orbital plates of the ethmoid, but are anchylosed with the sphenoid both in front of the foramen opticum and below, also with the vomer, and with a turbination of the ethmoid. In all three cases the sphenoidal spongy bones take part in the formation of the orbit, while the palatal has no orbital plate.

The shape of the posterior extremity of the vomer affords us a very distinctive difference between the skulls of the Orang and Chimpanzee, which I believe has not hitherto been noticed. In the Orang it is expanded, flat, and with irregular edges. In the Chimpanzee it is thick, comparatively narrow, with straight edges, bifurcated, and fitted into a groove so as to leave a canal between it and the sphenoid. These characters are well marked even in the extremely young skull. In the Gorilla the posterior extremity of the vomer has expanded irregular edges as in the Orang, while it is bifurcated and a canal is left above it as in the Chimpanzee* (fig. 6).

In the *Bat*, the intermaxillaries are extremely small; they do not come together in the middle line, and have no mesial-palatine processes; they therefore have no connexion with the vomer.

In the *Carnivora*, the vomer and ethmoid are quite typical in their arrangement. In the Cats, the Dogs, and the Bears, the mesial plate of the vomer reaches a short way back upon the palatals. In the new-born Kitten the sphenoid, vomer, and intermaxillaries are seen in a very instructive condition. The body of the presphenoid is not yet compressed into a septum, and the two projections from the wings of INGRASSIAS, which afterwards meet beneath so as to enclose the sinuses, are as yet but little processes; the vomer passes forwards in a straight line from the front of the body of the presphenoid; and beyond the vomer this line is continued by the intermaxillaries, while the surface for articulation of the vomer with the maxillaries is as yet a mere knob. This arrangement is calculated strongly to suggest the idea of a series of centra (fig. 7, D).

* In the engraving of the base of the skull of a Chimpanzee in the Trans. Zool. Soc. vol. iii. pl. 60, illustrating Professor OWEN's paper, "Osteological Contributions to the Natural History of the Chimpanzee," the canal along the middle line of the sphenoid is represented as completed behind the vomer by junction of the roots of the pterygoid bones, or in other words, by the vaginal processes meeting together in the middle line. This completely conceals from view any parallel-bordered part of the vomer which may be supposed to rest between these and the body of the sphenoid. The skull represented is that of a male. The representation of the vomer of the Gorilla in pl. 63, illustrating the same paper, agrees with the description given above. My sketch was taken from a Gorilla's skull in the Museum of the Jardin des Plantes, Paris.

In the skull of a young male Chimpanzee in Dr. ALLEN THOMSON's possession the pterygoids have already almost met in the middle line, as represented in Professor OWEN's drawing, while the vomer is compressed between them. Most probably this is a sexual characteristic.

The posterior extremity of the feline vomer, as I have observed it in the Lion, Tiger, Leopard, and Cat (fig. 7, A), is of a well-marked constant form, viz. the posterior margin of the ethmovomerine laminæ is a straight line across, and behind it the vomer sends back a narrow process between the palatals. In the canine family the ethmovomerine laminæ are very broad, and their posterior margins slope backwards and inwards to the posterior extremity of the vomer (fig. 8).

In the Bears and their allies the mesial-palatine processes of the intermaxillaries come in contact with the lateral plates of the same bones behind the incisive foramina, and in the middle line between them there is also a characteristic foramen (fig. 9). This foramen varies very much in different species, and also in individuals. The *Ursus ornatus*, in three specimens in the Museum at the Jardin des Plantes at Paris, has it larger than any other bear's skull in that collection. The *Ursus maritimus* has it always of considerable size. In *Ursus arctos* and *Ursus americanus* it is small and variable. The Coati Mondi has this foramen very large; and I observe it, but small, in the Weasel, the Marten, the Glutton, and the Hyæna, very small in the Badger, and not at all in the Civet and the Otter. The inferior margin of the vomer of the Weasel, notwithstanding the length of the hard palate, is very short; it articulates with the intermaxillaries, and only for a very short distance with the maxillaries.

In the Seal the ethmoid is so compressed that the elements of which it is composed are brought close together; so close that beneath the nasal bones the central plate, which is slightly flattened out above, comes in contact with and is soldered to the outer plates of the lateral masses (fig. 10). There is no need in this instance of a special ethmovomerine lamina to unite the vomer and lateral mass; they are in contact, and are anchylosed directly with one another. At an early age the central plate of the ethmoid of the Seal becomes closely connected with the vomer; yet even after it has extended down to the vomerine groove, one can mark this distinction, viz. that the central plate replaces the septal cartilage, while the vomer surrounds it. The vomer merely touches the intermaxillaries, and does not reach back to the palate bones.

The arrangement in the Hedgehog is interesting in this respect,—that, owing, I presume, to the unarched form of the head, the inferior margin of the central plate of the ethmoid intervenes for about a third of an inch between the vomer and presphenoid; so that if the maxillary and palate bones be removed from a hedgehog's skull, there are seen extending in a straight line forwards, the bodies of the occipital, postsphenoid, and presphenoid bones, the central plate of the ethmoid with a slightly flattened edge lying between the ethmovomerine laminæ, then the vomer, and lastly, the mesial-palatine processes of the intermaxillaries; the whole presenting the appearance of a series of homologous elements (fig. 11).

Ruminantia.—In the Ox, the Sheep (fig. 12), the Deer, the Camel, and the Alpaca, the vomer has the typical arrangement. The inferior margin is not prolonged back sufficiently far to articulate with the palate bones, and its anterior extremity is scarcely prolonged upon the intermaxillaries, but fits on to them in such a way as to make their

mesial processes continuous with it. In the Giraffe, however, there is an exception to the usual connexions; the superior parts of the palatals floor-in the greater part of the ethmoidal turbinations, and the alæ of the vomer only come in contact for a very short distance in front of the palatals with prolongations inwards from the framework of the lateral masses of the ethmoid, but are not anchylosed to them. The arrangement is extremely similar to that in the Horse, even in the shape of the back part of the vomer.

There are some interesting varieties in the intermaxillaries of ruminants. In the animals which we have hitherto examined, the angle of junction of the mesial-palatine and lateral plates of the intermaxillary has been the most anterior part of that bone, and the groove for the septal cartilage has been open in front, so that, as in the human subject, the cartilage could be prolonged forwards beyond the intermaxillaries. But in the Sheep there is a slight, a very slight inclination of the lateral plates to prolong themselves forwards beyond the points at which the mesial palatine processes come off from them (fig. 13). These points are at a little distance from the middle line, and the prolongations forwards of the lateral plates are inclined inwards, so that in front of the mesial plates there is a little space left in the middle line. Into this space the anterior extremity of the septal cartilage, slightly dilated, dips down, and the tendency of the tips of the intermaxillaries is to embrace it. This arrangement is so faint in the Sheep that it would appear unworthy of attention, were it not that in the Camel and the Alpaca it is carried out to a most distinct and unmistakeable extent. In them the mesial-palatine processes, which are but slender in comparison with the lateral plates, arise at such a distance from the middle line that at their bases they are first directed inwards to meet one another before they are directed backwards. The space for the extremity of the septal cartilage is large, and almost converted into a foramen by the prolonged tips of the lateral plates approaching the middle line in front of it; and as they do so, they turn their inner aspects downwards, and embrace the cartilage on its upper border. Thus there can remain no doubt that the portion of the cartilage which projects through the space into the palate is really its anterior extremity (Plate V. fig. 14).

In the Giraffe this relationship of the intermaxillaries to the septal cartilage is very distinct. The mesial-palatine processes are very large, while the lateral plates are comparatively slender; but the continuation forwards of the latter to embrace the septal cartilage is well marked (Plate IV. fig. 15).

Pachydermata.—In the Horse, as already mentioned, the connexion of the vomer with the lateral masses of the ethmoid is of an exceptionally slight description. The leaflets of the ethmoid, which in most instances lie in contact for a considerable extent with the ethmovomerine laminæ, are in this case floored-in completely by the superior parts of the palate bones. But even in these circumstances a slender lamina, immediately in front of the palate bone, and in contact with its nasal foramen, passes downwards and inwards on each side from the ethmoidal turbinations, and articulates with the ala of the vomer, though it is not anchylosed to it (fig. 16). These peculiar relations in the skull of the Horse seem principally to depend on the palate bones being

pushed further forwards than usual; a circumstance which is probably connected with the habitual nasal respiration of the Horse*, as, by the posterior nares being placed well forwards, the most direct and free connexion between the nasal passages and respiratory organs is obtained. The lateral plates of the intermaxillaries of the Horse meet for some distance in the middle line in front of the septal cartilage, in the same way as we have seen them tending to do in the Camel; but the mesial-palatine processes arise quite from the fore part, and are in contact with each other at their origins, so that the extremity of the cartilage is enclosed completely in a cul-de-sac, and does not project into the palate (Plate V. fig. 17).

In the Pigs, the vomer articulates with the intermaxillaries, maxillaries, and palatals below, and forms one bone with the lateral masses of the ethmoid, according to the typical mode. The posterior extremity of the vomer is always very narrow, and the intermaxillaries are as in the Carnivora. We may notice also the little bone developed in the anterior extremity of the septal cartilage, in connexion with the snout.

In the skull of a sucking-pig, I have observed to advantage some of the more important morphological relations of the vomer (Plate IV. fig. 18). The central and cribriform plates of the ethmoid have not yet begun to ossify; there are separate centres of ossification in a number of the leaflets of the lateral masses, and larger ones in the great turbinated processes; and on each side the vomer is continuous (with only a slight trace of the junction) with a mass of bone consisting of the ethmovomerine lamina and part of the lateral mass of the ethmoid. It is but fair to add, that already the vomer is anchylosed to the body of the sphenoid. The articular surface on the inferior edge of the vomer for the maxillaries is flat, with a raphe in the middle line, and abruptly ceases where the intermaxillaries fit on; and the inferior surface of the mesial processes of the intermaxillaries is continuous with the maxillary surface of the vomer; a state of matters exactly similar to what we have noticed in the young human subject, and which may also be seen in the skull of a new-born puppy.

The Hippopotamus has the vomer and intermaxillaries arranged like those of the Pig. The ethmovomerine laminæ are broad, but in the specimen which I examined, although it was a well-grown one, there were suture markings between them and the vomer.

In the young Elephant (Plate V. fig. 19) the intermaxillaries come in contact with one another by means of large triangular surfaces, which reach to a considerable height above the level of the floor of the nares behind. They have no mesial-palatine processes. There is a small anterior palatine canal between them and the maxillaries in the middle line. The vomer is but a slightly developed bone: in the specimen before me it presents in the greater part of its extent superiorly a mere edge to articulate with the septal cartilage, and no vestiges of alæ. At the posterior part, however, it is bifurcated, and comes in contact with the lateral masses of the ethmoid, but is not anchylosed to them. At its

* Pointed out by Sir Charles Bell in his 'Anatomy and Philosophy of Expression,' 3rd edit. pp. 126 and 134.

anterior extremity it presents two processes; one passing into the anterior palatine canal, the other inclined somewhat upwards behind the surfaces of contact of the intermaxillaries. The superior and inferior margins of the vomer are thus rendered continuous with those of the intermaxillaries.

In the two-horned species of Rhinoceros the intermaxillaries are very small, and consist merely of lateral plates, without vestige of mesial-palatine processes; while the anterior extremity of the vomer stops short on the palate plates of the maxillaries, some distance behind their anterior margin. But in *Rhinoceros indicus* (which unfortunately I have not examined) a process of considerable size is described as arising from behind the superior margin of the intermaxillary; and in *R. tichorhinus*, not only is the osseous septum of the nose rendered complete by this process reaching up to the nasal bone, but the thick nasals are prolonged downwards to the level of the palate, and articulate with the anterior extremities of the intermaxillaries, so as to form a complete arch of bone in front of the nostrils, the only instance of this among Mammals*.

In the Tapir we have another instance of intermaxillaries without any mesial-palatine processes, and the anterior extremity of the vomer stopping short upon the maxillary bones; the intermaxillaries, however, are of considerable size, and come in contact with one another for a considerable distance in front of and above the septal cartilage, exactly as in the case of the Horse, only much more extensively (fig. 20).

Rodentia.—In the Rodentia the vomer is remarkable for the very little tendency it evinces to articulate with the maxillaries. Its superior connexions are typical, *i. e.* it is continuous with the lateral masses of the ethmoid, and the margin of junction of the ethmoid and ethmovomerine lamina enters into the formation of the nasal foramen of the palate bone, as seen from within. In all those that I have examined the vomer is bifid in front, and fits on edge to edge with those parts of the mesial processes of the intermaxillaries which bound the groove for the septal cartilage. The mesial processes of the intermaxillaries are well developed, and are expanded laterally in connexion with Jacobson's organs in a very characteristic fashion. The edges for articulation with the vomer are sometimes, as for instance in the Hare, very minute: but in other species with strong incisors they are well developed; thus they are of considerable size in the Paca (fig. 21); while in the Porcupine they are remarkably elevated, and the vomer sends down a process from its mesial plate between them.

In the Rat, the Beaver, the Porcupine, and the Paca, the vomer comes in contact with the anterior extremity of the maxillary part of the palate plate by a little, slightly dilated point; in the Squirrel it scarcely comes in contact. But the most interesting condition is seen in the Hare and Rabbit: they have only one single large foramen incisivum; for although the mesial-palatine processes of the intermaxillaries project well backwards, the palate plates of the maxillaries do not come far enough forwards to meet them: the vomer does not even approach the maxillaries, but its posterior margin terminates inferiorly in a thickened angle, which articulates with the intermaxillaries in such a manner as to be

* Cuvier, Ossemens Fossiles, tom. ii. Meckel, Anatomie Comparée (traduit), tom. iv. p. 273.

continuous with their inferior margin; and between this angle and the point where the vomerine groove is continued on to the intermaxillaries, there intervenes a considerable space in which the vomer comes in contact with the intermaxillaries by the edge of a triangular development of mesial plate (fig. 22).

In the *Edentata* the vomer has its superior connexions typical. In *Orycteropus* its inferior margin reaches back to the palatals, in *Manis* and *Myrmecophaga* it does not; and in *Bradypus* it only articulates with the anterior part of the maxillaries. The intermaxillaries are very small. They are smallest in *Bradypus*. In *B. didactylis* and *B. torquatus* they are separate; in *B. tridactylis* they are fused into one little plate, which is in contact with the maxillaries by its posterior and lateral angles, and anteriorly turns upwards upon the septal cartilage. In *Myrmecophaga* the intermaxillaries are somewhat better developed, but the incisive foramina being large, the mesial-palatine processes do not quite come in contact with the vomer, even though the latter projects some distance forwards beyond the maxillaries.

The intermaxillaries of the Armadillo are very characteristic. Their lateral plates are broad, and meet together in the middle line of the palate behind the incisive foramina. These foramina are small; and the mesial-palatine processes which separate them are slender, but are prolonged back upon the superior aspect of the line of junction of the lateral plates, so that thus the vomer rests upon the mesial-palatine processes, according to the general rule, and does not come in contact with the lateral plates.

In the *Marsupiata* the relations of the vomer are normal. Sometimes, as in the Koala, the inferior margin articulates with the palatals; sometimes, as in the Kangaroo, it does not. The groove for the septal cartilage is open in front, as in the Carnivora. In the Kangaroo (fig. 23) the ethmovomerine laminæ are very broad: the vomer exhibits on each side a peculiar lateral ridge, which extends forwards from the point where the ethmo vomerine lamina comes off, and articulates in front with a very long prolongation backwards of the mesial process of the intermaxillary, which extends between it and the maxillary. This ridge exists also in the Wombat, the Phalangers, and the Opossum; probably it is a constant marsupiate characteristic.

Cetacea.—In the Manati and Dugong we again meet with the arrangement of the vomer and intermaxillaries which we found in the Tapir, viz. the vomer is quite unconnected with the intermaxillaries, and the latter have no traces of mesial-palatine processes. The lateral plates of the intermaxillaries are very well developed, and meet each other above and in front of the septal cartilage, as they do in the Horse and the Tapir. In the Manati the anterior extremity of the vomer projects beyond the anterior margin of the maxillaries; in the Dugong it falls considerably short of it. In the Dugong the turbinations of the lateral masses of the ethmoid are very slight; but their framework is strong, and united by ethmovomerine laminæ to the vomer (fig. 24). In the Manati the lateral masses of the ethmoid are considerably more developed; they also are united to the vomer.

In the carnivorous Cetacea the vomer is largely developed, and its inferior margin

extends back the whole length of the palate. The intermaxillaries prolong the vomerine groove for a greater or less distance according to the genus. The distinctness of the central plate of the ethmoid from the vomer is very clearly seen, inasmuch as the former terminates abruptly where it becomes continuous with the cartilage in front, while the grooved surface of the latter is continued smoothly forwards. In vertical sections of skulls of the Dolphin and Grampus in the Museum of this University, although many sutures are obliterated, a straight line passing forwards from the inferior margin of the presphenoid indicates distinctly the place of contact of the still separate vomer and central plate of the ethmoid (fig. 25). The olfactory apparatus and lateral masses of the ethmoid are entirely absent; but the posterior parts of the alæ of the vomer are enormously expanded to take their place, and pass up on each side of the central plate of the ethmoid in front of the frontals, even as far as the nasals, forming the whole posterior wall of the nares, viz. the whole of that wall which in other Mammals is formed by the ethmoid. I notice also in *Deductor globiceps* that the expanded margin of the vomer, with the assistance of an angle of the maxillary, completes for the palate bone its nasal foramen (fig. 26).

Monotremata.—In the Ornithorhynchus the inferior margin of the vomer extends quite back to the posterior extremity of the hard palate, which is formed by the meeting of the pterygoids. In the Echidna it does not extend back so far. The condition of the intermaxillaries in the Ornithorhynchus is extremely interesting. The large bones which extend forwards from the maxillaries at a considerable distance from one another, and which give the form to the broad flat bill, are beyond all question intermaxillaries, and correspond exactly to the intermaxillaries of the Bat, or any other animal in which the mesial-palatine processes of these bones are not developed. Utterly unconnected with these, in front of the vomer, and continuing on its upper aspect the vomerine groove, is the little bone which has been recognized both by Cuvier and Meckel as corresponding with the mesial-palatine processes of intermaxillaries, which it no doubt does; but it is very interesting to find these represented by a bone so distinctly separated from the lateral plates of the intermaxillaries. It leads us to the consideration of the remarkable arrangement which exists in cases of cleft palate in the human subject, contrasted with the natural arrangement.

In cases of *complete cleft palate in the human subject* the inferior margin of the vomer is free, and is seen in the middle line of the open roof of the mouth. The intermaxillaries, fused or separate, articulate with its anterior extremity, and continue forwards in the same straight line, and support the incisor teeth when they are developed. They are entirely disconnected from the maxillaries, and are developed on the under aspect of the septal cartilage, which is a mesial structure; while the maxillaries are developed in lateral laminæ which, in these instances, fail to reach the middle line. Yet in the normal condition, marvellous as it may appear, the intermaxillary grows from the same centre of ossification as the maxillary *, and therefore must be con-

* See on this subject, in the 'Comptes Rendus' (Dec. 1858 and Jan. 1859), various papers by M. Ed.

sidered as at least commencing from what had originally been a lateral lamina. On the one hand, in the normal condition, the mesial processes of the intermaxillaries are obviously present; and in the cleft-palate condition, on the other hand, there can be no doubt that the lateral plates are represented as well as the mesial processes, since the incisor teeth are developed on the mesial bone. Thus it is certain that, according as the palate is completely closed or remains cleft, the whole intermaxillary—both mesial and lateral portion—comes from one or other of two parts which at an early period are always quite distinct. This is a fact of great importance.

Morphological Conclusions respecting the Vomer, Ethmoid, and Intermaxillaries in Mammals.

I had desired that this paper should be as strictly observational and as little theoretical or controversial in its character as might be; but I find it is impossible, now that I have arrived at this point, to refrain from indicating in what direction the observations just made appear to tend. I am conscious that by entering on this theme I render myself liable to the charge of presumption, in asking to be heard upon matters which have been discussed by the greatest authorities. My excuse is, that it is the observational part of my subject which compels me into the theoretical: and in venturing an opinion upon certain segments of the skull, I shall endeavour to limit myself as much as possible to what seem to me to be deductions to which the facts discussed inevitably lead; facts, some of them at least, not hitherto known, or not previously collated, and therefore not till now at the disposal of the theorist; but a knowledge of which, I cannot help thinking, is indispensable to the just conception of the segments to which they relate.

I gladly embrace this opportunity of acknowledging my obligations to Professor Goodsir for the use of books and specimens placed at my disposal, and for valuable information bearing upon topics treated of in this paper; and if, for the reason just mentioned, I have taken it upon me to adopt conclusions differing from his in certain details (although not more than from those of others), I do not forget that to him entirely I owe my morphological training; nor am I the less sensible of the advantage which I have enjoyed in being frequently indoctrinated by him in those great principles of Morphology which he illustrated in his communications to the British Association in 1856 *.

1. The first proposition which I shall make is, that the whole septum of the nose is continued forwards from the line of centra formed by the basilar process of the occipital and the bodies of the postsphenoid and presphenoid †. This statement can be best verified in some animal in which the body of the presphenoid is more developed than in

Rousseau, showing that the intermaxillaries are at no period normally separate from the maxillaries; and by M. Larcher, showing that in cleft palate they are distinct.

* Edinburgh New Philosophical Journal, Jan. 1857, pp. 118–181.

† The embryological aspect of this proposition is considered below at page 315.

Man. Take, for example, the Cat (fig. 7, B & C). The posterior aspect of the body of the presphenoid in the Cat exactly corresponds to the anterior extremity of the postsphenoid, presenting an appearance exactly similar to what the body of a vertebra exhibits on removal of its epiphyses; viz. a part in the middle, broad below and narrow above, formed by the centrum, and the two superior angles formed by the alæ. If we look at the same bone from the front, we see plainly that the septum between the sphenoidal sinuses consists entirely of the anterior extremity of the centrum greatly compressed; and just as the posterior extremity of the presphenoidal centrum is continuous with the postsphenoidal, so is its anterior extremity continuous with the septal cartilage of the nose and the central plate of the ethmoid. This continuity is in some respects even better seen in the new-born Kitten, as above described. Now we know that, morphologically, it is of little importance whether cranial bones are developed in the primordial cartilage of the skull, of which the septal cartilage is the remains, or are developed round it. The basisphenoid in fishes, for example, is developed round it, while in mammals it is developed in it. Therefore the central plate of the ethmoid, the vomer, and the mesial processes of the intermaxillaries all claim from their position to be centra as much as the basioccipital and basisphenoid. With regard to the mesial processes of the intermaxillaries, that they play the part of a centrum is shown, not merely by their constant relation to the septal cartilage, but by their articulation with the vomer being of such a description as to make them continuous with that bone both on the upper and under aspect, and by their condition in cleft-palate.

2. The vomer, lateral masses of the ethmoid, and palate-bones belong to one segment. This seems to be an altogether unavoidable conclusion, from the constant connexion of these bones in so invariable a manner.

What meaning can be attached to the remarkable way in which, in the human subject, the vomer and lateral masses of the ethmoid, notwithstanding their altered forms and positions, preserve the relations to one another which they exhibit in other animals, but that they are members of one segment? And how else can we explain that the place of the flattened ethmoid and ethmovomerine laminæ of the Dugong is occupied in the Delphinidæ by expansions of the vomer, which both form the posterior walls of the nostrils and articulate with the palatals?

This conception of the construction of the ethmoidal segment, as well as the reasons just mentioned for considering that the mesial-palatine processes of the intermaxillaries play the part of a centrum, is at variance with the view of Professor Goodsir, that the vomer and the intermaxillaries are members of one "sclerotome*."

Also if the connexions of the vomer and lateral masses of the ethmoid recorded above are considered sufficient to prove that these bones are parts of one segment, it will at once appear evident that the lateral masses of the ethmoid and the ethmovomerine laminæ form an incomplete neural arch. In that case we must differ from Professor

* *Op. cit.* p. 138.

Owen, who considers that the central plate of the ethmoid represents in a coalesced condition the prefrontals of the fish or reptile*, and plays the part which we allot to the lateral masses in the neural arch of the vomerine segment. It cannot do so, for these reasons: 1st, that while the lateral masses of the ethmoid are continuous with the vomer, the central plate is never truly continuous, but only contiguous to the vomer; 2ndly, that the central plate must, for reasons above stated, play the part of a centrum; and 3rdly, that, as Professor Goodsir has shown, Professor Owen's view is inconsistent with the relation of the olfactory nerves to the central plate †.

3. The frontal and the central plate of the ethmoid belong to one segment. The fact of the central plate of the ethmoid having no early connexion with the vomer, as well as its tendency to remain distinct even after the vomer and it have come in contact, shows that it is not part of the same centrum as the vomer; therefore, inasmuch as we have already concluded that it is a centrum, it can form no part of the vomerine segment. On the other hand, the presphenoidal centrum is complete without it. In these circumstances it becomes apparent that the central plate of the ethmoid is the centrum of a segment intervening between the vomerine and presphenoidal. This opinion is strengthened by our remembering that in the Sheep the central plate of the ethmoid is anchylosed to the centrum of the presphenoid before uniting with the lateral masses, and that in the Hedgehog it appears on the base of the skull for a considerable distance between the presphenoid and vomer. The frontal forms the neural arch belonging to this centrum, and is in constant connexion with it. But if, as above proved, that margin of the os planum of the human subject which lies superiorly is morphologically posterior, then the margin of the frontal which articulates with it, viz. the inner edge of the orbital plate, is morphologically anterior; and therefore not only the foramen cæcum, as Professor Goodsir believes ‡, but also the space occupied by the cribriform lamina, lies within the arch formed by the frontal and the central plate of the ethmoid. Thus the cribriform lamina, which in point of development is a mere lateral expansion of the central plate, forms a screen across the entrance into the cavity of the ethmovomerine arch in the plane of segmentation, and has no further morphological importance than may be supposed to attach to the tentorium cerebelli. By the turning up of its anterior extremity to touch the frontals and nasals, the central plate of the ethmoid divides the neural arch of the segment to which it belongs, as well as those in front of it, into a

* Owen, 'On the Archetype and Homologies of the Vertebrate Skeleton,' pp. 131 & 135.

† *Op. cit.* p. 149. The strength of Professor Goodsir's argument rests in this: that, according to Professor Owen's theory, the olfactory nerves in the mammal are made to lie outside a neural arch, through which they pass in the reptile and fish; which involves the supposition that the points of egress of the olfactory nerves have been moved in the mammal one segment backwards. According to the theory advanced in this communication, although no doubt the bone corresponding to the mammalian central plate of the ethmoid passes upwards on the outside of the olfactory nerves, and does not rise up between them, yet in both fishes and mammals the olfactory nerves are contained within the neural arch of the segment.

‡ *Op. cit.* p. 142.

right and left portion, exactly as the vomer, by a prolongation downwards, divides the anterior hæmal arches.

4. The intermaxillaries, maxillaries, and nasals are the osseous elements of one segment. Of this we might find sufficient evidence in their relations in mammals; and indeed the proposition does not stand in need of much proof, if it be once considered certain that the vomer, lateral masses of the ethmoid, and palatals belong to one segment, and that the mesial-palatine processes of the intermaxillaries are the centrum of the segment following. But that we may fully understand the parts played by the intermaxillaries, maxillaries, and nasals respectively, we must defer our remarks on this subject till we have examined the arrangements in other classes.

We shall now glance for a single moment at the manner in which this explanation of the anterior segments of the skull affects the general view of the cranial segmentation.

That there is an occipital segment, and that the postsphenoid and parietals are portions of another segment, is generally admitted; and we have concluded that the frontal and central plate of the ethmoid are elements of a third segment. These then are the three segments which roof-in the cranial cavity. They are all complete above. Their centra are successively smaller in their order from behind; and the floor of the neural arches which they form gradually turns upwards as it passes forwards.

In front of the occipital segment lie the petrous bones*, and connected with them are the organs of hearing. In front of the parietal segment lies the presphenoid, connected with which are the organs of vision. In front of the frontal segment lies the ethmo-vomerine segment, and connected with it are the organs of smell. None of the three segments connected with the special senses has a complete neural arch. With the most posterior of them no centrum is connected; the second (the presphenoid) has a small centrum; and the most anterior has a larger centrum—the vomer: and the tendency of these centra, contrary to the tendency of the centra of the brain-protecting segments, is to curve downwards as they pass forwards. Thus there are two alternating sets of sclerotomes, which may be distinguished as the protective and the sensory; while foremost of all is an imperfect and peculiar seventh and terminal sclerotome—the facial, which may be considered as binding these two sets together. On this seven-segmented plan I believe the head to be formed in all vertebrata; and although it be true that in certain cases there are no sclerous elements of the cranium developed in connexion with either the ear or the eye, these organs are nevertheless themselves portions of segments lying between those in connexion with which the neighbouring protective sclerotomes are developed. The upward tendency of the protective, and the downward tendency of the sensory sclerotomes, is not seen at all in the fish, but becomes more and more observable as we ascend to reptiles, birds, and mammals; and is exhibited best of all in man, in whose structure the idea of vertebrate creation is completed (fig. 27).

These few and imperfect remarks have been necessitated by the consideration that no

* It is unnecessary for the purpose of this argument to discuss the positions of the interparietal, mastoid, or squamous.

segment of the skull can be properly viewed apart from the others. While on the other hand, were I to enter into further details, I should be led far beyond the subject of this paper, and such details are not required for the elucidation of the points under consideration.

Birds, Reptiles, and Fishes.

The Vomer and Intermaxillary in the Bird.—There is only one intermaxillary in the bird. It sends no mesial processes backwards on the palate. Its lateral plates articulate with the maxillaries exactly as the mammalian intermaxillaries do. Its distinctive peculiarity is, that from the place of union of its two halves in front it sends up long processes between the nostrils, which reach back to the roof of the skull. These processes close-in the septal cartilage above and in front, exactly as it is closed in by the intermaxillaries in the Horse, the Tapir, and the Dugong. If the parts of the intermaxillaries of the Dugong which meet in the middle line were prolonged back to the frontal, their arrangement would altogether resemble the intermaxillary of the bird, for they are situated as much in front of the nostril as it is. The arrangement also in *Rhinoceros tichorhinus* only differs from the beak of the bird, in that the bones roofing the nostrils are prolonged down to the intermaxillaries, instead of the latter passing up to the former. The nostrils of the bird, therefore, correspond to those of the mammal, and the bones which roof them (ethmoido-frontals of Goodsir) are the nasals.

Fortunately, opinions are agreed as to which bones are the palatals of the bird. They approach each other behind and come in contact in the middle line, and form by their junction a grooved surface, which glides backwards and forwards on the basisphenoid. But very frequently a small bone (the vomer of Cuvier, Owen, &c., the entopterygoid of Goodsir) is intercalated between them, so situated as to furnish a continuation of this groove along the inferior edge of the septal cartilage; and with this little bone the palatals articulate edge to edge. It is often absent, as in the Gallinaceæ, and in different birds it presents different shapes: thus it is broad in the Crow, and a vertical plate of considerable size in the Duck (fig. 28); and in the Gull and Guillemot, as well as in the Albatros, which has it of very large size, it bears a most striking resemblance to the mammalian vomer. That it belongs to the same segment as the palatals is as obvious as that the vomer, lateral masses of the ethmoid, and palatals in the mammal belong to one segment; and if that proposition has been proved by the foregoing observations, there can be no further doubt that this bone in the bird's skull is the vomer.

The framework of the lateral masses of the ethmoid of the mammal is not represented in the bird; but the turbinations for the distribution of the olfactory nerves, which in the mammal are attached to that framework, are in the bird, when they are ossified, usually in close connexion with the palatals, which are members of the same segment. This can be seen in the Gull and in the Albatros. In the Parrot there is an exception to the rule.

If the nasals of the bird correspond with the nasals of the mammal, then the frontals

of the bird (sphenoido-frontals of Goodsir) must correspond with the frontals of the mammal; and indeed they have the same relation to the cranial cavity, and are met, like those of the mammal, by the anterior extremity of a bone belonging to the series of centra. But if they are the frontals, then the bone which meets them, viz. the interorbital plate (prefrontal of Owen), corresponds to the mammalian central plate of the ethmoid, at least in the anterior part of its extent *.

The Vomer in Reptiles and Fishes.—Of the vomer in reptiles and fishes it is not necessary, for the purposes of this paper, to say much. If the vomer, palatals, and lateral masses of the ethmoid in the mammal form one segment, and if this segment is that to which the olfactory nerves belong, then it requires no argument to prove that these bones must be respectively represented in reptiles and fishes by the vomer and palatals of Cuvier and Owen, and the prefrontals; for these comply with all the necessary conditions. The prefrontals and palatals are always in contact, and sometimes, as in the Crocodilia, in such a way as to indicate in the strongest manner that they are parts of one sclerotome; while the vomer articulates always with one or other of these two pairs of bones, and sometimes with both. Moreover the vomer in reptiles and fishes always lies along the inferior margin of the septal cartilage (or middle frontal process of the primordial cranium). In reptiles, however, it presents a series of variations. While in the Turtle it is a single bone, and disposed much as in the mammal, except that inferiorly it appears prominently in the palate †, in other reptilia it is in two parts—a right and a left. In the Crocodile these parts lie side by side, and inferiorly articulate with the palate plates of the palatals, while superiorly they curve outwards and come in contact edge to edge with the superior extremities of the same bones. On the other hand, in the Serpents it is the inferior edges of the vomerine bones which curve outwards. In the Lizards they merely come in contact in the middle line; and in the Batrachia they do not even meet.

If it be asked, what corresponds in reptiles and fishes to the central plate of the ethmoid, I reply that it is the interorbital septum. This structure is frequently completely ossified in fishes so as to form a single distinct bone; and, as Professor Goodsir has pointed out ‡, it completes a neural ring with the great frontal. According to the theory now advanced, it differs from the central plate of the mammalian ethmoid only in that it stretches outwards and upwards to meet the frontal, instead of the frontal stretching downwards, spanning the whole arch to meet it; and in that it does not project upwards in the middle line so as to divide the arch into lateral halves.

According to this view the centrum of the frontal segment always lies between the

* The posterior part of the interorbital plate appears to belong to the presphenoid, as has been pointed out by Professor Huxley in his Lecture "On the Theory of the Vertebrate Skull," pp. 10 & 11. See Royal Society's Proceedings, Nov. 18, 1858.

† The vomer appears, however, in the palate of even some mammalia, viz. in certain Cetacea. In *Hyperoodon* it even appears at two different places.

‡ *Op. cit.* p. 158.

orbits; and the peculiarity of its position in the mammal consists merely in this,—that by the curving downwards of the frontal and vomerine neural arches, and the curving upwards of the frontal centrum, the latter is entirely concealed by the former.

The Intermaxillaries in Reptiles and Osseous Fishes.—We now approach a most interesting series of intermaxillaries, which afford the clue to the explanation of the segment to which these bones belong. In the Crocodiles the arrangement of the intermaxillaries is like that in the mammalia, but by the great elongation of the maxillaries they are far removed from the vomerine bones. The articulation of their anterior extremities is like that in the Horse; they pass up in front of the septal cartilage and close it in. In the Chelonians the nasals are absent; and the intermaxillaries are united into a single bone, which is placed between the anterior extremities of the maxillaries, and in the Tortoise articulates with the vomer. From the Chelonians we pass to the Lizards, and in them also we find the intermaxillary single and articulating with the anterior extremities of the maxillaries; but it differs from the chelonian intermaxillary by sending upwards and backwards in the middle line a process to articulate with the nasals, which corresponds to the processes projected upwards by the intermaxillary of the bird, and which is often the most developed part of the bone. In *Varanus* and others the nasals are represented by a single bone.

In the Serpents we again meet with a single intermaxillary which articulates in the middle line with the nasals; and as it is but loosely connected with the maxillaries, and forms the anterior extremity of the line of centra, it puts one strongly in mind of the arrangement in cases of cleft-palate in the human subject.

It greatly resembles also another bone, viz. the nasal in fishes (nasal of Owen); and the resemblance consists in this, that, like the reptilian intermaxillary, the nasal of the fish, when present, forms the anterior member of the series of centra.

Sometimes this relation of the nasal in fishes is very much disguised by the thickening of the vomer which takes place in connexion with the development of vomerine teeth,—a thickening which gives to the vomer the appearance of projecting downwards, and which often looks as if it formed the termination of the line of centra, while the extremity of the nasal appears to be situated altogether above that line. This is the case, for example, in the Cod; but when, in the fresh state, a vertical section is made through the nasal and vomer, it is seen that, not the vomer, but the extremity of the nasal bounds anteriorly the cartilaginous bar of the base of the skull (fig. 29). There is, in addition, a nodule of cartilage * attached in front of the nasal, in the Cod and many other fishes, which seems to be a portion of that bar, separated from the main part by the interruption of the nasal. But that the nasal lies in the line of centra is best seen in some of the fishes which have no vomerine teeth. It is beautifully seen in *Malapterurus*, in which the whole shape of the nasal singularly resembles the intermaxillary of *Boa* or *Python* (fig. 30).

One might well believe from its relation to the line of centra that the nasal of the fish

* Recognized as the vomer by Professor Goodsir, *op. cit.* p. 140.

was the true intermaxillary, were it not that the bones universally acknowledged as intermaxillaries in the fish have claims to the name which cannot be set aside, inasmuch as a complete chain of forms of intermaxillaries, of which they form one link, can be traced from reptiles to fishes through the Batrachia. The position therefore which I maintain is this, that in the sclerotome formed by the nasals, intermaxillaries, and maxillaries, when the centrum is represented by a bony formation, it is derived in mammals, birds, and reptiles from the intermaxillaries, and in fishes from the nasal: the Ornithorhynchus being the only animal, as far as I know, in which it is a separate bone.

If it be repugnant to any preconceived notion to believe that the same morphological element (the centrum) can be derived in one instance from one developmental element, and in another instance from a different one in the same segment, we have only to keep in memory the exactly analogous instance of the intermaxillaries in the human subject, which usually arise from the maxillaries, and yet in certain conditions of development take origin in connexion with the septal cartilage (see above, p. 303).

To return to the series of forms of intermaxillaries in reptiles and fishes: the intermaxillaries of the Frog are two small bones articulating with one another in the middle line, and externally with the anterior extremities of the elongated maxillaries; they give off processes towards the nasals, and very slight mesial-palatine processes. Exactly in the same way do the intermaxillaries of the Salmon articulate with one another, and with the maxillaries; the only difference being that the maxillaries hang downwards instead of being directed horizontally backwards. In most fishes the intermaxillaries have superior mesial processes, which pass backwards over the nasal, and evidently correspond with the superior mesial processes in frogs, lizards, and birds. Frequently, as in the Cod, the maxillaries and intermaxillaries are so loosely connected, that the idea is given of two separate arches; but, on the other hand, it is to be remembered that not only is this looseness of connexion (considered apart from the question of development, to be afterwards noticed) no reason why they may not be members of one segment, but also that the intermaxillaries are always in contact with one another in the middle line, while the maxillaries are not so; that the arrangement is often such as has just been mentioned as existing in the Salmon; and that sometimes the maxillary and intermaxillary are even more closely soldered into one arch, of which the intermaxillary forms the proximal, the maxillary the distal part.

Morphological Conclusions respecting the Construction of the Facial Segment throughout the Vertebrata.

Ere endeavouring to discover from the above data in what precise manner the maxillaries, intermaxillaries, and nasals unite to form the facial sclerotome, it will be well to state distinctly that they cannot be considered as forming in the mammal a continuous neural ring behind the nostrils, however tempting at first sight that idea may appear; for such an hypothesis would render inexplicable the union of the intermaxillaries above the anterior extremity of the septal cartilage, which occurs in the Dugong, Tapir, &c., besides that it could not be applied to other than mammalian forms.

The real constitution of the facial segment will become evident if we consider the series of appearances assumed by it, passing upwards from the fish. In the fish the intermaxillaries and maxillaries hanging downwards form a sufficiently evident incomplete hæmal arch; while the ideal cylinder formed by the series of neural arches is brought to a close in front by the nasal, which rests, like the nasals in the other classes, upon the cartilage in connexion with which the centra are developed; and when the centrum of this sclerotome is represented by a plate of bone, it is a process of the nasal which represents it. Obviously the nostril of the fish lies behind the facial segment, for both the intermaxillaries and maxillaries lie in front of the nostril (fig. 31).

The facial segment of the Frog differs from that of the fish in that the maxillaries are directed horizontally backwards, and come in contact with bones behind; and in that the centrum of the segment is represented by a process derived, not from the nasals, but from the intermaxillaries. Further, the nasals of the Frog differ from the nasal of the fish in being expanded to protect the nostrils, like those of most reptilia, birds, and mammals. The exact signification of this expansion we shall consider anon.

The relation of the nostril of the Frog to the facial segment is the same as in the fish, but inasmuch as it communicates with the mucous surface, and this communication lies between the maxillary and the palatal, we can now see distinctly that the nostril is a passage lying between the segments to which these bones belong,—what has been named by Professor Goodsir a "metasomatomic" opening*.

In lizards, serpents, and birds the facial sclerotome is constituted as in the Frog; and the hæmal arch has still, as in the Frog, its distal extremities directed backwards instead of downwards. According to this view the superior mesial processes of the intermaxillaries are to be considered morphologically as projecting not so much upwards as backwards, and as lying above the nostrils, while the hæmal arch lies in front of the nostrils.

Lastly, in the mammalia, by the union of the palate plates of the maxillaries the hæmal arch is completed, and the ring of bone which surrounds the incisive foramina is the anterior extremity of the ideal cylinder formed by the series of hæmal arches (fig. 27). This is best conceived of by looking at this ring in the Hare or the Rabbit. There we see the single incisive foramen bounded in the middle line in front by the mesial-palatine processes of the intermaxillaries, while extending backwards from them is the hæmal arch formed by intermaxillaries and maxillaries exactly as in other classes, except that it is complete.

The fact that in certain mammals the intermaxillaries limit the anterior extremity of the septal cartilage as much as they do in birds and lizards, embracing it at a point morphologically anterior to the nostrils, appears to me to prove that this is the true explanation of the facial sclerotome in the mammalia, and that the nostrils throughout the vertebrata are intersegmental openings lying between the two most anterior segments of the skull.

* To Professor Goodsir we are also indebted for the terms "sclerotome, myotome," &c. and other additions to explicit morphological nomenclature.

If these conclusions are correct, the connexion of the nasals with the intermaxillaries and maxillaries behind the nostrils, although so constant, is of altogether secondary morphological importance. But it is undeniable that by the great extent of this connexion in mammals, and by the non-development, in most of them, of any part of the intermaxillaries above the septal cartilage, and the separation of the nasals from the intermaxillaries in the middle line, where in birds and lizards they come in contact, there is a marked attempt, so to speak, on the part of Nature to convert the nasals, intermaxillaries, and maxillaries of the mammal into a neural arch behind the nostrils. This tendency reaches its maximum in the human skull, in which the hæmal ring of the facial sclerotome has almost, if not altogether, disappeared, and the whole bulk of the elements of the hæmal arch may be said to be devoted to the formation of what we may call the pseudo-arch behind the nostrils. It may be described as an effort to open up the closed extremity of the neural cylinder, and at the same time to close the open extremity of the hæmal cylinder*.

Still, however, the morphological interpretation of the mammalian nostrils is that they are intersegmental clefts. They are similar clefts to those in which the eye and ear are developed; and the alar cartilages pass round the olfactory clefts exactly as the tarsal cartilages pass round the optic clefts, and the pinna of the ear and the tympanic bone round the auditory clefts.

Thus these structures are morphologically as well as functionally comparable; they encircle openings which pass from the dermal to the mucous surface, and which primarily lie in the transverse plane; although in Mammalia the malar, which belongs to the facial segment†, passes backwards and sends up a process behind the orbit, and, in a similar manner, the intermaxillary and maxillary send up processes behind the nostril. That these processes, however, are not situated so entirely behind the nostril as might on first thoughts be supposed, will be perceived if we take into consideration that the arch formed by the intermaxillaries and maxillaries lies in the plane of the palate, and that processes pointing upwards at right angles to that plane are therefore directed ideally backwards. Thus the nasal process of the maxillary is ideally inferior to the nostril, in the same way as we have seen that the superior mesial process of the intermaxillary of the bird or lizard is ideally superior to the nostril.

Let it not be supposed that such a structure as the trunk of the Elephant presents any obstacle to the theory of the nostrils here offered. The fusion in the middle line of structures undoubtedly lateral in their ideal position is well exemplified in cases of sympodia; and in those monsters in which the head is only developed as far forwards as the ears, the pinnæ of the ears are united at their bases, forming a short tube.

* If we take into consideration that the facial segment is the segment especially engaged in expression, I think that I shall not be considered too fanciful in saying that the gradual closing of the hæmal cylinder and opening of the neural cylinder by the disposition of the bones of the facial segment is in harmony with the increasing development of innervation from fishes up to man.

† This statement, made in passing, is illustrated by the attachment to the maxillaries alone of the largely developed malars of the Sloth family and of the vestiges of malars in *Myrmecophaga*.

With regard to the nasals: to understand the exact morphological part which is played by them, we must have recourse to embryology. The only theory of the segmentation of the skull, as far as I know, in which the teachings of embryology have been taken into account and been sought to be explained, is that of Professor Goodsir. He is of opinion that the maxillaries are developed, not, as has been supposed, in the maxillary lobes, but in the lateral frontal processes of Reichart*; and certainly, if the maxillary lobes are homotypic with the pair of visceral laminæ behind them, this hypothesis presents the only escape from a most serious difficulty; for, while the maxillary lobe arises from behind the eye, all theories agree in representing the maxillary bone as belonging to a segment in front of the eye. But although it is with the utmost diffidence that I would express an opinion differing from Professor Goodsir's, I take courage to do so in the present instance, inasmuch as his view does not agree with the conclusions arrived at above from the consideration of adult forms, and because, from observations made on embryo lambs, I am convinced that embryologists have been right in considering that the maxillary bones are developed from the maxillary lobes. I believe that the true solution, and that which will be found to explain all the phenomena, is this: that the cleft between the maxillary lobe of the embryo and the lateral frontal process is not transverse in position, but longitudinal; that it does not separate an anterior from a posterior segment, but that it divides the inferior elements of more than one segment from the corresponding superior elements. It will be observed that if the maxillary bone belongs to a segment in front of the eye, and if it is really developed from the maxillary lobe, it follows as a necessary consequence that the nature of the cleft between the maxillary lobe and lateral frontal process is as now stated. For on one side of the cleft is the blastema in which the palatal and maxillary afterwards appear, while on the other side is that in which appear the frontal and nasal, and doubtless also the lateral mass of the ethmoid. Moreover, this idea of the fusion of segments by longitudinal cleft is by no means an unwarranted assumption, as may possibly be alleged. In support of it we observe,—

1st. The maxillary lobe, as it grows, runs alongside of the lateral frontal and middle frontal processes, and does not strike out at right angles to them.

2nd. The permanent severance of the pterygoid of the fish from the base of the skull is an instance of longitudinal fission of at least one sclerotome, remaining throughout life.

3rd. The hypothesis of fission of segments by longitudinal clefts is necessary to explain the peculiar condition of the elements of the face in cartilaginous fishes, if the head is to be considered as segmented at all; for no one can suppose that the upper jaw of the cartilaginous fish, and the parts which unite it to the skull, are all parts of one segment. According to my hypothesis, the condition is merely an arrestment of development; for I judge from the maxillary lobe in mammals being originally separate from the superior parts of the face, and afterwards united to them, that longitudinal fission is a symptom of degeneration at the extremity of the series of segments, which

* *Op. cit.* p. 120.

tends to disappear as the segments involved become more fully developed*. I believe, therefore, that the palatals and maxillaries combine to form the upper jaw of the cartilaginous fish (as may be seen to advantage in the Sturgeon), and that the extremity of the snout represents the intermaxillary†; and that thus, for example, the weapon borne by the Sawfish is an intermaxillary bearing teeth. Among mammals, to a certain extent similar is the arrangement in Cyclopian monsters: the nose is represented in them by a proboscis above the single eye, and the maxillary forms no superior connexion in front of the eye.

4th. Tendency to longitudinal fission of the anterior cranial segments is exhibited in the permanent duplicity, in many animals, of the vomer and of the intermaxillaries; also in the compression of the septum of the nose, and separation of the segments into two alternating sets, as above described. Moreover, the flattening at the base of the skull of the substance surrounding the chorda dorsalis, and its division into two trabeculæ where the chorda ceases, are phenomena, as it appears to me, indicating the same thing; and in this last case also the divided parts unite as development proceeds‡.

In deference to the opinions of Professor Huxley§, I may here state that I cannot think that the cessation of the notochord is proof of the immediate cessation of segmentation. The keenest advocate of the segmentation of the cranium will admit that in the face the segment or segments are of a degenerated description. So are those at the caudal extremity. But while in the caudal segments the arches degenerate before the centra, in the cranium the centra degenerate very soon. It is in harmony with this that the chorda dorsalis is one of the first elements to disappear in the cranial segments; while the division of the basis cranii into trabeculæ is to be accounted for in the way above shown‖.

* In this respect longitudinal and intersegmental fissures resemble one another. The parts of the embryo which are developed in immediate contact with the cavity of the ovum, and in which the great systems are fully represented, viz. the thorax and abdomen, are those in which the various tissues, the osseous, muscular, nervous, and vascular, ultimately exhibit the most complete segmentation; but it is only in the head and neck—an extremity of the embryo in which the systems are not typically developed, (and there it is only in the visceral walls, which have in the head and neck the most rudimentary development,)—that segments are for a time partially separated by fissures of the blastema. These intersegmental fissures are directly connected with the rudimentary condition of the parts which they separate, and disappear as the latter become more fully developed; and I have wished in the text to indicate that that also is the case with longitudinal fissures.

† At least that part of the intermaxillary which, as above shown, is in osseous fishes combined with the nasal.

‡ The foramen in the basioccipital of *Phoca vitulina* is doubtless a phenomenon of the same description.

§ *Op. cit.* p. 52.

‖ In justification of the plan pursued in this communication, in which the main argument is drawn from comparative anatomy and then shown to be in accordance with development, I shall here state why I cannot go so far as to concur in Professor Huxley's doctrine (*op. cit.* p. 5), that "the study of the gradations of structure presented by a series of living beings may have the utmost value in suggesting homologies, but the study of development alone can finally demonstrate them." Were that doctrine true, it

Conceiving, then, the development of the anterior part of the cranium to take place in the manner above stated, I consider that the nasals have the same relation to the segment to which they belong as the frontal has to its segment, inasmuch as they are continuous in position with the frontal, except in those lower vertebrata in which the prefrontals complete a well-developed intervening arch. They are more closely connected with the corresponding elements behind them than with the remaining elements of their own segment, as a natural consequence of the fission of segments. And now it will be understood how it is that the nasals are often so far sundered from the intermaxillaries: it is, that the preceding segment having a very elongated centrum (the vomer) and a very imperfect neural arch (the lateral masses of the ethmoid), the intermaxillaries are projected forwards in front of the vomer, while the nasals cling behind to the frontals.

With respect to the expanded form of the nasals in many reptiles, in birds, and in mammals, I apprehend that they are not to be considered as spreading downwards to form an imperfect arch of the neural series, but as spreading outwards to protect two intersegmental passages; for it is the nostrils, and not, in any sense, a continuation of the cranial cavity, which they protect. This will be best understood by looking at the Frog's skull, in which the osseous cranial cavity is closed-in at the fore part of the prefrontals, and the neural arch of the segment in front is reduced to zero, the position where it exists in certain fishes being represented in the Frog by the line of junction of the nasals with one another and with the mesial cartilage below; while the expansions of the nasals over the nostrils are like the projections outwards of two transverse processes, and are strictly similar to those elongated projections of the frontals which roof-in

would be impossible finally to demonstrate the correspondence of any two structures in different animals, since we do not see one developed out of the other. But, on the contrary, there is a vast number of such correspondences so plainly obvious as to need no demonstration. That which makes these correspondences so evident is simply the comparison of the bones; and it is the business of the anatomist, in cases of correspondence less obvious, to submit the structures which he compares to a careful scrutiny, until, by minute examination of their relations, and determining what is constant and what is variable, he is able to give a certain judgment upon points which appear to the uninitiated eye obscure. But if the prosecution of such researches renders certain the correspondence of a number of elements in different animals, will not the light thrown by the varied relations of these elements upon the laws which regulate their arrangement furnish as certain and secure data on which to build as can be obtained from the use of any scientific instrument whatever? It has not been the instruments used, but the manner in which they have been handled, that has led to the discrepancies of morphological theories. I have an interest to insist on this matter, because it is impossible that the questions discussed in the present communication can be settled by embryological evidence, for these reasons: viz. the maximum segmentation of the cranium is found, not in the embryo, but at the period of most characteristic development of the bones composing it; and the whole history of the sclerotomes to whose elucidation this communication is devoted is not given in any single species; but, on the contrary, were we to search in the mammal for the stages illustrated in the structure of fishes, we should find that at the period at which these stages would fall due to be represented, there is as yet no differentiation of tissues, and the osseous system whose history we seek to trace has not begun to appear.

the orbits in Whales. In other animals the arrangement is perhaps not so distinct, and there may be a difficulty, especially in mammals, to determine if the nasals do not, in part, at least, form a continuation of the roof of the cranial cavity; for in them the neural arch of the ethmoidal sclerotome is, as we have seen, a continuation of that cavity, and is open in front. But, inasmuch as in the mammal the olfactory-sense capsules are withdrawn into the interior of the ethmoidal neural arch, the cranial cavity is closed in by soft parts at a point posterior to the nasals; and I therefore incline to think that the nasals, protecting in this, as in other cases, the nostrils, are, like those of the Frog, similar in nature to the orbital processes of the cetacean frontals.

Note.—Since writing the above, I have deemed that it would be advisable to add here some further details as to the morphological structure of the cranium in the mammal, even though it is impossible in this place to give the arguments on which these details are founded. The descending process of the occipital bone, the mastoid, the squamous, the external angular process of the frontal, and the lacrymal are serially corresponding structures, belonging respectively to the occipital, petrous, parietal, frontal, and ethmovomerine segments, and tending to project from their neural arches. There are no hæmal arches in connexion with the protective segments. Into the composition of the hæmal arch of the petrous segment enter the styloid processes, the stylohyoid ligaments, and the small cornua of the hyoid; while the great cornua of the hyoid belong not only to a segment behind, but to the true splanchnic skeleton; that is to say, to the series of sclerous structures internal to the primary vascular arches, and which are best developed in the branchial arches of fishes. The other hæmal arches are, as has been stated, the pterygoid, palatal, and maxillary, belonging respectively to the presphenoid, ethmovomerine, and facial segments. The maxillary arch, having assumed in the mammal as much as possible the aspect of a neural arch, as has been above explained, although in reality hæmal, has a bone radiating from it—the malar, which tends to form connexions with the bones radiating from the neural arches behind, viz. with the lacrymal, external angular process of the frontals, and the squamous. The lower jaw is not a hæmal arch at all; that is to say, it is not an arch belonging to a single segment and corresponding to the palate-bones, or to a pair of ribs, but is a limb-arch. For I hold that Professor Goodsir has distinctly shown that the shoulder-girdle and pelvic girdle are not rib-arches, and that limbs do not belong merely to single segments; but seeing, on the other hand, that these girdles pursue an arched direction, I conceive that they are to be considered as arches external to the series of rib-arches, and supporting radiations, just as rib-arches sometimes do. Such a limb-arch is the lower jaw; but the visceral laminæ being very imperfect in the head, it is intimately connected with arches bounding the hæmal cavity. In the fish it supports extensive radiations, viz. the opercular apparatus, but in the mammal it bears no radiations. The quadrate jugal in the bird has every appearance of being a radiation homologous to the operculum; for not

the quadrate jugal, but the bone called mastoid by Professor Owen, corresponds to the mammalian squamous. The quadrate bone of the bird has been abundantly proved to correspond to the incus of the mammal. The elements of the limb-arch are the incus, malleus, and lower jaw; and its superior member, the incus, is, throughout the vertebrate series, connected with the mastoid and squamous bones. As for the stapes, it is a radiation of the petrous sclerotome, and corresponds to the structure which supports the eye of the Shark, and to the ethmoidal turbinations in the presphenoidal and ethmovomerine sclerotomes. While, on the one hand, the rib-arches belong to the individual segments of the body, on the other there are three limb-arches, corresponding to the three great regions of the body: one for the head—the region in which the highest development of organs of animal life takes place; one for the cervical region—the region in which is the highest development of the vascular organs; and one for the abdomen—the region in which the greatest development is found of organs of vegetable life.

Explanation of the Plates.

PLATES IV. & V.

Fig. 1. Bones from specimen referred to in the text as No. 1.

A. The vomer, ethmoid, sphenoidal spongy bones, and left palate and maxillary bones, from the skull of an infant, slightly enlarged, and viewed from behind:—*a*, orbital plate of the ethmoid; *b*, posterior extremity of the vomer; *c*, sphenoidal process of the palate-bone; *d*, orbital surface of the palate-bone, and, immediately above it, the orbital portion of the sphenoidal spongy bone: between the two processes of the palate-bone is the spheno-palatine foramen, completed above by the inferior portion of the sphenoidal spongy bone: *e*, the superior portion of the sphenoidal spongy bone; *f*, inferior portion.

B. Another view taken from the same specimen:—*a*, *b*, *c*, the parts of the inferior margin of the vomer which articulate with the palate, maxillary, and intermaxillary bones respectively; *d*, inferior aspect of the sphenoidal spongy bone; *e*, orbital plate of the ethmoid, seen in perspective; *f*, inferior turbinated process of the ethmoid.

Fig. 2. Bones from specimen referred to in the text as No. 2.

A. Ethmoid, with sphenoidal spongy bones attached:—*a*, cribriform plate; *b*, os planum; *c*, superior portions of the sphenoidal spongy bones, with space between them for the body of the presphenoid; *e*, *e*, orbital portions; *f*, *f*, inferior portions.

B. Sphenoid:—*a*, body of postsphenoid; *b*, body of presphenoid.

Fig. 3. Bones from specimen referred to in the text as No. 3.

A:—*a*, part of the orbital plate of the frontal; *b*, os planum; *c*, presphenoid;

d, external pterygoid plate; *e*, palate-bone; *f*, orbital surface of sphenoidal spongy bone; *g*, the space between the orbital and sphenoidal processes of the palate-bone, bridged over by the sphenoidal spongy bone.

B and C. Left sphenoidal spongy bone; viewed in B from its external, and in C from its superior aspect:—*a*, superior portion; *b*, orbital portion; *c*, inferior portion.

Fig. 4. A. Vomer, with the portions of the palate, maxillary, and intermaxillary bones with which it was articulated:—*a*, the process which rests upon the intermaxillaries; *b*, the process which descends behind the intermaxillaries.

B. Vomer from a fœtus:—*a*, *b*, *c*, the margins for articulation with the palate, maxillary, and intermaxillary bones respectively.

Fig. 5. Diagram of the vomer, ethmoid, and sphenoidal spongy bones of the human subject, with the vomer and ethmoid of the Sheep represented in dotted lines. The superior edge *a* of the human ethmoid corresponds to the edge *a'* of the Sheep's ethmoid; while *c* is the sphenoidal spongy bone rendered necessary in the human subject to fill up the space between *b'* and *b*.

Fig. 6. A. Posterior extremity of the vomer in the Orang.

B. The same in the Chimpanzee.

C. The same in the Gorilla.

Fig. 7. Bones from the Cat.

A. Vomer and ethmoid seen from below:—*a*, margin of vomer for articulation with the maxillaries; *b*, *b*, ethmovomerine laminæ.

B and C. Views of the presphenoid from the front and from behind.

D. Portion of base of skull of a new-born kitten:—*a*, intermaxillaries; *b*, vomer; *c*, presphenoidal centrum; *d*, postsphenoidal centrum; *e*, basioccipital; *f*, ethmoidal turbinations; *g*, pterygoid; *h*, tympanic.

Fig. 8. Vomer and ethmoid of the Fox:—*a*, ethmovomerine laminæ; *b*, margin of vomer for articulation with the maxillaries and palatals; *c*, margin for articulation with the intermaxillaries; *d*, groove corresponding to the upper part of the nasal foramen of the palate-bone.

Fig. 9. Intermaxillaries of *Ursus maritimus*.

A. Inferior aspect.

B. Superior aspect:—*a*, maxillary; *b*, mesial-palatine process of intermaxillary; *c*, incisive foramen; *d*, foramen peculiarly ursine; *e*, vomer.

Fig. 10. Vomer and ethmoid of *Phoca vitulina*:—*a*, vomerine groove; *b*, lateral mass of the ethmoid; *c*, central plate; *d*, expanded superior margin of central plate.

Fig. 11. Part of the skull of the Hedgehog:—*a*, frontal; *b*, nasals; *c*, left intermaxillary; *d*, vomer, with right ethmovomerine lamina removed; *e*, flat inferior margin of the central plate of the ethmoid; *f*, presphenoid; *g*, groove on the left ethmovomerine lamina corresponding to the nasal foramen of the palate-bone.

Fig. 12. The vomer and lateral masses of the ethmoid of the Lamb, seen from below:—*a*, the inferior margin of the vomer, rough posteriorly for articulation with the maxillaries, and smooth anteriorly where it comes in contact with the intermaxillaries; *b*, *b*, the grooves which complete the nasal foramina of the palate-bones. The spaces between these grooves and the margins of the vomer represent the ethmovomerine laminæ; and on the outer aspects of the grooves are the small orbital surfaces of the ethmoid.

Fig. 13. Intermaxillaries of the Sheep:—*a*, *a*, incisive foramina; *b*, *b*, portions of the maxillaries; *c*, space for the anterior extremity of the septal cartilage.

Fig. 14. Intermaxillaries of the Alpaca. Letters as in fig. 13.

Fig. 15. Intermaxillaries of the Giraffe. Letters as in fig. 13.

Fig. 16. Part of the skull of the Horse:—*a*, vomer (the anterior half removed); *b*, part of the palate-bone; *c*, part of the pterygoid; *d*, ethmovomerine lamina; *e*, nasal foramen of the palate-bone.

Fig. 17. Section of the fore part of the skull of the Horse:—*a*, fore part of the vomer, resting on the maxillaries, and articulating in front with the intermaxillaries; *b*, incisive foramen; *c*, cul-de-sac for the anterior extremity of the septal cartilage.

Fig. 18. Vomer, &c. from the skull of the sucking Pig:—*a*, presphenoid; *b*, portion of the vomer which rests on the intermaxillaries; *c*, ethmovomerine lamina and part of the ethmoid; *d*, various distinct ossifications belonging to the ethmoid.

Fig. 19. Vomer, &c. of the young Elephant:—*a*, *b*, surfaces by which the intermaxillary and maxillary come in contact with their fellows of the opposite side; *c*, incisive foramen; *d*, *e*, processes of the vomer, by which its inferior and superior margins are made continuous with the corresponding margins of the intermaxillaries.

Fig. 20. Anterior nares of the Tapir:—*a*, vomer; *b*, maxillary; *c*, palate-plate of maxillary; *d*, intermaxillary; *e*, foramen corresponding to both incisive foramina and the space for the point of the septal cartilage in the Camel &c.; *f*, lateral mass of the ethmoid; *g*, expanded superior margin of the central plate of the ethmoid; *h*, united nasals.

Fig. 21. Part of the skull of *Cœlogenys*:—*a*, *b*, mesial borders of the maxillary and intermaxillary; *c*, vomer.

Fig. 22. View of the articulations of the vomer in the Rabbit. Above are the vomer and ethmoid forming one bone. Beneath are the bones of the upper jaw of the left side, and a portion of the intermaxillary of the right side adherent:—*a*, anterior extremity of the vomer grooved for the cartilaginous septum of the nose; *b*, the part of the vomer which articulates with *c*, the extremity of the expanded mesial processes of the intermaxillaries forming turbinations in connexion with Jacobson's organ.

Fig. 23. Part of the skull of the Kangaroo:—*a*, presphenoid; *b*, frontal; *c*, ethmovome-

r'ne lamina; *d*, nasals; *e*, central plate of the ethmoid; *f*, lateral ridge of the vomer peculiar to marsupiate animals.

Fig. 24. Part of the skull of the Dugong:—*a*, frontal; *b*, intermaxillary; *c*, vomer; *d*, maxillary; *e*, *e*, palatals; *f*, *f*, ethmovomerine laminæ and lateral masses of the ethmoid.

Fig. 25. Section of skull of *Delphinus orca*:—*a*, intermaxillary; *b*, maxillary; *c*, *c*, vomer; *d*, palatal; *e*, central plate of the ethmoid.

Fig. 26. Skull of *Delphinus globiceps* from above:—*a*, intermaxillary; *b*, *b*, maxillary; *c*, palatal; *d*, pterygoid; *e*, nasal foramen of palate-bone; *f*, central plate of the ethmoid; *g*, expansion of the ala of the vomer; *h*, the part of the frontal which would have been concealed by the nasal, had not that bone been removed.

Fig. 27. Diagram of the two most anterior segments of the mammalian skull, and of the neural arches of the other segments. The dotted part represents the extent of the septal cartilage. The centra are darkened. Two dotted lines and arrowheads show the direction of the anterior hæmal arches:—*a*, occipital segment; *b*, auditory segment; *c*, sphenoparietal segment; *d*, optic segment; *e*, frontal bone; *f*, central plate of the ethmoid; *g*, lateral mass of the ethmoid; *h*, vomer; *i*, palatal; *k*, nasal; *m*, intermaxillary; *n*, maxillary.

Fig. 28. Vomer and palatals of the Duck, viewed from above.

Fig. 29. A. Section of the skull of the Cod. The cranial cavity is represented dark; the cut surfaces of bones are marked with oblique lines, and the cartilages are left white, while the interorbital septum is dotted:—*a*, intermaxillary; *b*, vomer; *c*, basisphenoid; *d*, basioccipital; *e*, nasal; *f*, frontal; *g*, supra-occipital.

B. A similar section through the nasal and vomer, after the cartilage has been removed:—*b*, vomer; *e*, nasal; *h*, prefrontal.

Fig. 30. A. Intermaxillary and vomer of *Python*:—*a*, intermaxillary; *b*, vomer.

B. Nasal and vomer of *Malapterurus*:—*a*, nasal; *b*, vomer.

Fig. 31. Diagram of the two most anterior segments of the skull of the fish:—*a*, prefrontal; *b*, vomer; *c*, palatal; *d*, nasal; *e*, intermaxillary; *f*, maxillary; *g*, a dotted circle indicating the position of the nostril. The centra are darkened. The dotted lines and arrowheads show the direction of the hæmal arches.

XVII. *On the Properties of Electro-deposited Antimony* (concluded). *By* G. Gore, *Esq.* *Communicated by Professor* Stokes, *Sec. R.S.*

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Second variety of active Electro-deposited Antimony.

92. In addition to the variety of electro-deposited antimony obtained from a solution of teroxide of antimony and hydrochloric acid, a second variety may be obtained from a solution of terbromide of antimony in the following manner.

93. Dissolve one part of teroxide of antimony in 10 parts of hydrobromic acid of sp. gr. about 1·3, filter the solution through a funnel loosely plugged with asbestus, and electrolyse it by means of three Smee's elements and an anode of antimony in the usual manner, at a speed of deposition of about 3 to 5 grains per square inch per hour. The exact proportion of the ingredients of the solution is not a matter of great importance.

94. The electrolytic conduction is free, and a coherent metallic deposit is quickly obtained. The deposit, when thin, is occasionally quite bright and black, and very similar in appearance to that obtained in the chloride solution, but generally it is of a much lighter colour and quite dull in aspect; it is at first scaly, exhibiting the unequal cohesive action already described (17.), but in a less degree than the first variety, and emitting but slightly audible crackling sounds during its formation. During the process of deposition solitary bubbles of gas occasionally adhere to the receiving surface, and cause deep conical holes in the deposit, especially if the solution is not sufficiently dilute; in escaping, these bubbles sometimes emit a chirping sound; in some instances the holes were so numerous as to give to the substance the appearance of a metallic sponge.

95. The deposited substance is firm and moderately hard, but rather less so than that formed in the chloride solution; its fractured surface has also a bright metallic lustre similar to that of the first variety. The specific gravity of two specimens was 5·415 and 5·472 at 60° Fahr.

96. By contact with a red-hot wire, it exhibits a similar molecular and thermic change to that manifested by the chloride variety (19.). A piece $\frac{1}{4}$th of an inch thick, at 60° Fahr., touched by a red-hot wire, exhibited strong molecular action at the points touched, but the action did not spread throughout the mass. A second similar piece, heated to about 200° Fahr. in a porcelain capsule upon the surface of boiling water, when touched by the heated wire manifested stronger and more extended change, but did not discharge the whole of its heat except by numerous contacts of the wire. A third piece in a platinum dish upon the surface of a solution of chloride of calcium at 268° Fahr., evolved all its heat instantly with explosive violence and projection of pieces of the metal by a

single contact of the red-hot wire. Pieces $\frac{1}{10}$th of an inch thick in a platinum dish upon a solution of chloride of calcium at 295° Fahr., did not discharge their heat by repeated scratching with a sharp steel pointer, but instantly discharged it by contact of a heated wire, and shattered themselves to pieces.

97. On placing a small fragment upon melted fusible alloy, and gradually heating the latter, the fragment discharged all its heat suddenly and powerfully, and shattered all to pieces, when the alloy obtained a temperature of 318° Fahr. A similar piece upon the surface of mercury discharged itself similarly when the mercury attained a temperature of 322° Fahr. And another piece, $\frac{1}{8}$th of an inch thick, weighing 33 grains, immersed in a solution of chloride of calcium at 272° Fahr., suddenly discharged its heat and shattered to pieces at the end of four minutes, when the solution was at 275° Fahr. A fragment $\frac{1}{10}$th of an inch thick, formed three years and four months previously (viz. in December 1858), was put in a platinum dish upon a solution of chloride of calcium at 268° Fahr., and touched with red-hot wire; it instantly discharged its heat with violence, scattering fragments of the metal in all directions. The temperature of sudden discharge appears from these circumstances to lie between 270° and 300° Fahr.; but depends upon several conditions, especially upon the degree of rapidity with which the heat is carried away in relation to that with which it is generated. The fractured surface of pieces suddenly discharged is brighter than that of the unchanged substance.

98. Thick pieces of the active substance may be easily reduced to powder in a mortar without causing them to discharge their heat. A small quantity of the powder formed into a train upon the surface of glass or metal at 60° Fahr., and touched by a red-hot wire, only exhibited the change near where it was touched; but if the powder was previously placed in a platinum dish upon a solution of chloride of calcium at 265° Fahr., and then touched at one end by the heated wire, it discharged its heat gradually, and the change, attended by evolution of fumes, was slowly propagated to the distant extremity.

99. The active substance was *gradually* discharged of its heat in the following manner:—A piece barely $\frac{1}{8}$th of an inch thick, and weighing 40·05 grains, was suspended during thirty minutes in a solution of chloride of calcium at 252° Fahr.; when taken out, washed with water, dried at 240° Fahr., cooled and weighed, it had lost 2·45 grains =6·117 per cent. It was then reheated in a platinum dish upon a solution of chloride of calcium at 250° Fahr., and touched six or eight times with a red-hot wire; not the slightest sign of molecular change occurred, and its weight remained but slightly altered: on breaking, it was found to have become less brittle and of stronger cohesion; its fractured surface was quite dull and earthy, and exhibited the appearance of a dark inner core surrounded by a layer of a lighter colour, as if an action or change had proceeded from the outer surface towards the centre.

100. 20·3 grains of it, fused in an analysis-tube (82.), gave 17·20 grains in the form of a metallic button, =79·52 per cent. of metal, and 20·48 per cent. of volatile matter, in the original unchanged substance; and there distilled into the bent part of the tube a

quantity of colourless buttery substance, which was slightly semifluid at 60° Fahr., and doubtless consisted of terbromide of antimony and a little aqueous hydrobromic acid, but was not further examined. In another case, two portions of the unchanged substance lost respectively 18·42 and 20·40 per cent. of volatile matter by fusion in the analysis-tube; the two portions were parts of a single piece of the deposited substance, the one losing 18·42 per cent. being from the upper part, and the other from its lower part, as it hung in the electrolyte.

101. Another mode of *gradually* discharging its heat was as follows:—A piece $\frac{1}{4}$th of an inch thick was placed upon the surface of mercury at 260° Fahr. during $1\frac{1}{4}$ hours, at the end of which time it was found, by repeated contacts of a red-hot wire, to have entirely lost its heating power; its cohesion was greatly increased, and its fractured surface was dull and earthy in appearance; its odour and taste were also acid: a portion was reduced to powder; the powder also possessed an acid taste, and strongly reddened damp litmus paper, whilst the powder of the *unchanged* substance did not. Another piece $\frac{1}{15}$th of an inch thick, weighing 23·52 grains, was placed upon platinum foil upon the surface of mercury at 265° Fahr., and kept at that temperature forty-five minutes, then cooled and weighed; it had lost 1·38 grain = 5·86 per cent.; it was then reheated on the foil and mercury to 270° Fahr., and touched several times with a red-hot wire, but no signs of the change occurred.

102. The electro-chemical equivalent of this substance was examined in the following manner:—Two similar depositing cells, one filled with the chloride solution (90.) and the other with the bromide liquid (93.), and containing anodes of antimony and previously weighed cathodes of sheet platinum of equal size, were connected in a single line with five Smee's elements, and deposits simultaneously formed upon them during two days; they were then taken out, wiped dry, allowed to stand twelve hours to further dry the one from the bromide solution on account of its porosity, and then weighed; then replaced in the liquid to receive further deposits during two days, and again dried and weighed. In two determinations of this kind there were obtained respectively 50·07 and 50·11 parts of the bromide deposit for every 42·5 parts of the chloride deposit, or for every single equivalent, or 32·2 parts of zinc consumed; and in two other determinations 51·2 and 51·4 parts of bromide deposit were obtained. Each of these quantities of deposit contains the same amount of metallic antimony, viz. 40 parts, or $\frac{1}{3}$rd of an equivalent. The variation in the numbers obtained was probably caused by the porosity of the deposit formed in the bromide solution.

Third variety of active Electro-deposited Antimony.

103. A third variety of heat-giving electro-deposited antimony remains to be described, and was obtained in the following manner:—Dissolve 1 part by weight of teroxide of antimony in 15 parts of hydriodic acid of sp. gr. 1·25; filter the solution from any oxide that may remain undissolved, and electrolyse it by two or three Smee's elements very feebly excited and an anode of antimony in the usual manner, and at a speed of deposition not

exceeding 1 grain per square inch per hour. The exact proportion or strength of the ingredients is not important, provided the acid is not too concentrated, sufficient oxide is dissolved in it, and the electric current is very feeble.

104. The deposit is generally at first scaly, especially if the voltaic current is rather strong; but afterwards it is dull in appearance and grey like that formed in the bromide solution, and the tendency to the evolution of gas is much greater than in that liquid. It is also liable to cohesive action and crackling (17. 94.), but in a less degree than the other kinds. Its cohesion is very feeble, and it is much more friable than the bromide variety; its fractured surface is dull and earthy in appearance, and it has a much less metallic character than either of the other kinds, unless it has been deposited with extreme slowness; it is then sometimes bright and black like a blackleaded surface of iron; its cohesion is then also greater, and its fractured surface black and bright. The specific gravity of one specimen which had been formed moderately slowly was 5·27 at 60° Fahr.

105. Repeatedly in depositing this variety in a coherent state at a moderate speed, I observed at the lower part of the deposit on each side of the cathode horizontal lines about $\frac{3}{4}$ths of an inch apart, of a lighter colour than the rest of the deposit. The contrast of colour was caused by a black powdery deposit adhering to the other portions and not upon the lines, and appeared to be more prominent at the lines; but what was the cause of the lines I have not ascertained.

106. On immersing dry pieces in water a hissing sound is produced, evidently by powerful absorptive action, and numerous bubbles of gas are evolved from the whole of its surface during about five or ten seconds, and a few small ones afterwards.

107. A small piece, $\frac{1}{4}$th of an inch thick, placed upon platinum foil upon mercury at 200° Fahr. and touched by a red-hot wire, exhibited no signs of evolving heat; but with the mercury at 338° Fahr., and touched as before, it discharged its heat feebly and slowly, and evolved iodide of antimony. A second similar piece, heated from 300° Fahr. upwards upon platinum foil upon mercury, discharged its heat with evolution of iodide of antimony when the mercury had attained a temperature of 350° Fahr.; but the thermic action was feeble, and occupied about 20 seconds. Its temperature of sudden discharge appears therefore to be about 340° Fahr., and the amount of heat evolved is apparently very much less than with the other varieties. If exposed to strong sunlight during two hours, it acquires a reddish-brown colour externally.

108. The dried substance, fused in an analysis-tube (82.), gave 77·76 per cent. of a metallic button, and a solid, red, easily fusible sublimate, together with a little moisture, manifestly teriodide of antimony, and a little aqueous hydriodic acid.

109. The electro-chemical equivalent of this variety was examined in the same manner as the preceding one (102.). In two determinations, first with slow action (0·5 gr. per square inch per hour), 50·39 parts of deposit were obtained; and second, with very slow action (0·2 gr. per square inch per hour), 48·07 parts were obtained for every single equivalent, or 42·5 parts in the chloride solution. The numbers were lessened in con-

sequence of a small quantity of the deposited substance, which was not firmly deposited, falling off in a state of powder to the bottom of the electrolyte.

Further particulars respecting the first variety.

110. Several experiments (of which the following is an example) were made to determine the influence of speed of deposition upon the state of aggregation, &c. of the deposited metal. Five parts of pure oxychloride of antimony were dissolved in 20 parts of pure hydrochloric acid of sp. gr. 1·15, and the solution (contained in one vessel) electrolysed simultaneously by two separate currents from two single SMEE's elements, one of which contained plates 6·3 times the size of the other, each charged with the same exciting liquid, viz. 1 measure of acid to 18 of water: the antimony anode of the large battery possessed six times the amount of surface of that of the other, but the cathodes were of equal size. The speed of deposition by the small battery was 0·75 gr. per square inch per hour, and by the large one 3·22 grs., or 4·26 times the amount. The rapid deposit exhibited much scaly cohesive action (17.), but the slow one did not. The slow deposit consisted nearly wholly of grey crystalline metal, which did not possess the thermic power, whilst the rapid one was nearly wholly black and amorphous, and possessed the usual heating quality. Owing to the decline of the current, the deposit upon both sides of the slow one was at first black and amorphous, and then became grey; and that upon the back of the rapid one consisted of alternate layers, black, grey, and black, produced by variations of the electric current, by gradual decline, and additions of acid.

111. It is particularly worthy of notice, that whether the alteration of electric power was gradual, as by slow exhaustion of the battery liquid, or sudden, as upon the additions of acid, the change of aggregation and of chemical composition of the deposited metal was sudden in both cases, in accordance with that rigid distinction or want of gradation between the two varieties of deposit already noticed (3.). The cohesion between the alternate layers was much more feeble than between the parts of the layers themselves, and the layers could be readily separated by means of a knife; the lines of junction were quite distinct and definite, although the layers were in perfect contact, and no films of depositing liquid appeared to be enclosed between them. These sudden changes of aggregation are particularly interesting.

112. It appears from this and other similar experiments, that with a suitable liquid, &c., when the quantity of the electric current is small in relation to the amount of receiving surface, the deposited metal is grey and crystalline, and possesses no heating power; and that when it is relatively large the metal is dark-coloured and amorphous, and has the peculiar thermic properties; other circumstances, such as high temperature of the solution, or excess of acid (9. 10. 11.), will also produce grey metal.

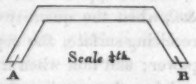
113. In several instances, when the active variety had been *rapidly* deposited, it was less bright and smooth, its colour was lighter, its fractured surface was coarse, and on

discharging its heat by a heated wire, the thermic change was unusually rapid and strong, shattering the metal throughout into scaly particles with almost explosive violence. The shattered mass consisted of parallel scaly layers, possessing an angular position with regard to the receiving surface, their horizontal edges pointing upwards at an angle of about 40° towards the surface of the liquid.

114. Two deposits were also simultaneously formed, both of the black amorphous kind, but at *equal* and nearly uniform rates of deposition, viz. 0·55 gr. per square inch per hour, upon cathodes of equal size and at equal distances from the anodes, by two separate currents, in a similar solution to that of the previous experiment (110.) contained in a single vessel, one current being generated by one SMEE'S element of large surface, and the other by twenty of small surface, the exciting liquid in each battery being the same, and the plates the same distance asunder. Both the deposits appeared alike, except that the one formed by the single element had a few small nodules of grey metal formed upon it during the latter part of the process. The active deposit formed by the twenty elements lost 7·4 per cent. by fusion in the analysis-tube (82.), and 2·7 per cent. by discharging it in the air at 60° Fahr. by a heated wire; and that formed by the single element lost respectively 7·1 and 1·86 per cent. under similar circumstances. These differences of results in the two cases may have been due to other circumstances than the difference of number of elements.

115. It has been already shown (66. 75. 83.) that the discharge of heat may take place gradually, or even in fragmentary portions, by careful management of the temperature in an air-bath, or easily by immersing the substance for a greater or less period of time in boiling water. By suitable treatment, different parts of a given piece of the active substance may also be either wholly or partly discharged of their heat. In one instance about $\frac{3}{4}$ths of an inch of the lower end of an active bar, 2 inches long, $\frac{1}{4}$ an inch wide, and $\frac{3}{16}$ths of an inch thick, was immersed in cold water, the water heated to boiling in 6 minutes, and kept boiling 20 minutes; the lower end was then found to have lost its heating power, whilst the upper end remained unaltered; and in several other similar experiments similar results were obtained. In these and all other cases where the active substance was slowly discharged, the substance became much harder and much more difficult to break.

116. The thermo-electric relation of the changed to the unchanged substance was examined as follows:—Two similar active bars were made of the shape of the annexed figure, containing small silver studs (A and B) in their extremities, enclosed by the antimony during the process of deposition; one bar was discharged of its heat gradually by immersing it in boiling water one hour;



it was then placed parallel to the other, but separated by bits of cork and india-rubber; the two studs at one end of the compound bar were connected together by silver wire, and that end immersed in a solution of chloride of calcium at 60° Fahr.; the other studs and ends, being previously connected by silver wires with a galvanometer, were

wrapped in oil-silk and immersed in water at 60° Fahr.; the chloride-of-calcium solution was now heated to 212° Fahr. in 35 minutes; a gradually increasing deflection, amounting at its maximum to 42½ degrees, occurred, the discharged bar being thermo-electro-positive to the active one; the temperatures were maintained during 2 hours and 40 minutes, during which time the deflection decreased to 38½ degrees. A second similar experiment was made with a pair of bars like the last, one of them being wholly active, and the other previously made inactive at one end only, viz. the end most distant from the galvanometer, by gradual discharge in boiling water during 30 minutes. With one end of this compound bar in water at 50° Fahr., and the other in boiling water, a deflection of 28 degrees occurred, the discharged end being thermo-electro-positive to the active end of the adjoining bar. A similar experiment with a single active bar of the above angular shape, with its ends dipping into two vessels of water, one at 57° and the other at 212° Fahr., and connected with the galvanometer by silver wires, gave a deflection of 42½ degrees, which decreased to 40 degrees in 1 hour and 25 minutes.

117. In numerous experiments the suddenly and also the gradually discharged substance was found in every instance to be decidedly electro-positive to undischarged portions of the same pieces in the following liquids—pure sulphuric, hydrochloric, or nitric acids, freely diluted with water; also in aqueous solutions of ammonia, sesquicarbonate of ammonia, hydrate of potash, and carbonate of soda.

118. A bar of the unchanged substance placed axially in a copper wire helix, the ends of which were connected with a galvanometer, on having its heat suddenly discharged by contact of a heated wire with one of its extremities excited no appreciable electric current in the wire.

Comparison of the three varieties.

119. They each require a solution containing an excess of free acid in addition to the amount necessary to dissolve the teroxide, the proportion of free acid necessary being greatest with the chloride and least with the iodide.

120. The rate of deposition admissible to obtain a firm deposit, and also the range of speed of deposition, is greatest in the chloride and least in the iodide solution; being about 0·5 to 10·0 grains per square inch per hour in the chloride, 3 to 5 grains in the bromide, and 0·5 grain or less in the iodide. The tendency to the evolution of hydrogen gas at the cathode is least in the chloride and greatest in the iodide: this tendency frequently causes the iodide deposit to be completely disintegrated.

121. Each of the deposits exhibits the phenomenon of unequal cohesion and cracking (17.), but that formed in the chloride solution exhibits it the most strongly. The metal obtained from the iodide solution is the most friable, and that from the chloride the least so. The chloride compound is the most metallic in appearance, and the iodide one the least. The specific gravities of the three varieties are also in series; that of the chloride deposit is 5·8, the bromide one 5·44, and the iodide one 5·25.

122. They each possess the power of evolving heat, and this quality appears at

different temperatures in the three substances, being at about 200° Fahr. in the chloride deposit, about 280° in the bromide, and about 340° in the iodide. The manifestations of heat evolved also vary, being considerable in the chloride deposit (73.), apparently less in the bromide one, and but very small in the iodide compound. They each become much less metallic in appearance by a gradual discharge of their heat, and the gradual discharge is also attended by an increase of cohesive force, most in the chloride and least (if any) in the iodide deposit. The iodide compound is also the most affected by solar light.

123. The chloride deposit contains about 6·3 per cent. of saline matter, the bromide about 20, and the iodide about 22·2. The pseudo-electro-chemical equivalent (if I may apply this term to such cases) varies in each variety, and is dependent upon the amount of saline matter which unites with the true equivalent of metallic antimony deposited; it is about 42·5 with the chloride deposit, 50 with the bromide, and 51 with the iodide.

Conclusion.

124. A probably correct explanation of the formation and properties of these several metallic deposits is as follows:—The electric current in passing decomposes the salt of antimony, setting free 1 equivalent of chlorine, bromine, or iodine, and $\frac{1}{3}$rd of an equivalent of antimony for every single equivalent of zinc consumed (48. 102. 109.). The antimony in the act of depositing, being in what is termed the "nascent" state, unites chemically, in a comparatively feeble or unstable manner, with the elements of the electrolyte, combining with them in an indefinite and somewhat variable proportion. The $\frac{1}{3}$rd equivalent, or 40 parts of antimony, carry down from $2\frac{1}{2}$ to 3 parts from the chloride solution, about 10 parts from the bromide liquid, and about 11 parts from the iodide solution, and thus occasion about 42·75 parts of deposit in the first liquid, about 50 parts in the second, and 50·5 parts in the third solution, for each equivalent of zinc consumed.

125. Another explanation, which has nearly an equal weight of evidence in its favour, is that the antimony is deposited in the "*amorphous*" state, and the chloride or other salt is enclosed mechanically in it during the process of deposition, and that the change consists in the assumption by the metal of the crystalline state, whereby it is converted into an inconceivable number of crystals of insensible size, and the imprisoned salt is set free.

126. Antimony is not the only metal that manifests this property of uniting during electrolysis with portions of the electrolyte; several other metals also show it: it is well known that silver deposited by electrolysis from a solution of the double cyanide of silver and potassium containing a little bisulphide of carbon, is harder and very much brighter than that deposited from the same solution without that ingredient; this bright deposit is not wholly metallic silver, but contains a small proportion (about 1 per cent.) of some other ingredients of the electrolyte.

I have not examined the laws or conditions that regulate the proportions of the substances that unite by this species of combination.

127. The substance formed in either of these solutions of antimony may be viewed as a feeble chemical compound of metallic antimony with a salt of antimony: that it is not a purely mechanical mixture is rendered very probable by the fact of its powder not reddening moistened litmus paper, especially in the case of that obtained from the bromide solution, which contains as much as 20 per cent. of saline and acid matter, whereas immediately after the gradual change has occurred it has the power of reddening litmus strongly (101.), and the salt of antimony contained in it may be much more readily extracted by dilute hydrochloric acid (83.); these facts also indicate that the change is, at least in part, a case of chemical decomposition, the substances remaining together after the decomposition in a state of mechanical mixture.

128. The compound deposited is evidently not a *direct* result of electrolysis, otherwise it would be deposited in the proportion of its electro-chemical equivalent; nor is it a *definite* chemical compound, because there is no equivalent or atomic proportion between the quantity of metallic antimony and that of the salt with which it is combined, and because the proportions of these two ingredients are in each case somewhat variable.

129. The decomposition of these deposits, like that of peroxide of hydrogen, is attended by evolution of much heat; it cannot, however, like the decomposition of that substance, be viewed as a result of a tendency of the separated ingredients to assume the gaseous or even the liquid state, because the compound set free is in each case in a nearly solid condition, but must be referred to some other and at present unknown cause.

130. The union of the nascent antimony with the salt of antimony is evidently dependent upon the rate of deposition; for when the speed of deposition in the chloride solution fell below about 0·75 or 0·5 grain per square inch per hour, the antimony lost its power of combining with the antimonial salt, and was deposited alone in the grey or crystalline state (110.). It is also dependent upon the temperature of the liquid (11.), and upon the proportions of the ingredients of the solution (9.).

131. Numerous and varied electrolytic experiments were made with solutions of arsenic, and small portions of scaly deposit were obtained from the aqueous fluoride, which exhibited in a comparatively feeble and indistinct degree (like the third variety of active antimony) a similar thermic property to that manifested by electro-deposited antimony.

XVIII. The Bakerian Lecture.—*On the Total Solar Eclipse of July 18th, 1860, observed at Rivabellosa, near Miranda de Ebro, in Spain.*

By Warren De La Rue, *Esq., PhD., F.R.S., Hon. Sec. Royal Astron. Soc., Treasurer Chem. Soc., &c.*

Received January 30,—Read April 10, 1862.

My attention was first called to the Solar Eclipse of 1860, in the latter part of the year 1858, on the occasion of my visiting Russia, when Dr. Mædler placed in my hands a copy of his anticipative pamphlet, entitled "L'Eclipse Solaire du 18 Juillet, 1860."

This paper contained a Map of Spain, with certain lines indicating the position of the central path of the moon's shadow, the limits of totality, and its epoch at various localities; and it occurred to me, on perusing it, that, if circumstances should permit of my observing the eclipse, Santander would be very convenient for the disembarcation and erection of the instruments I should, in all probability, require for photographic observations, to the prosecution of which my successful researches in astronomical photography led me to think I ought to devote myself. On communicating my plans to Mr. Vignoles, he strongly recommended me to cross to the southern side of the Pyrenees in order to avoid the mists which are caused by the condensation of vapours from the ocean against the northern slopes of the mountains. Subsequently Mr. Vignoles published an eclipse-map of Spain on a very large scale, and I selected Miranda for my station; but he suggested that I should place my observatory at Rivabellosa, about two miles from that town.

It is fortunate that I changed my station from Santander to Rivabellosa, as many of those astronomers who selected the former place were prevented by the state of the atmosphere from observing the eclipse.

On my journey to Russia, I stopped at Königsberg and made the acquaintance of Dr. Luther, who showed me the Daguerreotype of the total eclipse of 1851, which had been taken by Dr. Busch with the Königsberg heliometer. Great credit is due to Dr. Busch for that successful pioneering experiment, more especially when due allowance is made for the uncertainty then existing as to the brilliancy of the prominences, and for the state of the photographic art at that epoch. In the interval of seven years, however, astronomical photography had made great progress; and I recollect being much struck with the very indifferent definition of the protuberances in the Daguerreotype, from which I inferred the impracticability of deriving any conclusive evidence respecting the nature of such appearances from photographs, unless more distinct ones could be obtained. The inspection of the Königsberg Daguerreotype subsequently exercised some influence on my plan of procedure. Discarding all thoughts of employing the Daguerreo-

type process, because the collodion process was far more sensitive and convenient, I chose the latter as best suited to my purpose, although I knew perfectly well, from experience, how frequently the collodion-film is rendered defective by specks, streaks, and even minute holes. It was open to me to employ an achromatic or a reflecting telescope of ordinary construction, and to place the sensitive plate in the principal focus; but I was aware that the largest telescope I could possibly take with me would only give an image of a very moderate size, and that any of the before-named defects in the collodion might fall over and obliterate, or so confuse the impression of any prominence in one photograph, as to render its identification with its impression in a subsequent photograph a matter of impossibility. These considerations led me to think that it would be very desirable to employ the Kew photo-heliograph, because in this instrument the primary focal image of the sun is enlarged from about half an inch in diameter to nearly 4 inches, which is a scale amply sufficient to counterbalance the disadvantages of the collodion process; but, on the other hand, the light is thus attenuated sixty-four times, besides being absorbed to some extent in passing through the two lenses composing the secondary magnifier, an ordinary Huyghenian eyepiece; and this question consequently presented itself, Would it be possible with such an enfeebled image to get even a single impression during the whole duration of the totality? This was an extremely doubtful matter. By employing the Kew heliograph one would evidently run the risk of returning without any pictures of the *totality*, however many might be procured of the other phases of the eclipse*.

At the meetings of the Astronomical Society, and on other occasions, I made inquiries of those astronomers who had witnessed the eclipse of 1851, respecting the intensity of the light of the corona and red flames, as compared with that of the moon, and the relative brightness of the one to the other; but their answers did not tend to increase my hopes in respect of the possibility of procuring photographs of the totality by means of the Kew instrument. The general impression I formed from the information thus derived was, that the light emitted by the corona and red flames, taken together, was about equal to that of a full moon—less rather than greater; but no one recollected precisely the brightness of the prominences as compared with that of the corona. With this imperfect information as a guide, an attempt was made at Kew to photograph the moon, but not the slightest impression could be procured of our satellite by an exposure of the sensitive plate, during one whole minute†, to its image in the heliograph. My expectation of success in getting pictures of the totality was not great after this trial; nevertheless I still thought it desirable to carry on the experiment to the end, on account of the value of the results if I should fortunately succeed. It occurred to me several times to fit up also a photographic apparatus to an achromatic telescope, but I finally concluded that to attempt too many things would be sure to result in complete failure. I endeavoured, however, to stimulate other astronomers to

* Report on Celestial Photography, by the author, in the Reports of British Association, 1859, p. 152.

† While this paper was passing through the press a very faint impression of the moon was procured with the Kew heliograph in three minutes, with chemicals which gave a very strong impression of it in four seconds in the focus of my reflector of 13 inches aperture and 10 feet focal length.—August 1862.

make photographic experiments in the manner which I have indicated, as offering a greater probability of at least a partial success, so that the chances of obtaining pictures might be multiplied. With this object, I circulated as rapidly as I could my Report to the British Association on "Celestial Photography," which passed through the press in May 1860, and of which copies were extensively sent both to English and foreign astronomers at the latter end of May and the beginning of June.

I have now in this narrative to go back some months earlier than the period just alluded to, in order to connect it with the Himalaya Expedition, an expedition originating solely with, and organized by, the Astronomer Royal. When the year 1859 was drawing to a close, and I was turning my thoughts to the preparation which would be required for July 1860, Mr. AIRY mentioned that, if I had any intention of observing the eclipse, he might possibly be in a position to afford me some facilities for so doing, as he had it in contemplation to make an application to the Admiralty for a ship to convey intending observers to Spain, in the event of a sufficient number of astronomers expressing their willingness to join the expedition which he intended to organize. I expressed the satisfaction I felt in learning that he, the official head of astronomy in England, was willing to take the matter in hand, because I felt persuaded that, under his general direction, the expedition would prove a successful one; and I at once volunteered to form one of his party.

It was intimated to me that, in the event of my taking charge of the Kew heliograph, I should not be expected to entail upon myself the expenses of fitting it up for the object contemplated, or the personal expenses of the assistants who might accompany it, it having been from the first intended, that a grant from the Government Fund should be asked for to defray these charges. When, therefore, I had finally decided on taking charge of the instrument, I was requested to propose such a sum as I thought fully adequate to the purpose, and I named £150, which was granted. The entire expenses of the photographic expedition amounted to more than three times that sum, the balance being defrayed by myself.

The actual preparations were commenced at the latter end of January 1860, first of all by Mr. BECKLEY, the mechanical assistant of the Kew Observatory, and were continued, so far as his other occupations would permit, until the month of June; but, in spite of every exertion on his part, so much remained to be done, that in June I engaged Mr. REYNOLDS (now my private photographic assistant) to aid in completing the arrangements. My party, besides myself, was, after a few changes, thus finally constituted:—Mr. BECKLEY, Mr. REYNOLDS, Mr. DOWNES, and Mr. E. BECK.

Among the preparations to be made was a stand for the telescope, the cast-iron pedestal of the Kew heliograph being too heavy for convenient transport. It was necessary, moreover, to make some contrivance for supporting the frame of the polar axis in a position adapted to certain limits of latitude, within which I might fix my station; and it was thought that this could be best arranged by making a new cast-iron pedestal composed of several pieces which took apart for the convenience of carriage*.

* This iron stand has been left *in situ*, and thus marks the precise locality of my observatory.

Originally, merely a temporary tent in which to develope the photographs was procured; but when it was known that H.M.S. 'Himalaya' would be placed at the disposal of the Astronomer Royal, I put this aside, and caused a complete photographic observatory to be constructed, part to contain the heliograph with a removable roof, and part divided off and fitted up as a photographic room, with a cistern, to be filled from the outside, a sink, and with tables and shelves to hold the apparatus and photographs. This observatory took to pieces, and every part was marked when in its place, so that no time need be lost in putting it together again in its destined position. Besides the ordinary roof, there was another covering, consisting of strong canvas, supported at the distance of about three feet from the walls and roof of the developing-room. The object of this was to prevent the overheating of the photographic room, a circumstance most detrimental to photography. This canvas was kept wetted with water, in order that the evaporation might lower the temperature of the stratum of air between it and the observatory, and it fulfilled the object perfectly. The canvas, when the observatory was not in use, was drawn over the room containing the heliograph, and protected the instrument from rain.

The print exhibits the arrangement of the observatory, when secured after the day's work. The position of the canvas, and the simple arrangement for maintaining it at the proper distance from the house, and also the outside cistern, are well shown in the picture, which is copied from a photograph taken on the occasion of Mr. AIRY's visit to my station. When the observatory was at work, the canvas was removed from the front (the south side) and tied back as far as the upright which is seen on the western side, the top boards also being removed. The front boards were of a height which admitted of our observing the sun above them whenever it was desirable to do so.

Photographic chemicals were prepared in duplicate; part of the collodion intended to be used was mixed with the iodizing solution in London, and after subsidence was carefully decanted previous to packing, in order to avoid the defects before alluded to; but collodion and iodizing solution were also taken separately, so that some might be prepared on the spot, and used, if found free from defects, in that state of extreme sensitiveness which exists in collodion freshly iodized with the potassium iodizer. Nitrate-of-silver baths, prepared in the ordinary way with crystallized nitrate of silver, were taken, and were used in depicting the several phases of the eclipse, with the exception of those of totality. In taking the latter pictures the baths used were made with nitrate of silver which had been fused carefully in my own laboratory, and were so extremely sensitive that they would give photographs of the full moon in the focus of my reflector in less than a second of time, while with the usual bath five seconds were barely sufficient to give a picture of similar intensity.

As few astronomers perhaps are aware of the number of materials required for such an expedition, I here give the list of contents of one of the boxes of chemicals.

Packages.	Contents.	Packages.	Contents.
3	Half-pint bottles of Collodion.	1	½ oz. Iodide of Potassium.
1	Four-ounce Bottle of Collodion, iodized.	1	Ounce Measure.
		1	Gallon Distilled Water.
1	Half-ounce Bottle of Pyrogallic Acid.	1	Set of Scales and Weights.
		3	Plate-drainers.
1	Six-ounce Bottle of Acetic Acid.	1	4 oz. of Tripoli.
1	1½-pound Bottle of Hyposulphite of Soda.	1	Packet Cotton Wool.
		1	Glass Funnel.
1	Case containing Oxide of Silver and dilute Nitric Acid, in separate bottles, for correcting the bath, in case of need.	1	Retort Stand.
		1	Lantern.
		3	Bottles of Varnish.
			Test Papers.
			Filtering Paper.
2	24 oz. of Nitrate-of-silver Bath.	4	8-oz. Mixing Glasses for Collodion.
1	2 oz. Crystals Nitrate of Silver.		
1	4 oz. fused Nitrate of Silver.		

The apparatus, when completed, weighed 34 cwt., and was made up into thirty pack-

ages for convenience of transport. Among the miscellaneous requisites were included:—distilled water weighing 139 lbs.; engineers' and carpenters' tools weighing 113 lbs.; lanterns, lamp-oil, spirit-lamp, and spirits of wine, weighing together 73 lbs.; a small stove and kettle for boiling water, and, lastly, some preserved provisions, in case the party should be compelled to encamp for a few days. Owing, however, to the excellent arrangements most kindly made by Mr. VIGNOLES, the latter were quite unnecessary.

As my plans became matured, it occurred to me that it would be desirable to make determinations of geographical position; and I therefore borrowed from the Kew Observatory a small transit theodolite with 6-inch altitude and azimuth circles, both reading with the verniers only to one minute of arc. The optical part of the instrument was found to be very indifferent, and the readings of the altitude circle were, from some cause, not so accordant from time to time as they ought to have been. I took with me also three chronometers—a box chronometer, a pocket chronometer, both indicating Greenwich mean time, and my journeyman sidereal chronometer.

I was induced to make preparations for eye-observations (which I did not originally contemplate), partly in order that I might be in a position to interpret from my own sketches and recollections the results of the photographs, and partly because in case I should fail in making photographs, I might still be able to contribute something to the series of optical observations. I therefore took with me a beautiful achromatic of 3 inches' aperture, which Mr. DALLMEYER had kindly lent me for the occasion. This telescope was mounted by Messrs. TROUGHTON and SIMMS on a most convenient and steady altazimuth-stand, designed by the Astronomer Royal. The equatorial movement was effected by the joint action of two radius bars, which enabled me to keep the sun exactly within a tangential square, which I had had ruled upon glass and placed in the focus of an eyepiece to be described hereafter.

Lastly, through the kindness of Messrs. ELLIOTT and Mr. CASELLA, I obtained the loan of some meteorological instruments. Messrs. ELLIOTT lent me one of their excellent aneroid barometers. Mr. CASELLA lent me a marine barometer and a standard thermometer, both verified at Kew, the readings of which were used in the reductions of the astronomical observations.

The apparatus was sent to Plymouth on the 5th of July, whence we set sail on the 7th; on the 9th we reached our destination. Mr. VIGNOLES met the 'Himalaya' in a small steamer which he had chartered to convey ashore the astronomers who intended to land at Bilbao, with their apparatus and luggage; but I am placed under a further obligation to him, not only for his kind and liberal hospitality during my stay at Bilbao, but also for dispatching my apparatus, as soon as it was landed, to Rivabellosa, which is situated at a distance of seventy miles from the port of Bilbao, and is only accessible through a pass difficult for the transmission of heavy baggage.

On the evening of the 10th we left Bilbao in a diligence which I had engaged to convey my party to Rivabellosa, at which place we arrived on the next day, after a journey very trying to our chronometers.

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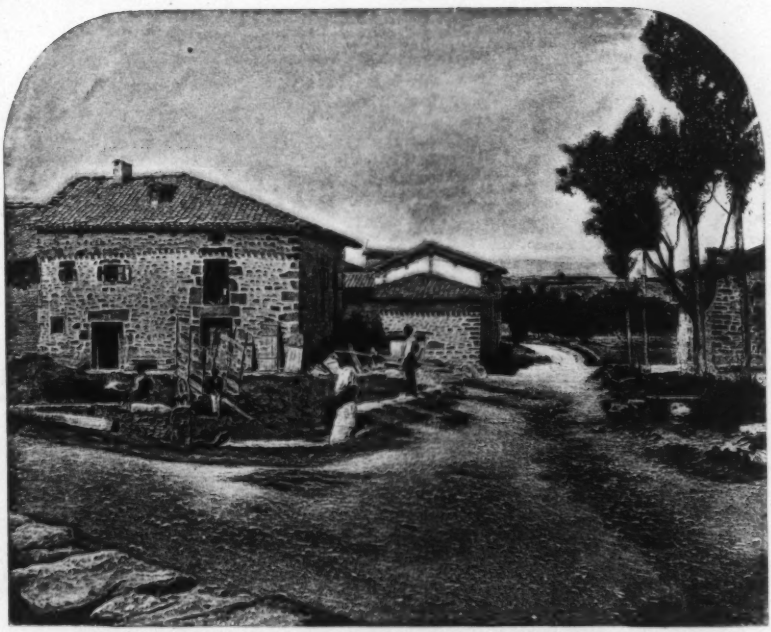


THE ALCALDE'S HOUSE AND THRASHING FLOOR. MR. DE LA RUE'S LODGINGS.

THE VILLAGE OF RIVABELLOSA.

Printed by the ordinary Letterpress, from a Block produced by means of Photography and Electrometallurgy.
Absolutely untouched by the graver.

The instruments reached Rivabellosa on the evening of the 11th; the previous part of the day had been occupied in taking a general survey of the country around the village, with the object of selecting a site whereon to erect our observatory, and I at length settled on one of the thrashing-floors* which are to be seen in great numbers in that country in the open fields. It was about 60 feet in diameter, and close to the road, which we found to be a great convenience, inasmuch as the water required for our use had to be brought from a distance. Moreover, it was level, and extremely hard and dry. I had hardly selected this site when I learned, with some concern, that the harvest had commenced, and that the proprietor intended to make use of the floor on the morrow for his thrashing operations, which it is the custom of the country to complete immediately after the reaping. However, Don Simon, land-surveyor of the Bilbao and Tudela Railway, who explained this to me, kindly undertook to negotiate for the hire of the station; but the owner, when informed that his thrashing-floor was the best adapted for my purpose of any place I had seen, at once said that it was quite at my disposal, and, although he had to convey his grain to a distance, refused any remuneration.

The instruments were conveyed to the thrashing-floor, and the transit theodolite unpacked in time to make an observation of the sun soon after 10 o'clock on the morning of the 12th. By the evening the observatory was erected and in actual operation, and a photograph of the sun was obtained on the 14th. To my staff was at last added Mr. S. Clark, who had acted as interpreter, and who kindly volunteered his assistance during the eclipse. And I am greatly indebted to the gentlemen composing my staff for their most efficient assistance. With a self-denial hardly to be expected under the circumstances, each carried out steadily his allotted task, and thus contributed to a result which is most gratifying, after so much preparation and trouble.

Upwards of forty photographs were taken during the eclipse, and a little before and after it,—two being taken during the totality, on which are depicted the luminous prominences, with a precision as to contour and position impossible of attainment by eye-observations.

Photographs of the sun were taken on the two days succeeding the eclipse, namely, the 19th and 20th, and the instruments were then taken down and packed.

On the 26th the Himalaya Expedition re-embarked on board the 'Himalaya,'—myself, staff, and baggage being reconveyed to the vessel from Bilbao through the kindness of Mr. Vignoles, who accompanied us to England. On the 28th we landed at Portsmouth. I must here take occasion to express my best thanks to Captain Seccombe and the officers of the 'Himalaya' for their great kindness during the outward and homeward voyage. My thanks are also due to Señor Montesino, Chairman of the Bilbao and Tudela Railway, for assistance rendered; also to Don Simon, and to Mr. Bennison and Mr. Preston, gentlemen belonging to Mr. Vignoles's staff.

* In the accompanying print a thrashing-floor, similar to the one I employed, is represented. The Plate is inserted partly with the object of calling attention to Mr. Paul Pretsch's phototype process, which I have recently employed to furnish representations of sun-spots. (See Monthly Notices Roy. Ast. Soc. vol. xii. p. 278.)

After having obtained the photographs and got them home safely, my work was only begun. I knew that they contained in themselves most valuable records; but it did not, in the first instance, appear so clearly how I could turn them to account. The two totality pictures presented the most interest, and to them I first turned my attention; and as it was evident that no measurements ought to be made on the originals, I then bethought me of the best means of multiplying them. In the first instance, I got some enlarged positive copies photographed by Mr. Downes, and, made some little progress in measuring them; but I soon found that I should require others, and, on attempting to have some made a little later in the year, I experienced an amount of difficulty I never could have anticipated. The original negatives proved to be so extremely intense, that nothing short of unobscured sunlight would penetrate them and reveal their details; so that it was only by working for many selected days with Mr. Downes, that I succeeded in getting a sufficient number of positive copies for my purpose. Those who remember how remarkably dull and wet the year 1860 was in England will readily understand that the selected days were few and far between, so that it was fast approaching winter before I had got on much with the work on the photographs. I also had recourse to the albumen process, and obtained a few copies of the size of the original by superposition, without the intervention of the camera. These were made in my presence, at Messrs. Negretti and Zambra's, and from these positives some negatives were taken. Although these copies did not aid me greatly, it was fortunate they were taken; for in course of time the original negative No. 26, the second of totality, gave indications of decay, and on attempting to save it by revarnishing it, the collodion expanded, and crinkled up so much that, except as a record of what was done, the original negative is spoiled. I have protected it from further injury by cementing it with Canada balsam to a second glass plate; but one of the albumen negatives must now supplement it, if more copies are ever taken by direct superposition. Enlarged negatives, however, exist; but, as something is always lost in copying, the damage to the original negative is unfortunate*.

Measurements of the enlarged positives on glass of the totality pictures soon proved to me that the most accurate results could be obtained by measuring the photographs of the other phases, and that these results would indicate the path of the moon, and the position of the centres at the epoch of totality, independently of any determinations of geographical position. For this object it became necessary to measure the original negatives, because the slightest deviation of the optical axis of the copying camera from a right angle to the plane of the sensitive plate, or the least eccentricity, would cause a distortion of the cusps. I had, therefore, to devise an instrument for measuring the photographs; and having considered how the object could be effected, I put one in hand with Messrs. Troughton and Simms, who constructed it for me with their usual skill and precision. After the instrument had been made, it was found to be convenient that some of the parts should be provided with a means of adjustment; so that although it was commenced in February of 1861, it was not until July 18th that it was

* The whole of the original negatives have been deposited with the Astronomer Royal.

completely finished and ready for use. Since that time, every spare moment has been devoted to the final accomplishment of this work; and, taking into account the interruptions I am subject to, I feel convinced that it could not have been done in less time, although I candidly confess that the delay in sending in this Report must appear scarcely warranted.

Determination of the Geographical Position.

Previous to starting for Spain, I made certain preparations for facilitating the after-operations which I might have to carry out; for I knew that the time allowed for getting the instruments into position would not be more than sufficient, even if each day permitted of observations being made. For this reason, I computed, with the formula

$$\frac{\sin(\alpha \mp \lambda)}{\sin \alpha} \quad (\alpha = \text{the polar distance}, \lambda = \text{the colatitude}),$$

a series of star constants for all the stars likely to be visible in my instrument. I also computed a table of corrections, to be applied to the times of equal altitudes of the sun for intervals of two, three, four, five, and six hours, for each day from July 12 to July 22; and similar corrections to the azimuths of equal altitudes, to enable me to lay off at once a meridian line and erect a mark. My star constants did not, however, prove of much service; for it was rarely that I could get a glimpse of the stars; so that on only two occasions was I enabled to make any observations at all, and then only with the greatest difficulty, although I watched patiently for opportunities through or between the clouds.

I have already stated that I took with me three chronometers. My journeyman sidereal chronometer, Leplastrier 2915, is a very old one. In consequence of the wear of the fusee, this chronometer, which is an eight-day one, varies in its daily rate from 8·6 seconds to 17 seconds, losing; but during long periods the rate is pretty uniform, as will be seen from the following observations:—

Leplastrier 2915.

	sec.
From June 30 to July 3, losing daily	10·36
From July 3 to July 30, losing daily	12·51
From July 30 to August 7, losing daily	12·06
From August 7 to August 17, losing daily	11·29
On July 3, $21^h\ 24^m$, it was fast of Cranford sidereal time	194·8
Cranford observatory is west of Greenwich in longitude	97·5
Hence, on July 3, $21^h\ 24^m$, Leplastrier No. 2915 was fast of Greenwich sidereal time	97·3
	194·8
On July 30, $6^h\ 30^m$, it was slow of Cranford sidereal time	138·0

The pocket chronometer, Frodsham No. 9768, was found to have the following rates and errors:—

	sec.
From June 26 to June 29. its daily rate was losing	1·05
From June 29 to June 30, its daily rate was losing	2·84
From June 30 to July 3, its daily rate was losing	2·00
	h
On June 26, 15, it was slow of Greenwich mean time	3·0
On July 3, 15, it was slow of Greenwich mean time	15·0

From these data, the mean daily rate was 1·71 seconds losing. On the 17th or 18th of July this chronometer tripped, most likely in consequence of its having been touched by the inmates of my lodgings; and it was therefore useless to make any determination of its error on my return from Spain.

From a comparison with the box chronometer to be next mentioned, for the loan of which I was indebted to the kindness of Mr. Frodsham, I estimated the error on Greenwich mean time of Frodsham 9768 to be 3·2 sec. slow, for July 3, 15^h, instead of 3 sec., as found by observation. The comparison was made on June 25, 22^h: two hours later Mr. Ellis found the box chronometer, Frodsham

	sec.	
No. 3094, to be on June 26, 0^h	1·9	slow of Greenwich mean time.
In two hours 3094 would have gained	0·08	
At the time of comparison 9768 was slow of 3094	0·50	
Between the comparison and the ascertaining of the error of 9768, namely 17 hours, the latter would lose	0·74	
Estimated error of Frodsham 9768, on June 26, 15^h	3·22	

This difference between 3·2 sec. and 3·0 sec. might have been occasioned by the journey of No. 3094 to Greenwich, but in any case was so small that, for the subsequent calculations, I retained without correction the error afterwards determined by observation on July 3, 15^h, when Frodsham 9768 was slow of Greenwich mean time 15·0 sec.; whence on July 5, 0^h, its error would have been 17·3 sec. slow of Greenwich mean time.

The box chronometer, Frodsham No. 3094, was (through the kindness of the Astronomer Royal) compared at Greenwich by Mr. Ellis, whose determination of its errors are given below.

	h	m	sec.	
June 26 . . .	0	0	1·9	slow of Greenwich mean time.
June 27 . . .	0	0	1·0	
June 28 . . .	0	0	0·5	
June 29 . . .	0	0	1·3	fast of Greenwich mean time.
June 30 . . .	0	0	2·3	
July 1 . . .	0	0		
July 2 . . .	0	0	3·6	
July 3 . . .	0	0	4·5	

The average daily rate of this chronometer was therefore gaining 0·91 sec.; and assuming this rate to have continued, on July 5 0^h its error would have been fast of Greenwich mean solar time 6·3 seconds.

On the supposition that the pocket chronometer would continue to lose 1·71 sec. daily, and that the box mean-time chronometer would continue to gain daily 0·91 sec.

box chronometer Frodsham 3094 would gain over Frodsham 9768 . . 2·62 secs. daily.

The subjoined Table contains the assumed errors of each chronometer, the estimated difference between the two chronometers, and the difference actually observed as nearly as possible at noon of each day—the observation being reduced to noon:—

Date.	Assumed error of Frodsham 3094.	Assumed error of Frodsham 9768.	Estimated difference between 9768 and 3094.	Observed difference between 9768 and 3094.
	sec.	sec.	sec.	sec.
July 5.	fast 6·3	slow 17·3	−23·6	−24·5
July 6.	fast 7·2	slow 19·0	−26·2	
July 7.	fast 8·1	slow 20·7	−28·8	
July 8.	fast 9·0	slow 22·4	−31·4	
July 9.	fast 9·9	slow 24·1	−34·0	−35·0
July 10.	fast 10·8	slow 25·8	−36·6	−37·0
July 11.	fast 11·7	slow 27·5	−39·2	−37·5
July 12.	fast 12·6	slow 29·2	−41·8	−37·5
July 13.	fast 13·6	slow 30·9	−44·5	−38·5
July 14.	fast 14·5	slow 32·6	−47·1	−40·0
July 15.	fast 15·4	slow 34·3	−49·7	−41·5
July 16.	fast 16·3	slow 36·1	−52·4	−42·5

On examining the two box chronometers immediately after our arrival at Rivabellosa, Leplastrier 2915 was found to be apparently uninjured, but I was chagrined to find that Frodsham No. 3094 had been most severely disturbed by the joltings of our vehicle, notwithstanding the protection of its outside padded case, and an extra precaution I had taken to press shavings into its own case, to keep it firm in its place. The cap of the glass had become unscrewed, the glass had shaken out, and the chronometer itself, shifting from its normal position, had risen out of its seat; fortunately, however, the glass could not move far, on account of the wadding, and the hands were consequently uninjured. I succeeded in replacing the chronometer, and in putting the glass into its frame; but it thenceforward took up an entirely new rate, as was evident on comparing the differences between its readings and those of the pocket chronometer. An inspection of the foregoing Table shows that up to the 10th the two chronometers maintained the average rate assigned to each; for example, the computed difference minus the observed difference on that day amounted to only −0·4 second. After my return to England, chronometer No. 3094 was, with the Astronomer Royal's kind permission, again compared by Mr. Ellis, who found the following errors from Greenwich mean time:—

For Frodsham 3094.

	h	m	secs.	
August 10	0	0	2·8	slow.
August 11	0	0	3	slow.
August 13	0	0	2·7	slow.
August 14	0	0	3	slow.
August 15	0	0	2·2	slow.
August 16	0	0	2·6	slow.
August 17	0	0	2·8	slow.
August 18	0	0	2·7	slow.
August 20	0	0	2·5	slow.

its average daily rate being losing 0·03 second, whereas, before starting, it was gaining 0·91 second.

On the 16th of July, Mr. Otto Struve, Dr. Winnecke, and Lieut. Oom visited my station; and I took advantage of their doing so, to make a comparison of chronometers. Dr. Winnecke estimated the error of Frodsham No. 3094 to be fast of Greenwich mean time 7·5 seconds; consequently as the pocket chronometer Frodsham No. 9768 was slow of No. 3094 42·5 seconds, it follows, from this comparison, that it was −42·5 seconds +7·5 seconds =35 seconds slow of Greenwich mean time, which differs by only −1·1 second from the error assigned to it, by applying its mean daily losing rate of 1·71 second.

It appears, therefore, that the pocket chronometer could be relied on up to the 16th of July. On correcting the assumed error from Greenwich mean time of No. 3094 by comparisons with the pocket chronometer, we obtain the following:—

Frodsham No. 3094.

	Computed error from Greenwich mean time, taking No. 9768 as the standard.	Assumed error from Greenwich mean time applying its average rate.
July 9	fast 10·9 seconds	fast 9·9 seconds
July 10	fast 11·2 seconds	fast 10·8 seconds
July 11	fast 10·0 seconds	
July 12	fast 8·3 seconds	
July 13	fast 7·6 seconds	
July 14	fast 7·4 seconds	
July 15	fast 7·2 seconds	
July 16	fast 6·4 seconds	

With the advantage of the comparison made on the occasion of Mr. Struve's visit, I have confidence in assuming the error of Frodsham 3094 to have been from +6·4 to +7·5 seconds, say +7 seconds, on July 16 at noon.

On the 19th the Astronomer Royal and his party honoured me with a visit, and I had the great satisfaction of hearing from our leader that he was well satisfied with the suc-

cess of my photographic operations, and also with the arrangements of the observatory, and the many preparations which had been made to secure the result.

At $2^h\ 30^m$, a comparison was made between Mr. AIRY's pocket chronometer Molyneux No. 1007 and the box chronometer Frodsham No. 3094; Molyneux was slow of Frodsham 3094 43 seconds.

Molyneux 1007 was slow of Greenwich mean time,

	h	sec.	
On July 16 . .	23	28·8	losing daily 7·2 seconds.
July 17 . .	22	35·7	gaining daily 2·7 seconds.
July 18 . .	22	33·0	

Applying the latter rate, Molyneux would appear to have been, on July 19, $2^h\ 30^m$, 32·5 seconds slow, and consequently Frodsham 3094 10·5 seconds fast of Greenwich mean time. I believe that Molyneux 1007 could not be greatly depended on; but, the comparison of chronometers having been made, I place the result on record, although I am not able to make it accord with the other observations within several seconds.

Observations.

The following observations were made with the transit theodolite. During the day the instrument had to remain exposed to the sun; and this caused the several parts to expand very unequally, and kept the bubble in the level always in motion—a circumstance which proved very troublesome.

Estimation of Longitude.

July 12. Four pairs of reduced observations of equal altitudes of the sun showed that at local mean noon

Frodsham 3094 was fast of local mean time . . . 11 min. 51·9 sec.

July 14. Two pairs of reduced observations of equal altitudes of the sun showed that at local mean noon

Frodsham 3094 was fast of local mean time . . . 11 min. 51·3 sec.

With reduced observations of the azimuths of equal altitudes of the sun on the 12th and the 14th, northern and southern adjustable meridian marks were placed, the first against a building, the second against some trees; both sufficiently distant to give distinct vision of the mark, which was a cross × of wood, moveable in a top and a bottom groove in a wooden frame.

Attempts were made on the night of the 13th to obtain observations of stars; but the weather was too cloudy; by dint of perseverance, however, I did manage to get, through breaks in the clouds, the meridian altitude of α Lyræ, and an altitude of the pole star out of the meridian, presently to be referred to in the determinations of latitude.

On the night of the 14th I was more fortunate, and was able to obtain observations of a high and low star, and finally to adjust the meridian marks by means of an observation of δ Ursæ Minoris on the meridian. As soon as the observation of δ Ursæ Minoris

was made, an assistant at each of the marks held a lantern near it, so that being illuminated I might see it with the theodolite and thus be enabled to make signals to them for the permanent adjustment of the marks, which was successfully accomplished. The following day being Sunday, no work was done; but on Monday Mr. PRESTON volunteered to aid me in projecting my meridian towards the station of the Astronomer Royal at Pobes. Using, as a signal flag, a bed sheet, which was not at all larger than was necessary in order to be well seen, I was able to direct Mr. PRESTON where to erect a staff with a similar appendage, in the line of my meridian, towards the north, and at a distance of about seven miles from Rivabellosa. Mr. O. STRUVE, as I before said, visited us on the 16th and undertook to work out the geodetic survey, which it was agreed should be made to connect the Pobes and Rivabellosa stations.

July 14. Observations of α Scorpii and ζ Herculis, when reduced, showed that at 16 hour 28 min.

Leplastrier 2915 was fast of local sidereal time . . . . 11 min. 1·7 sec.

The weather afforded no other opportunity for star observations for longitude determinations until the night of the 18th, when I was too much fatigued to avail myself of it.

July 16, 0 h. A transit of the sun showed that

Frodsham 3094 was fast of local mean time . . . . . 11 min. 49·9 sec.

July 20, 0 h. A transit of the sun showed that

Frodsham 3094 was fast of local mean time . . . . . 11 min. 41·5 sec.

July 20, 8 h. 0 min. sidereal time. The transit of the sun being observed simultaneously with the sidereal chronometer, showed that

Leplastrier 2915 was fast of local sidereal time . . . . 9 min. 55·8 sec.

From the foregoing observations are derived the following results.

Frodsham No. 3094.

	Fast of Rivabellosa, mean solar time.			Daily rate.	Error on Greenwich mean time as estimated by comparison with Frodsham 9768.
	h	m	sec.	sec.	sec.
July 12th . .	0	11	51·9		Fast 8·3
				Losing 0·3	
July 14th . .	0	11	51·3		Fast 7·4
				Losing 0·7	
July 16th . .	0	11	49·9		Fast 6·4
				Losing 2·1	
July 20th . .	0	11	41·5		

whence the Longitude was West of Greenwich

	m	sec.
July 12 . . .	11	43·6
July 14 . . .	11	43·9
July 16 . . .	11	43·5
Mean . . .	11	43·7

Taking Dr. WINNECKE's estimate of the error of Frodsham No. 3094 on the 16th, namely, 7·5 seconds fast of Greenwich, we get for the longitude

West 11 min. 42·4 seconds

Applying the average daily losing rate of 12·51 sec. since July 3rd, 21 h. 24 min. for Leplastrier, we derive the following results from the observations made with that chronometer :—

Leplastrier No. 2915

	h	m	Was fast of local sidereal time. m	sec.	Error on Greenwich sidereal time by applying mean daily rate.	m	sec.
July 14 . .	16	47	11	1·7	Slow	0	37·9
July 20 . .	8	0	9	55·8	Slow	1	48·4

whence the longitude was West of Greenwich

	m	sec.
July 14 . .	11	39·6
July 20 . .	11	44·2
Mean .	11	41·9

By combining all the foregoing results, and taking the arithmetical mean, we obtain for my Observatory at Rivabellosa

	m	sec.
	11	43·7
	11	42·4
	11	41·9
the longitude . . . . . West	11	42·7

Observations for Latitude.

July 12. The reduced zenith distance of the sun's upper limb, when on the meridian, gave as the latitude of Rivabellosa

N. 42° 42′ 37″

July 13. An observation of α Lyræ on the meridian and of *Polaris* out of the meridian, when reduced, gave for the latitude freed from error of the level,

N. 42° 42′ 19″.

July 14. An observation of the zenith distance of the sun's upper limb previous to the meridian passage at the hour angle 24° 5′ 15″ gave, when reduced, the latitude

N. 42° 41′ 48″.

July 20. The reduced zenith distance of the sun's upper limb, when on the meridian, gave the latitude

N. 42° 41′ 17″.

Combining the foregoing determinations of latitude, we obtain

	°	′	″
July 12. Observations of sun	42	42	37
July 13. Observations of stars	42	42	19
July 14. Observations of sun	42	41	48
July 20. Observations of sun	42	41	17
as the mean . . . . . Latitude N.	42	42	

The foregoing numbers are not so accordant as I could desire; but they are, I believe, as good as could have been obtained with the instrument, particularly under the circumstance of its continual exposure to the sun during its employment in the day-time.

Elevation of the Station.

Before leaving Bilbao on the 10th, the aneroid barometer was read off, when it stood at 30·019 in., temperature 71° Fahr. On arriving at Rivabellosa it indicated 28·473 in., the temperature being 65° Fahr. With these numbers I estimated the height to be 1481 feet above the ground floor of Mr. VIGNOLES' house at Bilbao, which is several feet above the mean sea-level.

Mr. PRESTON, however, was so kind as to connect my station by levelling with a normal point on the railway, and made its height to be 1572 feet 4 inches above the mean sea-level.

Recapitulation.

The geographical position of my observatory at Rivabellosa was, therefore,

Latitude N. 42° 42′, Longitude W. 11 min. 42·7 sec.,

and its height above the mean high-water mark 1572 feet 4 inches. Mr. STRUVE has communicated to me that the geodetic connexion of Rivabellosa and Pobes showed the geographical position of my observatory to be

Latitude N. 42° 43′ 24″, Longitude W. 11 min. 41·3 sec.

Lastly, I estimate the error of the mean-time chronometer, Frodsham No. 3094, July 18th, 0^h, to have been 4·6 seconds fast of Greenwich mean time, by assuming a progressive increase in its losing rate from July 16th to July 18th, and taking the mean between Dr. WINNECKE's and my own estimate for its error on July 16th.

After the return of the expedition, Mr. CARRINGTON kindly made some extensive calculations to admit of a direct comparison of my observed results with the demands of theory. To Mr. FARLEY I am also much indebted for special computations of the moon's position in respect of the sun's, in a form the most convenient for comparison with measurements hereafter to be mentioned.

Abstract of the Results of Mr. CARRINGTON'S *Calculations for Rivabellosa.*

Assumed position of station:—

Geographical latitude	42° 42′
Longitude W. of Greenwich	11^m 42ˢ·7.
Height above sea-level	1572 feet.

Whence the following elements:—

Geocentric latitude	=42° 30′·5.
Log. distance from earth's centre . . .	9·9993676.

The true positions of the sun and moon are those of LE VERRIER'S and HANSEN'S

Tables, as derived from the Special Circular issued by the Superintendent of the Nautical Almanac.

For the totality the apparent positions were calculated for 3 h. 0 m. and 3 h. 5 m. Greenwich mean time, with the following results:—

Zenith distance of sun's centre	40° 33′.
Angle at sun between pole and zenith . . .	47° 59′.
Moon's semidiameter	16′ 33″·0.
Sun's semidiameter	15′ 44″·8.
Totality began at 3 h. 0 m. 37·4 s. at 101° 38′ on the sun.	
Totality ended at 3 h. 4 m. 1·6 s. at 312° 56′	,,
Direction of motion from 297° 17′ to 117° 17′	,,
Relative motion during totality	92″·8.
Nearest approach of centres	13″·0.
Duration 3 min. 24·2 secs.	

For the first contact the apparent positions were calculated for 1 h. 40 m., 1 h. 50 m., and 2 h. 0 m. Greenwich mean time, with the following results:—

Zenith distance of sun's centre	28° 49′.
Angle at sun between pole and zenith	35° 50′.
Moon's semidiameter	16′ 34″·6.
First contact at 1 h. 47 m. 56·0 s. at 296° 54′ on the sun.	
Rate of approach of centres per minute . . .	24″·87.

For the last contact the apparent positions were calculated for 4 h. 0 m., 4 h. 10 m., and 4 h. 20 m. Greenwich mean time, with the following results:—

Zenith distance of sun's centre	52° 45′.
Angle at sun between pole and zenith	51° 36′.
Moon's semidiameter	16′ 30″·7.
Last contact at 4 h. 10 m. 15·2 s. at 117° 20′ on the sun.	
Rate of retreat of centres per minute	29″·85.

The formulæ used in computing the moon's parallax and apparent semidiameter were

$$\theta = \text{R of zen.} \qquad m = \frac{\varrho \cos \phi' \sin p}{\cos \delta} \qquad n = \frac{\varrho \sin \phi' \sin p}{\sin \gamma}$$

$$\alpha' - \alpha = -\frac{1}{\sin 1''}\left\{ m \,.\, \sin \overline{\theta - \alpha} + \tfrac{1}{2}\, m^2 \sin 2 \,.\, \overline{\theta - \alpha} + \tfrac{1}{3}\, m^3 \sin 3 \,.\, \overline{\theta - \alpha} \right\}$$

$$\delta' - \delta = -\frac{1}{\sin 1''}\left\{ n \,.\, \sin \overline{\gamma - \delta} + \tfrac{1}{2}\, n^2 \sin 2 \,.\, \overline{\gamma - \delta} + \tfrac{1}{3}\, n^3 \sin 3 \,.\, \overline{\gamma - \delta} \right\}$$

$$\tan \gamma = \tan \phi' \,.\, \frac{\cos \tfrac{1}{2} \,.\, \overline{\alpha' - \alpha}}{\cos \left(\theta - \tfrac{1}{2}\overline{\alpha' + \alpha}\right)}.$$

$$R' = R \,.\, \sin (\gamma - \delta') \,.\, \operatorname{cosec} (\gamma - \delta).$$

Mr. FARLEY'S *Elements of the Eclipse for Rivabellosa.*

The following calculations are based on the same latitude and longitude as those of Mr. CARRINGTON.

Greenwich mean time.	Apparent distance of ☉ and ☾ centres.	Angle of line joining centres, N. towards E.	Angt. S. D. or Radius of ☾.	Ratio of Lunar to Solar radius.
d. h. m.	′ ″	° ′	′ ″	
July 18. 1 45	33 32·8	296 54	16 34·5	1·0526
1 55	29 22·0	296 54	16 34·4	1·0525
2 5	25 8·9	296 52	16 34·3	1·0524
2 15	20 53·0	296 47	16 34·1	1·0523
2 25	16 34·4	296 38	16 33·9	1·0521
2 35	12 12·2	296 23	16 33·7	1·0518
2 45	7 47·2	295 48	16 33·4	1·0515
2 55	3 19·3	293 40	16 33·1	1·0511
3 5	1 13·8	126 59	16 32·8	1·0508
3 15	5 48·2	119 20	16 32·5	1·0505
3 25	10 26·7	118 22	16 32·2	1·0502
3 35	15 9·1	117 57	16 31·9	1·0498
3 45	19 55·2	117 43	16 31·6	1·0495
3 55	24 45·0	117 32	16 31·3	1·0492
4 5	29 39·0	117 22	16 30·9	1·0488
4 15	34 37·2	117 14	16 30·5	1·0484

	h	m	sec.	
Time of first contact . . .	1	47	57	at 296° 54′ N. towards E.
middle . . .	3	2	20	duration of totality 3 min. 20 sec.
last contact . . .	4	10	15	at 117° 18′ N. towards E.

Nearest approach of centres 0′ 12″·7.

OBSERVATIONS OF THE ECLIPSE.

I. *Observations with the unassisted Eye, and with the Telescope.*

A splendid day on Sunday the 15th was succeeded by one of the grandest and most awful thunder-storms I have ever witnessed; and the 16th was cloudy, almost without intermission. The day previous to the eclipse had been completely overcast, with the exception of a short interval about noon; but even then the sun could only just be seen through a cloud somewhat thinner than those which obscured the rest of the heavens. The climate had therefore proved anything but propitious, and every interval of fine weather had to be diligently made use of for the adjustment of the instruments and the prosecution of observations. Fortunately an opportunity had presented itself on two days for practice in observing the sun with the Dallmeyer between 1 h. 30 min. and 4 P.M., and for special practice at about 3 o'clock. It was ascertained that during the progress of the eclipse the radius bars would have to be changed from one leg of the tripod-stand to the other, and arrangements were made to prevent the necessity for doing this during or near the period of totality.

To this instrument I had fitted an eyepiece of my own contrivance, which I described,

verbally, at one of the meetings of the Astronomical Society, and which was in consequence adopted by Mr. PRITCHARD. No account having been published of this appendage, and experience having proved its value in eclipse observations, I think it desirable to describe it here. It will be remembered that Mr. HODGSON some time ago proposed that a piece of polished glass should be used as a diagonal reflector in observing the sun; and "Hodgson's solar eyepiece" has been generally adopted, and is a most convenient and efficient instrument. It occurred to me, that if the glass reflector were made in the form of a parallelogram, of such dimensions that a moiety of its surface would suffice for the field of the telescope, one-half of the upper reflecting surface might be silvered and the other left plain, and that the addition of a suitable contrivance would enable the observer to draw into position the unsilvered or the silvered surface, according as either partial or total reflexion might be required. The silver film is so extremely thin that it in no way affects the focus, yet it is susceptible of the highest possible polish. It was not a convenient plan to silver only half the mirror; so, when the whole had been silvered *, one-half of the silver was neatly removed by means of a cloth, wetted with cyanide of potassium, strained over the forefinger. The roughened back of the reflector was freed from silver, and the plate then washed thoroughly with distilled water and allowed to dry. A little pad of wash-leather, well charged with dry rouge by rubbing it on a second piece of leather on which some rouge-powder had been placed, very soon removed the peach-like bloom from the silver surface, and produced a perfect polish.

The construction of the eyepiece will be readily understood by reference to the accompanying wood-engravings, wherein the same letter refers always to the same part.

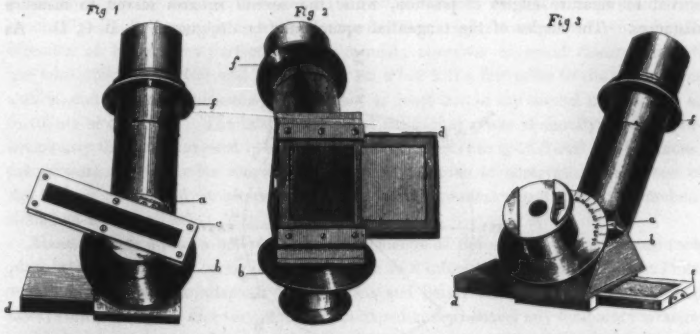
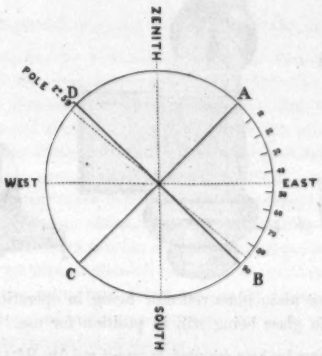


Fig. 1 is a front view, the plain glass reflector being in operation. Fig. 2 is a view of the under side, the plain glass being still in position for use. Fig. 3 shows the glass

* A method for silvering glass has been described by myself and Dr. MÜLLER in the 'Monthly Notices of the Astronomical Society,' vol. xix. p. 171.

reflector drawn out so as to place the silvered surface in the field. *f* is the socket-adapter, which screws into the telescope; it has a slit cut into it to receive a pin fixed on the sliding tube, the object of which is to keep the position-lines of the eyepiece, when once adjusted, in their proper position; *e* is the glass reflector, the half towards *d* being silvered, that towards *e* plain; *d* is a covering to protect the sliding reflector; *b* is a circle fixed to the body of the eyepiece, having one quadrant divided into nine spaces of 10° each; *a* is an index attached to a positive eyepiece, which can be moved by it through an arc of 90°; *c* is a graduated sun-shade, composed of a wedge of dark glass and another of white glass, reversed in position, so as to form, when combined, a parallel plate. This is held firmly in its place by means of a spring, shown in fig. 3, which, while it holds the shade firmly in any required position, also allows its instant removal at pleasure. The glass reflector *e*, as soon as the observer desires to use the silvered surface, can be drawn forward in a small fraction of a second, without disturbing any other part of the instrument.

In the focus of the positive eyepiece was fixed a piece of parallel glass on which were etched several lines; this micrometer-plate was carried round with the eyepiece whenever the index, *a* (figs. 1 & 3), was moved. A reference to Plate VI., which contains a fac-simile of my hand-drawings, and also a representation of the position-lines, will render clear the following explanation. Four principal lines on the glass plate formed a tangential square calculated to enclose exactly the moon's disk, which in fact it accomplished with great precision; four other lines surrounded the first square at the distance of exactly 1′ of arc; and a third series formed a third square at the same distance from the second. Joining the angles of the squares were two diagonal fainter lines, which served to measure angles of position, while the several squares served to measure distances. The angles of the tangential square may be designated A, B, C, D. As



soon as the axis of the telescope-stand had been adjusted in a vertical position by

means of screws affixed to the feet of the tripod-stand, and of a level attached to the vertical axis of the telescope, a distant mountain peak was made to run along one of the lines D A or C B by causing the telescope to turn on the vertical axis when the index of the eyepiece stood at zero; and the position of the whole eyepiece was regulated by means of a slight axial adjustment of the sliding tube until the line in question, and the course of the object horizontally across the field, showed an absolute coincidence. The diagonal lines D and A were then each 45° from the zenith; and the angle between the pole and the zenith at the period of totality being 47° 59′, the wire D, to the west of the zenith, was 2° 59′ east of the pole. This will readily be seen by the annexed diagram, which shows the apparent positions of the zenith and of the east, west, and south points in the field of the eyepiece, the zenith being represented in its natural position, but the east and west points reversed, right for left; so that, in referring any measurement to D, 2° 59′ must be added to reduce them to positions from the north point towards the east.

At last the eventful 18th of July arrived, and appeared hopelessly cloudy. The sky was watched with the most intense anxiety by us all; and I am free to confess that *my* nerves were in the most feverish state of agitation. Not the slightest break in the clouds or mist was visible until about 10 o'clock, when a streak of clear sky gave us the first faint gleam of hope. At noon the sky began to clear very generally, not so much by the clouds being carried off by the wind, as by their melting away in the air. An attempt was made to get an observation of the sun; but the clouds were so dense that it was only just as he was passing away from the field of the telescope that his following limb could be made out. Soon after this the clouds, which had been dissolving and gradually becoming thinner, disappeared all at once; and we had a magnificent sky, absolutely cloudless, except near the horizon. The heights towards the north in the direction of Pobes were perfectly free from mist; moreover we could discern through the telescope Mr. Bonomi and Mr. J. Beck on a low hill a few miles to the west of our station, and it gave us pleasure to know that at least two of the several parties were as fortunate as ourselves. The mists surrounded the higher peaks of the Pyrenees so pertinaciously that we expressed to each other great fears for our good friend Mr. Vignoles; for we were aware that he intended to carry out his plan of observations from one of the highest peaks, and we afterwards heard with the greatest regret that our apprehensions had been realized.

About twenty minutes before the commencement of the eclipse, an occurrence took place which very nearly brought all our labours to a calamitous termination. Mr. Preston had placed at our disposal his excellent and handy servant Juan, whom we had always found obliging, and very ingenious in expedients whenever any temporary arrangements had to be made. In order that he might have an opportunity of looking at the eclipse, I smoked a piece of glass for him with a wax lucifer-match, and he then, on his own account, prepared several pieces for the bystanders in a similar way. The demand soon increased so much, that he was scarcely able to keep pace with it, and at length became so excited that he threw away the matches in all directions without extinguishing them, and some, falling in the standing corn, set it on fire. The corn was very thin,

and fortunately the wind was blowing from the position of the fire towards the thrashing-floor; otherwise but a few minutes only could have elapsed before the conflagration would have assumed such dimensions as to be beyond the power of man to control. Happily, a few seconds after the occurrence, the crackling sound and the smell of burning straw drew my attention to the spot, and, water being at hand, the fire was got under before it had spread more than a few feet.

The Alcalde of Miranda had intimated to me, a few days before, that he was instructed to place at my disposal as many of the civic guard as I might think necessary to prevent interruption; but my experience of the consideration evinced by the Spaniards was such, that I replied that one or two would be quite sufficient. Shortly before the commencement of the eclipse, there arrived five mounted guards, who were of great use in preventing the crowd from encroaching on the thrashing-floor, which an excusable curiosity to watch our proceedings tempted them often to do. It is right to add that I could not persuade the guards to take any present whatever, their reply being that their orders on this head were imperative, and that, moreover, they had felt pleasure in being of service. When we were on the point of commencing our observations, about 200 persons had assembled round our observatory, and, although they conducted themselves perfectly well in other respects, their talking quite prevented my hearing the beats of the chronometer. They seemed to think that the eclipse could only be seen from my station; and it was with some difficulty that a number were persuaded to go to an adjoining height, whence the effects on the landscape and the progress of the shadow could really be better observed. I explained this, through the kindness of a gentleman from Miranda who spoke French, and who showed his faith in what I stated by leading the way. The Alcalde of Rivabellosa, Civilo Guinea, to whom I was indebted for facilitating my operations, explained to those who remained around the station the necessity for silence, and they thenceforth carried on their conversation in a tone which caused us no inconvenience. It is indeed impossible to speak too highly of the good feeling manifested throughout by the Spaniards of all grades, who endeavoured in every way to promote our objects.

Owing to my pocket chronometer having tripped, and become many minutes fast of Greenwich mean time, some confusion arose about the period of first contact, and a photograph, which it was intended to procure as close as possible to that event, was lost in consequence of the plates being prepared too soon, and none being ready when it actually took place. The error of the pocket chronometer was only discovered when it was too late, and it was then found to be faster than Frodsham 3094 by 8 min. 11 sec.

The first contact was observed at 1 h. 56 min. 55 beats.

			h	m	sec.
5 beats to 2 seconds			1	56	22
	m	sec.			
Error, fast of Frodsham	8	11			
Error, Frodsham fast of Greenwich mean time		4·4		8	15·4
Making the observed Greenwich mean time of first contact to be			1	48	6·6

	h	m	sec.
The occultation of the largest solar spot, which I call *c*, occurred at 2 h. 10 m. 125 beats	=2	10	50
Deducting the difference between the pocket Frodsham and Frodsham 3094		8	11
It gives	2	2	39

as the time by Frodsham 3094. This agrees within 0·5 second of the time noted by Mr. Beckley when a photograph of the phenomenon was taken in accordance with my signal, namely, 2 h. 2 m. 39·5 secs. Attention is called to the very exact accordance of the times recorded by myself and by Mr. Beckley, because I shall hereafter have to draw attention to certain conclusions which I have deduced, the soundness of which is dependent in part upon the epochs of the photographs having been accurately registered by Mr. Beckley.

After the occultation of the spot *c*, nothing worthy of record occurred until about 2 h. 12 m., when a cluster of clouds formed very rapidly and unexpectedly in the immediate neighbourhood of the sun, and completely put a stop to both optical and photographic observations. The clouds melted away about six minutes after they had formed, and thenceforward until the end of the eclipse all went on without interruption.

I had never before witnessed so great an obscuration of the sun as that presented by this eclipse many minutes even before the totality occurred, and I was particularly struck by the change of colour in the sky, which had been gradually losing its azure blue and assuming an indigo tint, while at the same time I remarked that the surrounding landscape was becoming tinged with a bronze hue, which to my mind suggested the idea that the light of the sun near the periphery is not only less intense than, but possibly different in quality from, that of the centre*. Spectrum experiments at future great eclipses, when the sun's crescent is reduced to an extremely narrow line, would set this question at rest, and might also have an important bearing on the line of investigations so ably inaugurated by Kirchhoff and Bunsen, into the composition of the sun's atmosphere. Another phenomenon could not fail to attract attention. When the sun's visible disk was reduced to a very narrow crescent, the shadows of all near objects became extremely black and sharply defined, whilst the lights, by contrast, assumed a peculiarly vivid intensity, the aspect of nature strongly recalling to mind the effects produced by the illumination of the electric light. Several minutes before the totality, the whole contour of the brown-looking lunar disk could be distinctly seen in the heavens.

Only a few brief seconds, unfortunately, could be spared from the telescope after the totality had actually commenced; but when I had once turned my eyes on the moon encircled by the glorious corona, then on the novel and grand spectacle presented by the surrounding landscape, and had taken a hurried look at the wonderful appearance

* In connexion with this remark, compare Sir John Herschel on the Chemical Rays of the Spectrum. Philosophical Transactions, 1840, Art. 82.

of the heavens, so unlike anything I had ever before witnessed, I was so completely enthralled, that I had to exercise the utmost self-control to tear myself away from a scene at once so impressive and magnificent, and it was with a feeling of regret that I turned aside to resume my self-imposed duties. I well remember that I wished I had not encumbered myself with apparatus, and I mentally registered a vow, that, if a future opportunity ever presented itself for my observing a total eclipse, I would give up all idea of making astronomical observations, and devote myself to that full enjoyment of the spectacle which can only be obtained by the mere gazer.

Although, possibly, not more than twenty seconds were devoted to observations with the unassisted eye, the phenomena remain strongly impressed on my memory, and at the time of writing this account, sixteen months after the event, I have it now pictured before me mentally, as vividly as if it had but just occurred. The darkness was not nearly so great as I had been led to expect from the accounts which I had read of former total eclipses; and although I had a lantern at hand, I did not require it, either in making my drawings or for reading the divisions of the micrometer quadrant on the eyepiece. The illumination was markedly distinct from that which occurs in nature on any other occasion, and certainly was greater than on the brightest moonlight night; and yet, at the time, the light appeared to me less than what I remembered of bright moonlight. It was only by subsequent trials, in endeavouring to make out details of the drawings which I had made of the phenomena, and to distinguish between colours under various circumstances of moonlight and twilight, that I was able to form a proper appreciation of the amount of light; and the best account I can give of it is, that it most resembles that degree of illumination which exists in a clear sky soon after sunset, when after having made out a first-magnitude star, other stars of less brilliancy can be discerned one after another. The light was good enough and sufficiently polychromatic to enable me to distinguish the colours of near objects; but those in the distance appeared to be illumined by the most unearthly hues.

Immediately surrounding the corona, the sky had an indigo tint, which extended to within about thirty or twenty-five degrees of the horizon, while lower down it appeared to me to be modified by a tinge of sepia. It became red as it approached the horizon, close to which, and just above the mountains, it was of a brilliant orange. The mountains appeared, by contrast, of an intensely dark yet brilliant blue. I saw two stars to the east of the sun, which by the aid of Mr. Hind's diagram I have since identified as Jupiter and Venus; but I had not time to search for more, or, most probably, I should have seen others. These planets, and also Castor, were made out by Messrs. Robert Swanson, Harry Edmonds, and Matthews, in the employ of Messrs. Brassey and Co., and were identified by them in my copy of Mr. Hind's diagram.

The effect of totality upon the bystanders was most remarkable. Until the beginning of totality, the murmur of the conversation of many tongues had filled the air; but then in a moment every voice was hushed, and the stillness was so sudden as to be perfectly startling; then the ear caught the sound of the village bells, which had been

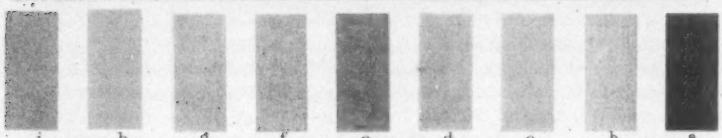
tolling unheeded during the eclipse, and this circumstance added much to the solemn grandeur of the occasion.

The time I could spare was far too short for any exact observations of the corona; however, I knew that Mr. PRITCHARD, Mr. OOM, Mr. BONOMI, and other observers intended to make special delineations and measurements of that phenomenon, and I therefore confined my attention to its general characteristics. It appeared to me to glow with a silvery-white light, softening off into a very irregular outline, while from its general boundary shot out several long streamers. It extended generally to about 0·7 or 0·8 of the moon's diameter beyond her periphery. Close to the moon, and reaching not further than 2′, the light was very brilliant, and several zones of gradations of brightness appeared to exist, but the very bright zones would all be comprised in a circle about 0·25 of the moon's diameter.

The observations just recited were made in the brief interval I could afford between the telescopic observations, which I will now proceed to describe. In order to facilitate my operations, I had prepared two diagrams exactly representing the appearance of the micrometer lines in the telescope, and, by chance, I had made the tangential square of such dimensions as to include a circle precisely 4 inches in diameter, which had been coloured to render it more readily distinguishable, and which represented the moon. Four inches happened to be almost exactly the diameter of the moon's disk on the screen of the heliograph; so that, later on, the photographs and the two drawings made during the totality, were readily compared by the superposition of each upon each.

On the diagrams I had painted fifteen streaks of various tints, some of which I believed might resemble the colour of the prominences, and some I knew would be useful as a contrast, to enable the eye to form a more correct judgment. The chromatic scale I here insert contains a selection of the tints painted round my diagram.

SCALE OF COLOURS WITH WHICH THE PROMINENCES WERE COMPARED.



Several minutes (probably five) before totality, I entirely removed the dark glass, and found that the sun's image might be looked at without the slightest inconvenience after reflexion by the plain glass. I could then see in the telescope, as I had shortly before seen with the naked eye, the whole of the lunar disk, which appeared of a deep sepia brown, nearly, but not quite, black, and, to my great surprise, I perceived a luminous prominence, about 20° to the west of the zenith, shining with great brilliancy, although, on account of the plain-glass surface being then in use, the greater part of its light

passed through the glass, and was therefore not reflected to the eye. I then, cautiously, but rather quickly, brought into action the silvered surface, and beheld with delight that the luminosity of the prominence, which I will call A*, was so great that there could be very little doubt of our obtaining the much wished-for photographic pictures.

I now watched carefully for the so-called Baily's Beads, but no such phenomena presented themselves,—at which, however, I felt no surprise, for I had always believed that they arose, in all probability, from the atmospheric disturbance of an image formed by a telescope wanting in perfect definition. The Dallmeyer I used was so perfect that I did not think I should see anything of the kind.

To the east of the zenith, about 20°, a floating cloud, quite detached from the moon's limb, and distant from it more than 0'·5, next attracted my attention. This cloud, which I will call C, appeared about 1'·5 long, and was inclined about 50° or 60° to the moon's limb. It had two curvatures, both convex on the edge most distant from the moon, and was decidedly of a rose tint, but of a much paler hue than the published accounts of previous eclipses had led me to expect. I compared the prominence carefully with my scale of tints, and found that it very nearly matched the colour marked *c*. It must therefore have been of a yellowish pink (approaching a salmon); for *c* on my chromatic scale was a mixture of carmine and cadmium yellow. This prominence (C) presented a great amount of detail, and reminded me of the aspect of a cirrus cloud glowing with the illumination of a setting sun. I should here remark that, in comparing my scale of colours with the luminous prominences, I depended on the general light of the heavens, and that I did not employ my lamp, which, I found, completely changed their appearance.

The prominence A was generally more brilliant, and did not seem to me to be so pink as the detached cloud; I could, moreover, detect a tinge of yellow in its brilliant light. It also showed considerable structure, appearing to consist of several streaks, curved inwards, while from the summit came two peach-coloured faint streamers, bending over in opposite directions downwards towards the moon's limb.

I paid most particular attention to the prominence A, because I knew from its position that it was critically placed for the observation of any change of position-angle in reference to the moon's centre; and I also remarked carefully the prominence C, and sketched all that I could make out by the most careful scrutiny. On comparing my drawings with the photographs, it will be perceived that a certain boomerang-like prominence in the photograph is wanting in my hand-drawings, and that there are also other prominences visible in the photograph which are not shown in the drawings. This is a curious circumstance, hereafter to be more particularly dwelt upon; but it is right to mention it here, because it affords me the opportunity of saying that, at all events, as regards the boomerang, I am certain that it was not visible in the telescope; for I observed so carefully in the neighbourhood of the floating cloud, that it is next to an impossibility that such an object could have escaped detection.

* See Index Map, Plate XV.

In the eastern quadrant (in reference to the zenith) a long line of prominences, extending over 70° on the moon's limb, was visible at the commencement of the totality, but before the end of totality it was covered by the lunar disk. This streak (which I call I) terminated in a hook, about 1′·5 high, bent upwards, and was *much indented on the concave side*, where it was in contact with the moon's edge. It was extremely brilliant, and, although it presented in parts a pink colour, was not uniformly so coloured, but to my eye had here and there a considerable admixture of yellowish white. In the first photograph of the totality is depicted a curious branching prominence, not unlike the fallen stump of a tree, which I did not observe, and therefore did not record in the sketches. I do not state so positively that this prominence was not visible, for this reason, that I did not pay such special attention to that part of the field, my eye being directed more particularly to the prominences A and C; but I have a strong impression that it was not visible. Just about the part where this would be, the corona appeared to me, in the telescope, to be particularly bright; but, besides a mere sheet of brilliant light, I saw nothing to delineate.

About half a minute after the commencement of totality, the progress of the moon uncovered, in the western quadrant (in reference to the zenith), a small peak, like that of a mountain, which I will call R. As the eclipse progressed, this prominence became more and more uncovered, and another smaller peak appeared, the whole contour reminding me somewhat of the hull and masts of a ship in full sail. Just before the reappearance of the sun this prominence reached apparently about 1′·5 beyond the moon's limb.

Extending from the southern base of this prominence, there came into view, about a minute before the end of totality, a long streak of prominences much *indented and irregular on the concave side*. This streak extended over fully 50° on the moon's limb, when it had been fully uncovered by her onward course. It was pretty generally of a decided rose tint. Just previously to the reappearance of the sun, I remarked a sort of carmine glow near that part of the moon's limb where the crescent of the sun was first re-formed.

Plate VI. is a most exact fac-simile of the two drawings, black representing white, which I made during the totality, and it is desirable that I should make a few remarks about them.

Figure 1 was begun, as nearly as I can recollect, about thirty seconds after the commencement of totality. As a preliminary step, the moon's disk was brought exactly within the tangential square, and the position of the prominence A, in respect of the line D, was noted first of all, and at once marked down on the left-hand diagram; the hook I was then referred to the line B, and the mountain R to the line D, the latter being registered with great care. The floating cloud and the other prominences were then filled in, possibly not quite so carefully. The details were next drawn in, black representing white, and the first diagram was completed as rapidly as possible, yet as faithfully as the short time at command would permit. I was aware, whilst so occupied,

that by the addition of detail after detail in the several prominences I was exaggerating their dimensions; but there was too little time to spare to rub out and commence anew.

When the first drawing was completed, about a minute and a half after the commencement of the totality, I looked away from the telescope in order to make the eye observations which I have already described, and before I resumed my work at the telescope an interval of half a minute may have elapsed, but certainly not more. The next thing I did was to measure the angular position of the prominence A; and after bringing the moon well into the tangential square, I moved the wire D through the arc necessary to bring it into contact with the side of that prominence nearest to D, which brought the index to an exact coincidence with one of the divisions on the quadrant; I noted down 10° for the angle moved through; but this is an evident error, for the angle was as nearly as possible 20°, which, added to 2° 59′, makes the position-angle of the western boundary of that prominence 22° 59′, from the north towards east, which is not far from its true position at that time.

Whilst measuring this prominence, I asked Mr. Reynolds, whose allotted task it was to develope the photographs after their exposure in the heliograph, whether anything could be seen on the first plate of the totality; and learning, with a thrill of intense pleasure, that the operation had completely succeeded, I made no further measurements, knowing full well that I should get them far better in the photographs.

Immediately after this, I commenced my second drawing, given in Plate VI., and noted down the position of the prominences A and R very exactly, by referring them to position-line D; and I then filled in the other details. As very little time remained for the completion of the drawing, I devoted my attention chiefly to the prominence R and a faint hooked prominence about 45° to the west of the position-line D, which did not imprint its image on the second photograph to the extent I should have expected from its dimensions in my sketch.

Between the completion of the first sketch and the commencement of the second, I estimate that there was an interval of about one minute, and that the second sketch was therefore commenced as nearly as possible 2½ minutes after the beginning of totality.

Thus, before commencing sketch No. 1, there elapsed,

	min.	sec.
From the beginning of total obscuration	0	30
To complete No. 1 sketch it required	1	0
Time consumed by eye observations, away from the telescope.	0	30
The measurement of prominence A occupied	0	30
Interval elapsed from the beginning of totality to the commencement of sketch No. 2	2	30

By placing a horn protractor on the original sketches, the following measurements were made:—

Protuberance.	Synonym.	Part measured.	Distance from position line D towards east, in degrees and decimals of a degree. First drawing.	Second drawing.
A.	Cauliflower. Wheatsheaf.	First boundary, a* .	23°·0	20°·0
		Second boundary, a' .	28·0	24·5
		Middle	25·5	22·25
C.	Detached cloud.	First point, c . . .	58·7	49·0
		Last point, c' . . .	69·0	58·5
R.	Mountain peak.	First point, r' . . .	347·0	343·5

In order to reduce these measures to position angles from the North towards East, it is necessary to add to them 2° 59′, say 3°, which will give us—

	First drawing.	Second drawing.	Apparent angular motion in the interval.
A. Middle	28°·5	25°·25	3°·25
C. First point, c . .	61·7	52·0	9·7
R. First point, r' . .	350·0	346·5	3·5

As I before stated, the positions of A and R were laid down with great care; and it will be hereafter seen that their deviations from the positions given by measurements of the photograph are remarkably small. The measurement of all the details, however, do not agree so well, because the same care could not be devoted to the laying down of their positions.

These drawings show that there was a decided angular shifting of the luminous prominence A, and of others, in reference to the moon's centre; and taking into account the probable interval between the two drawings, namely two minutes, the amount of angular motion of A is a very near approximation to the angular change which must actually have occurred. As mentioned above, there is in the drawings an exaggeration of the dimensions of the prominences, which renders them unfit for the precise determination of the moon's actual progression in the line of motion during the period of totality; nevertheless they afford excellent evidence that there was, in fact, a covering and an uncovering of prominences, which, taken in connexion with the change in the position-angle of the protuberance A with reference to the moon's centre, can only be explained on the assumption that these extraordinary appendages belong to the sun, and not to the moon.

Furthermore, it would be quite possible to make out, with considerable although not with absolute accuracy, from these drawings, the direction of the moon's motion, and the extent to which the prominences first seen were obscured by the progress of the lunar disk, and others uncovered on the opposite side as the moon continued her course. For instance, it will be remarked on inspection, that the streak of prominences, almost 1′ in

* The letters refer to the index map, Plate XV.

height, depicted in the eastern quadrant of fig. 1, Plate VI., is almost entirely covered in fig. 2, and that the difference of position of the moon in the two pictures, when measured by a suitable scale, indicates a motion of about 50″ in the interval of two minutes which they include,—a result very near to the truth, for the actual progression in that period was 54″·5. The photographs, however, as will be hereafter seen, are so much better adapted for such determinations as these, that it is not worth while to dwell more upon the conclusions to be derived from the hand drawings.

In the two coloured drawings, Plates VII. and VIII., I have depicted the result of my telescopic observations; to facilitate my doing which at some future convenient time, I made a coloured sketch on the afternoon of the eclipse. This coloured sketch, the black-and-white drawings made at the telescope and shown in fac-simile in Plate VI., together with my photographs, which I have not hesitated to use to correct any errors of position or dimension in the sketches, have enabled me to give in these drawings what I believe to be a very truthful representation of the appearance of the prominences, immediately after the commencement and just before the end of totality. The corona I do not give as an absolutely true representation of that phenomenon, but as fairly resembling its general appearance. It has been derived from the photographs, so far as they show it.

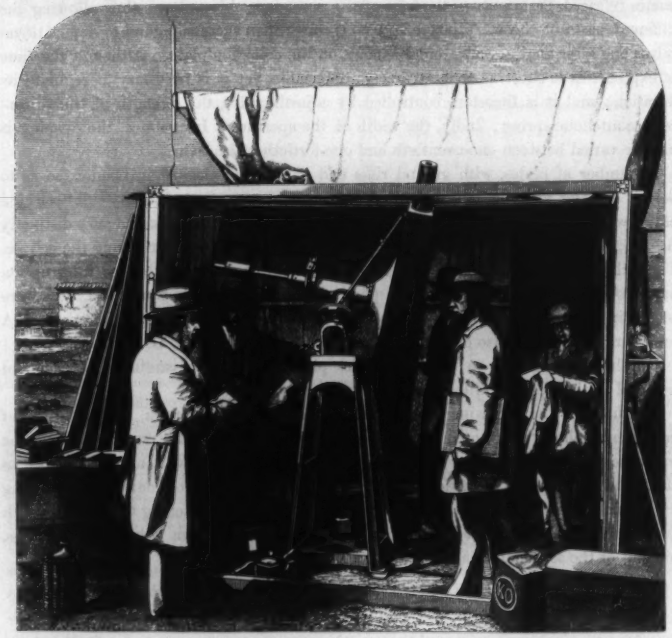
II. *Account of the Photographic Observations.*

The Kew heliograph, with which the photographs were obtained, is represented in the accompanying engraving *. It was devised by myself, for the special object of making photographs of the sun's disk, at the request of the Council of the Royal Society, in accordance with a recommendation to that effect by Sir John Herschel.

It has an equatorial mounting of the ordinary form, after the so-called German model, to which is attached a clock-work driver. The tube is square in section, and larger at the lower end than at the upper or object-glass end. The object-glass has 3·4 inches' clear aperture and 50 inches' focal length; the primary focal image of the sun at his mean distance is 0·466 inch in diameter; but before it is allowed to fall on the sensitive plate, it is enlarged to about 3·8 inches by means of an ordinary Huyghenian eyepiece. In the plane of the focus of the posterior lens of this eyepiece are attached two position-wires, which cross at right angles, and which were adjusted, approximately, into a position at an angle of 45° to a parallel of declination. The object-glass is so constructed as to ensure the coincidence of the chemical and visual foci; but this coincidence being somewhat disturbed by the Huyghenian secondary magnifier, the amount of adjustment required to effectuate the best chemical focus was ascertained very carefully by a series of experiments.

* The engraving was copied from a photograph taken at Rivabellosa. The front boards of the observatory were taken out in order that this might be done. Mr. Downes, who was charged with the preparation of the plates, is standing in the doorway leading to the developing room. Mr. Reynolds has a plate-holder ready to place in the heliograph, and Mr. Beckley is observing the time with the chronometer.

For sun-pictures, and the photographs of the several phases of the eclipse, the aperture of the object-glass was reduced to about 2 inches in diameter by means of a stop; but the light of the sun is so extremely powerful, that, even with this small aperture, combined with the enlargement of the primary image and its consequent reduction in intensity by 64 times, the shortest exposure possible with the ordinary means of uncovering and covering the object-glass would be far too long, and would give none but solarized pictures. For this reason the instrument has attached to it an instanta-



neous apparatus of a peculiar construction. It consists of a sliding plate with a square aperture sufficiently large to permit of the passage of all the rays; this aperture is fitted with a sliding piece, actuated by a screw which projects through and a few inches beyond the telescope-tube; by means of this screw the aperture may be completely opened, closed, or reduced to a slit of any required width; a divided scale being affixed to the screw for that purpose. The projecting screw connected with the slide is shown in the engraving, on the underside of the tube.

Previous to taking the picture, the sliding plate is drawn up just so high that the unperforated part of it completely shuts off the sun's image; it is held in this position by means of a small thread attached to it at one end and looped at the other, the loop being passed over a hook on the top of the tube; and the slide is pulled downwards, in opposition to the thread, by means of a spring of vulcanized caoutchouc attached to the inferior side of the tube. When the picture is about to be taken, the retaining thread is set on fire*, and the rectangular aperture, as soon as the sliding plate becomes released, flashes across the axis of the secondary object-glass—thus allowing the different parts of the sun's image to pass through it in succession, and to depict themselves one after another, after enlargement, on the collodion-plate. Although the time of exposure is so short as to be scarcely appreciable, yet it is necessary to regulate its duration; and it is therefore controlled by adjusting, 1st, the strength of the vulcanized caoutchouc spring; 2ndly, the width of the aperture. In practice, the opening is usually varied between one-twentieth and one-fortieth of an inch.

A number of plates, with ground rims and edges, were cleaned in London, so as to permit of their examination, and all defective ones were rejected; forty-eight selected plates were then numbered consecutively, and arranged in boxes marked very distinctly A, B, C, D, so as to ensure their being taken out in the proper order during the eclipse. The heliograph was furnished with three plate-holders, in order that no interruption might occur in the succession of the photographs; and as these were filled, they were placed in such a way that each plate was sure to be exposed in its numerical order. A few spare plates were also cleaned, and marked A, B, C, D, E, F, G, H, I, &c.

On the day previous to the eclipse the plates were again carefully cleaned, and replaced in their proper order in their respective boxes.

On the 18th the following plates were placed in the heliograph, and the time of taking each photograph noted by Mr. Beckley, with any requisite remarks. The time was observed with Frodsham No. 3094, whose error at Greenwich mean noon was, as already stated, fast of Greenwich mean time 4·6 seconds, and whose daily rate was losing 2·1 seconds. The exact time of depiction was ascertained by listening to the click which the instantaneous slide made in striking home upon a stop, when it had flashed across, in front of the secondary magnifier.

* Mr. Clark, who undertook this task, is represented in the engraving with a lighted taper in his hand.

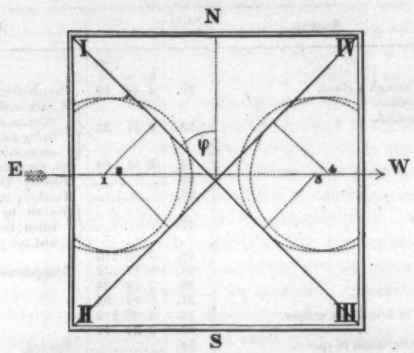
No. or letter.	h	m	sec. ½	Remarks.
1.	23	38	1	Through a cloud.
2.	.....			Spoiled.
3.	.....			Spoiled.
4.	0	23	105	
A.	0	29	45	
B.	0	34	34	
C.	0	42	24	
D.	0	49	93	
E.	1	3	23	
F.	1	13	85	
G.	1	25	45	
H.	1	29	76	
I.	1	41	76	
5.	1	45	3	
6.	1	47	96	
7.	1	52	71	The first of the eclipse.
8.	1	56	15	
9.	2	2	79	Occultation of spot *c*.
10.	2	3	41	
11.	2	7	21	
12.	2	11	31	
13.	.....			Clouds.
14.	2	20	24	
15.	2	22	97	
16.	2	27	103	
17.	2	33	36	
18.	.....			Spoiled.
19.	2	36	43	
20.	2	41	55	
21.	2	46	46	

No. or letter.	h	m	sec. ½	Remarks.
22.	2	48	60	Occultation of spot *b*.
23.	2	51	53	A solarized picture, in consequence of the full aperture being used and the instantaneous slide being detached.
24.	2	55	88	Id., the time uncertain.
25.	.....			Totality; time not noted.
26.	.....			Totality; time not noted.
27.	.....			Shaken by wind; time not noted; the instantaneous slide had not yet been replaced.
28.	3	13	116	
29.	3	17	32	Reappearance of spot *c*.
30.	3	20	24	
31.	3	24	35	
32.	3	26	112	
33.	3	34	12	
34.	.....			Spoiled.
35.	3	37	48	
36.	3	41	46	
37.	3	44	4	
38.	.....			Forgot to uncover the plate.
39.	3	51	116	Spot *a* uncovered completely.
40.	.....			Forgot to uncover the plate.
41.	3	59	5	Reappearance of spot *b*.
42.	4	2	96	
43.	4	5	98	
44.	4	10	75	
45.	4	16	40	

The diagram shows the appearance of the cross wires when projected on the glass screen. The image of the sun, being twice reversed, is finally depicted on the screen in its natural position, north being at the top, south at the bottom, east to the left hand, and west to the right hand. In the positive photographs of the eclipse, printed from the negatives, the pictures are likewise erect, and the points similarly situated. Calling the wires I., II., III., IV., I. would have approximately the position-angle of 45°, II. 135°, III. 225°, and IV. 315°. In the measurements, hereafter to be described, the several photographs were so placed on the measuring instrument as to cause its circle to read respectively one or other of these angles, according as either I., II., III., or IV. was employed in adjusting the photograph, the correction to the measured angles, necessitated by the deviation of that wire from its assumed position in reference to a parallel of declination, being subsequently applied.

The wires were found not to be absolutely at right angles.

	°	′
IV.— I. measured	89	59·3
I.— II.	89	52
II.—III.	90	8·7
III.—IV.	90	
	360	0



At $4^h\ 35^m$, when the heliograph was pointed to the west of the meridian, observations were made to determine the deviation of the position-wires, from an angle of 45° to a parallel of declination, by the method described by Mr. Carrington in the 'Monthly Notices' of the Astronomical Society, vol. xiv. p. 153; and the observations were repeated on July 19 at $11^h\ 55^m$, when the heliograph was pointed east of the meridian.

July 18, $4^h\ 35^m$, by Frodsham 3094 uncorrected.

The sun's limb made contacts with wires I. and III.

	sec.
at an interval of	194
" "	195
" "	193
Sum	582

and with wires II. and IV.

at an interval of	188·5
" "	191
" "	188
	567·5

582 log = 3·7649
567·5 log = 3·7540
Angle of φ 45° 43′·5 log tan. = 0·0109

July 19, $11^h\ 55^m$.

The sun's limb made contacts with wires I. and III.

	sec.
at an interval of	189·5
" "	188
Sum	377·5

and with wires II. and IV.

at an interval of	193·5
„ „	193
Sum	386·5

whence angle $\varphi = 44° 19'·5$.

	h	min.				
At	4	35	P.M.	$\varphi =$	45′	43″·5
At	11	55	A.M.	$\varphi =$	44	19·5
Interval	4	40	diff.	+	1	24·0
Interval . . . . . =	280			= +	84	

$\frac{+84}{280} = +0'·3$ the change of the angle φ in 1 minute.

	h	m	sec.
No. 6 photograph was taken at	1	47	48
Deducting	11	55	0
	1	52	48 = 112·8 min.

$112·8 \times 0'·3 = 34'$, $44° 19'·5 + 34' = 4° 53'·5$, the position of wire I. in reference to a circle of R.A. at the epoch of No. 6 photograph; hence the correction to be applied to the assumed position of 45° is $-6'·5$.

With the foregoing data has been calculated the following Table of corrections, to be applied to the assumed position of the wires at the epochs of the several photographs.

No.	Correction to the wires.	No.	Correction to the wires.	No.	Correction to the wires.
6.	−6′·5	20.	+ 9′·5	32.	+23′·2
7.	−5·1	21.	+11·0	33.	+25·4
8.	−4·0	22.	+11·7	35.	+26·4
9.	−2·0	23.	+12·6	36.	+27·6
10.	−1·8	24.	+14·0	37.	+28·4
11.	−0·7	25.	+15·5	39.	+29·8
12.	+0·5	26.	+16·5	41.	+31·9
14.	+3·2	27.	+18	42.	+33·0
15.	+4·0	28.	+19·4	43.	+33·9
16.	+5·5	29.	+20·3	44.	+35·3
17.	+7·1	30.	+21·2	45.	+37·0
19.	+8·0	31.	+22·5		

In these corrections I have not taken into account a small error in the computed angle of φ, which arises in consequence of the wires not being at right angles; for, on examining them, I found that the heat of the sun had caused a curvature in one, and I could not, without much trouble, have ascertained the correction with precision. It was computed that it would not, however, amount in any case to more than $+2'$. My measurements of position-angle, hereinafter given, are moreover liable, from the difficulty of adjusting the photographs, to discordances to the like extent, as will be easily

conceived when it is stated that 2′ on the sun's limb do not occupy more than the space of $\frac{1}{1000}$th of an inch on the photographs obtained with the Kew heliograph.

At the moment of taking the photographs, the collodion was in a soft and moist condition, but subsequently, when the measurements were made, it had become dry. It became, therefore, not only a matter of interest, but of fundamental importance, to ascertain whether there had been any contraction of the collodion while drying, and, if so, whether the contraction had been uniform. Much care and attention were necessary in order to determine this point. By observing the positions of specks on the glass in respect of markings on a photograph while wet, it could be seen whether they retained their relative positions when the collodion had dried. The result, however, proved that there was no appreciable contraction, except in thickness, and that the collodion film did not become distorted, provided the rims of the glass plate had been well ground. I cannot show this more strikingly than by citing the measured radius of the sun on two photographs, namely, Nos. 6 and 45, and the measurement of the angles between the position-wires depicted on them. The radius of the sun No. 6 was found to be 1906·5, that of No. 45 1906·0 thousandths of an inch.

	Angle between IV. and I.	Angle between I. and II.	Angle between II. and III.	Angle between III. and IV.
No. 6	89° 59′·3	89° 52′	90° 8′·7	90° 0′
No. 45	90 2	89 53	90 6	89 58·5
Diff. 6—45 . .	—2·7	— 1	+ 2·7	+ 1·5

These differences, which are extremely small, do not exceed those obtained in measuring at different times the same photograph, and depend somewhat on the judgment exercised in causing the images of the position-wires to be bisected exactly by the wire of the microscope.

Photographs of the various phases of the partial eclipse, either previous to or after totality, exhibit a very curious phenomenon. The concave edge of the sun in immediate contiguity with the moon's limb, appears brighter than the other neighbouring parts of the crescent, while the convex limb of the sun bordered by the dark background of the sky, does not appear at all brighter than its proximate parts. This brightening of that part of the sun's disk which borders on the moon's limb, extends only for the space of a narrow line beyond the latter, but is remarkably conspicuous. As it cannot be accounted for by assuming the existence of a lunar atmosphere, it naturally excited a desire to trace out its cause. The Astronomer Royal, to whom I pointed out the fact, ascribed it to the effect of contrast, and I have subjected this hypothesis to the test of experiment in the following manner:—Having made some photographic prints of the sun's crescent on paper, which showed the appearance in a striking manner, I cut out about half of the crescent with sharp scissors, in such a way that the visible surface of the sun might be lifted up like a tongue, and replaced in its normal position within the background at pleasure; on smoothing the part so cut out, and causing it to occupy its original place,

the bright line was apparent, but it disappeared when the crescent was lifted up, and a sheet of white paper was interposed between it and the dark ground of the photograph. These phenomena occurred when the photograph was examined with the naked eye, with the aid of spectacles, or, from a short distance, with a sharply defining telescope by Ross. Viewed in either of these ways, the brightening was found to begin immediately beyond the edge of the white paper as it was introduced more or less under the crescent.

For the purpose of illustrating this paper on the occasion of its being read before the Society, I prepared a representation of one of the photographs of partial phase, 3 feet in diameter, in which, bearing in mind the well-known fact that there is on the solar disk a gradual diminution of the intensity of the light from the centre to the periphery, I carefully reduced the brightness of the solar crescent in due gradation towards the convex boundary. In the first instance the background was not painted in, and I expected that when it was completed a brightening would immediately occur. Such, however, was not the case.

On calling Professor STOKES's attention to this failure in producing the phenomenon of brightening by artistic means, he suggested that I should renew the attempt by using a real photograph of the sun and a dark disk for the moon*. On this plan I succeeded in making eclipse-pictures artificially, which showed the brightening very distinctly. From these experiments I am inclined to believe that Mr. AIRY's explanation is the true one, although it is a curious subjective fact that the parts possessing superior illumination exhibit to our perception an extremely bright line, bordering immediately on the dark limb of the moon, while the less bright parts towards the circumference present no such appearance, although they also are contrasted with the dark background.

In order to study other points connected with the photographs, I had made, on glass, some enlarged copies, in which the moon's disk was increased in some cases to 9 inches, in others to 13 inches in diameter. It was found that measurements could be made on these with considerable accuracy, by means of a graduated beam compass reading to thousandths of an inch, and I had proceeded to some extent in this way, when it occurred to me to have an instrument constructed expressly for measuring the original negatives. The study of the enlarged copies led, however, to a method of producing charts of the prominences with complete fidelity; and the plan will, I think, hereafter prove applicable to the production not only of astronomical, but also of other graphic representations derived from photographs. In order to carry out this method, a table had to be constructed with a square hole cut in it somewhat smaller than the glass positive to be worked upon; a recess surrounding the hole was made in the top of the table, just the size of the glass, and of a depth corresponding to the average thickness of the plates. Four spring clips served to hold the glass firmly in its seat. Parallel with

* For the lunar disk I employed photographic paper darkened to the same tint as the background of the solar photograph. These disks were in some cases neatly inserted in a circular hole in the solar picture, and in other cases pasted on it. In either case the surface was polished and made uniform by passing the picture through a rolling-press.

one side of the table were inserted two brass plates with long slots through which two screws worked. These screws passed through a straight edge, which could be adjusted so as to cause a right-angled drawing-triangle resting against it to assume any required position with respect to the image of the position-wires depicted on the photograph. A long glass mirror was attached to the frame of the table underneath the top, in such a way as to be adjustable to the angle best suited to reflect light through the transparent positive. In front of and above the table was placed an inclined screen formed of tissue paper, to diminish the direct light, a certain amount of which was required to show the position of the etching-point. Without the aid of this screen, the direct light would have been too powerful, and would have prevented the details of the transparent photograph from being seen by the light transmitted from below after reflexion from the mirror.

In the first place, the centre of the picture was found, and marked with a diamond point. A drawing-triangle, with one angle of 90° and two of 45°, was now placed over the photograph, with one of its sides resting against the adjustable straight edge, when its hypothenuse would coincide approximately in direction with the images of the wires. By adjusting the straight edge, the hypothenuse of the drawing-triangle was brought to exact coincidence with either of the wires, and the straight edge, against which it rested, was then (by means of the screws passing through it) clamped in position. By sliding the triangle along in contact with the straight edge, a line parallel with the wire was next set off passing through the centre, and marked slightly on the periphery of the picture by scratching with a diamond point through the collodion film. On taking in the beam-compasses a chord corresponding to 45° plus the known + or − error of the wires, a circle of right ascension, or a parallel of declination, could be made to pass through the centre, and, the points of its intersection with the lunar disk having been marked, any angles of position could be ascertained, by taking the chord between any part of a protuberance to be measured and the normal points thus set off.

If it were desired to produce an etching of any photograph, the outline of the protuberances or of the sun's disk and spots, or of the crescent of the sun, as the case might be, was traced very carefully with an etching-point through the collodion, with the aid of a lens. When this had been done, the plate was warmed by holding it before a bright clear fire, and a piece of composition, consisting of a mixture of paraffin and white wax, rubbed over it; the heat of the plate caused the waxy mixture to melt, and thus a very even, thin, and translucent etching-ground was laid on the glass. The outline was now traced a second time, in this instance through the wax, and a camel-hair pencil, wetted with liquid hydrofluoric acid, was rapidly run over the parts traced. In about a minute the acid was removed with blotting-paper, and the plate rinsed with water, and again dried with bibulous paper. When quite dry, the wax was melted by holding the plate before the fire, and wiped off with a cloth. If the etching proved satisfactory, it was again covered with an etching-ground, then centered on the circular dividing-engine, and degrees and subdivisions set off, starting from a normal point previously marked on

the plate. It was then put on the straight-line engine, and a scale of minutes and seconds of arc set off from the moon or the sun's periphery, in accordance with the previously calculated value in arc of subdivisions of an inch; both sets of division were then etched in the same way as the outline.

Sometimes, according to the position in which the photograph was taken*, the etching was performed at the back of the plate, to correspond with the previous tracing through the collodion on its face. In this case the collodion picture might be allowed to remain as "a witness" (as workmen call it) of the correctness of the etching. In other cases, if the original negative had been purposely turned over, so as to present the opposite face to the camera, then the etching was made through the collodion, which had to be removed before the subsequent operations about to be described were performed.

An etched glass plate, if filled with printing ink, could be made to give a print by placing india proof-paper over it, and, after superposing a sheet of glazed paper upon this, rubbing the latter carefully with a burnisher; but it would not be advisable to attempt to take many impressions in this way. However, by the well-known processes of electrotype, copper duplicates of the glass plate can be procured, which can be printed from in the ordinary copper-plate press; and as the glass plate is only used for furnishing the matrices, and is not injured thereby, the printing-plates may be procured without practical limit as to number. In this way Plates XIII., XIV., XV., XVI. and XVII. were obtained. The original glass plate of Plate XV. was, however, made in a somewhat different manner from the others. Originally, it was a photograph of the sun; after the outline of the sun and his spots had been etched, and the normal line marked thereon, the collodion was entirely removed, to permit of the plate being superposed, accurately, first over Plate XIII., and then over Plate XIV. Previously, however, Plate XV. was coated with the transparent etching-ground, and the luminous prominences depicted on Plate XIII. traced off, care being taken to ensure the parallelism of the normal line of one plate with that of the other, and internal contact between the peripheries of the sun and moon respectively. The same thing was done with Plate XIV., the prominences visible in the two pictures being placed in coincidence. In this way the pictures of the prominences could be made to assume their proper position around the sun's picture. In order to facilitate this operation, a positive picture had been previously taken with the enlarging-camera, from both the original totality negatives laid one over the other, and combined suitably together, so as to form in one picture a correct representation of the whole of the prominences. When the two totality pictures had been traced off on Plate XV., a line was drawn to join the two positions of the moon's centre, which had been set off from Plate XIII. and Plate XIV. respectively; this line was then prolonged to show the path of the moon's centre during the period of totality; lines were also drawn to join these positions of the moon's centre, and the sun's centre, and prolonged to the periphery of the sun, to indicate the points of disappearance and reappearance of the sun's limb. When etched, this plate was

* By placing the original negative in the copying-camera with the collodion film either turned towards the lens or away from it, the picture produced was either in its natural position or reversed right for left.

angularly divided concentrically with the sun, and a scale of minutes and seconds of arc etched, starting from the sun's limb, by which means the prominences were referred to the sun's centre, and their angles of position and heights above his periphery could be read off with a fair degree of accuracy.

In the three Plates, XIII., XIV. and XV., a wrong correction was, however, applied for the errors of the wires in determining the zero of the angular divisions, namely $+23'$ for both totality pictures, instead of $+15'{\cdot}5$ for the first totality picture and $+16'{\cdot}5$ for the second; so that in taking angles of position of the prominences, the readings on Plate XIII. must be corrected by applying the number $-7'{\cdot}5$; those on Plate XIV. by applying the correction $-6'{\cdot}5$, and those on Plate XV. by applying $-7'{\cdot}0$.

Moreover, a small error in determining the centre in Plate XIV. also interferes with the absolute correctness of the position-angles and of the heights of the prominences above the moon's periphery. Subsequently to this being etched, I discovered the fact that the centre should have been placed about $5''$ of linear space nearer 270°, in a direction from 90° to that point, and $4''$ nearer 360°. The angular positions of some of the principal prominences, determined by measurement of the original negatives, will be hereinafter given, so that no difficulty will be experienced in correcting the position of the other prominences as read off from the Plates. The prominences in these Plates are represented in their natural (erect) position, and this is also the case with the sun-spots in Plate XV.; the position-angles are laid down from North towards East. The North point (360°) is consequently at the top, the East point (90°) is on the left hand, the South point (180°) is at the bottom, and the West point (270°) on the right hand.

In order to facilitate reference to the prominences, I have designated them on Plate XV. by capital letters, commencing with the prominence situated at right angles to the path of the moon across the sun's disk, which I call A; and I then follow on towards the east with the other capital letters, the small letters being employed, either alone, or with one or more dashes, to mark the subordinate parts.

The three principal sun-spots are marked a, b, c in the order of increase of their several position-angles.

In Plates XIII. and XIV. the details were drawn in on the back of the glass plate, and the collodion pictures still remain intact; Plate XV. was drawn on the face of the enlarged positive, which had been taken intentionally in a reversed position, by reversing the original negative in the copying-camera. The correction in the position of the glass negative on account of its thickness was duly made; that is to say, the totality pictures having been copied with the collodion turned towards, and the sun-picture with the collodion turned from the lens, the collodion was in this way carried from the lens a quantity equal to the thickness of the glass plate. The holder supporting the original negative was therefore moved towards the lens a similar quantity, and the relative sizes of the pictures, as a matter of course, remained undisturbed.

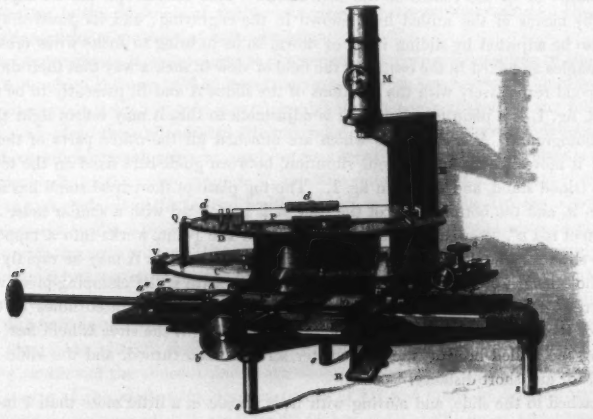
Plate XV. will be found useful as an index map of the prominences, and will facilitate comparisons of the results obtained by the various astronomers who observed the eclipse. Moreover, the position of the sun's axis being given on it, an idea may be formed by

mere inspection, of the general distribution in heliocentric latitude of these appendages. Plate XVI. was obtained by etching a positive copy of No. 22 photograph, and Plate XVII. by etching one of No. 28 photograph. These photographs were obtained with the same setting of the copying-camera as the sun-picture, the original of Plate XV. They show the contours of the sun and moon, and furnish a means for comparing the concave outline of the luminous prominences with the profile of the moon which was in juxtaposition with it.

Reserving, for the present, an account of the measurements actually made on the engraved-glass originals of Plates XIII., XIV., and XV., I will proceed to describe a new measuring-instrument, and the measurements made by its means, not only of the totality-pictures, but also of the original negatives of the sun taken before and after the eclipse, as well as of the different phases of the eclipse. These consist of the direct measurement of the sun's radius, the direct measurement of the moon's radius (where possible), the measurement of the chord joining the cusps, the measurement of the distance between the chord and the peripheries of the sun and the moon, and also the distances of these peripheries, the determination of the angles of position of the cusps, and their angular openings, and, lastly, the position-angles of the luminous prominences. The heights of the prominences could not be determined by means of the micrometer, in consequence of an inadvertence on the part of the makers, who by mistake made it somewhat too small; here, however, the engraved-glass originals of Plates XIII., XIV., and XV. supply the numbers with sufficient accuracy, so that this oversight is of no practical moment.

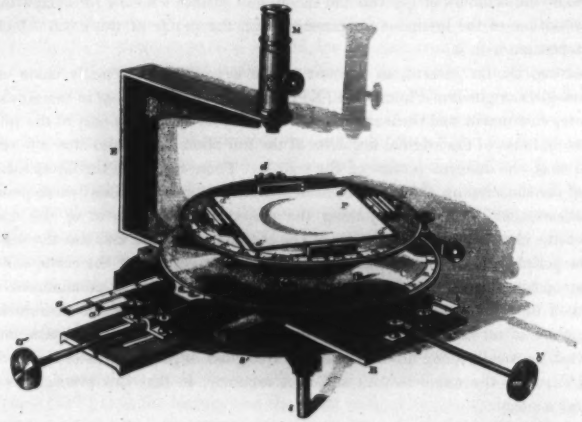
Figures 1 and 2 represent the micrometer in two different positions.

Fig. 1.



In both figures the same letter is employed to designate the same part, capital letters being used for the principal parts, and the same small letter, either alone or with one or more dashes, for the subordinate details attached to it. S is a tripod stand sup-

Fig. 2.



ported on the legs *s*; to this stand is firmly fixed the arm E, which supports the fixed microscope M in the centre of the instrument. This microscope can be adjusted to focus by means of the milled head shown in the engraving; and its positive eyepiece can also be adjusted by sliding it up or down, so as to bring to focus wires crossing at right angles and fixed in the centre of the field of view in such a way that their directions correspond respectively with the positions of the slides A and B, presently to be spoken of. R, fig. 1, is a plain mirror, which is adjustable so that it may reflect light through the photograph. A is a slide to which are attached all the other parts of the apparatus; it moves freely, and without vibration, between guide-bars fixed on the top plate of the tripod stand, and shown in fig. 1. The top plate of the tripod stand has a round hole in it, and the bottom plate of the slide A is perforated with a similar hole.

A steel rod a'', screwed for a certain distance at the point, works into a tapped hole in the slide A; by taking hold of the milled head a'', the slide A may be rapidly moved the whole length of the guide-bars, the rod carrying with it the clamping-piece with its two screws a''', a'''. These clamping-screws slide through two slotted holes in the top plate of the stand; when the nuts a''', a''' are screwed tight the slide is held fast, but by turning the milled head a'' the micrometer screw is also turned, and the slide can be moved for any short distance.

Attached to the slide, and moving with it, is a scale a, a little more than 4 inches in

length, and divided into inches, tenths, and $\frac{1}{20}$ths, the inches being numbered 0 to 4. On the stand is fixed the vernier a' (fig. 2), which reads to $\frac{1}{1000}$th of an inch, and by estimation to $\frac{1}{5000}$th.

On the slide A are two guide-bars, between which the slide B works at right angles to A; the guide-pieces are adjustable by means of set screws a''' (fig. 1), to ensure the rectangularity of the slide B. Slide B is moved by means of the steel rod b'', screwed at one end, and carrying its clamping-bar and screws b''' in the same way as the slide A. It has also an attached scale b, and a vernier b', fixed to the slide A, which reads with the scale b to the same quantities as the vernier a' of the scale a.

On the slide B, which is perforated, is a hollow axis, somewhat more than 4 inches in diameter, in which works the hollow axis of the divided circle C, which reads by means of its vernier V to 10″ of arc. The circle C is clamped by the clamp c, and can then be moved by means of the milled head c', attached to a tangent screw. The vernier V and clamp c are fixed to the slide B.

On the hollow axis of the circle C is a second divided circle D, which reads to minutes of arc, and is divided into four quadrants; the vernier Q (fig. 1) being attached to the lower circle C. The circle D is only used for axial adjustments, to bring the position-wires depicted on the photographs to parallelism with the wires in the microscope M.

The circle D has four dogs fastened on it, through which work the screws d, which carry along with them four pressure plates, with two projecting wires in each, to act against the photographic plate P, and make it central with the instrument. The photograph rests on the four ivory studs d'.

The photograph to be measured is in all cases placed with the collodion side downwards, in order to ensure a constant distance from the microscope; if placed upwards, any variations in the thickness of the various glass plates would necessitate a change in the position of the microscope at each operation.

In the first instance it is necessary to determine that particular position of the slides A and B in which the axis of the circle C corresponds with the centre of the cross wires of the microscope, which is very accurately and rapidly accomplished in the following manner:—A glass plate of suitable size has ruled upon it with a writing-diamond two lines which intersect each other as nearly as may be at right angles: this plate is placed face downwards on the ivory supports d', and the slides A and B are brought approximately to and clamped in the central position, which is at about the division 2 inches on their respective scales; by means of the screws d, the cross on the glass is made to coincide with the cross wires of the microscope. The circle C is now turned through half a revolution, when the cross on the glass plate will be found to have shifted. Half this deviation, according to its direction, is corrected by the screws a'' and b'' of the slides A and B, and half by the centering screws d of the circle D. After a few trials, the cross on the glass plate will not shift during the rotation of the circle C; when this is the case, the verniers of the slides A and B are observed, and the readings noted down.

The centre of the circle C (and consequently of D) was found to coincide with the cross of the microscope when A read 2·0025 inches and B 1·981 inch; and these posi-

tions were found not to alter materially for several months, during which they were from time to time tested.

The next thing to be ascertained was the rectangularity of the two slides, which was done in this way. By means of the upper circle D, one of the ruled lines on the glass plate was made to coincide with a wire of the microscope after the lower circle C had been clamped to read 360°. The slide B remaining central, slide A was unclamped and drawn out to the full extent; any deviation between the line on the glass plate and the centre of the cross in the microscope was then corrected by moving the tangent screw c', and, the slide being again pushed back to the centre, a few trials soon brought about an exact coincidence of the line on the plate and the wire of the microscope during the travelling of the slide A. The vernier V was next read off accurately, and the slide A brought to its normal position, 2·0025 inches. The circle C was then moved through exactly 90°, and clamped securely, and the slide B, having been unclamped, was pulled out to the extent of its path. If the same line on the glass plate maintained its coincidence with the cross of the microscope during the travelling of slide B, the slides were necessarily at right angles; if not, the error was corrected by moving the slide B, with respect to A, by means of the adjusting-screws a'''. If the deviation was considerable, it became necessary, after its rectification, to re-ascertain the normal central position of slides A and B. The greatest deviation from rectangularity amounted, after final adjustment, to 0° 0′ 10″, a quantity which could in no way affect the measurements.

The thickness of the wire of the microscope was ascertained to be 0·0003 inch, amounting to about 0″·15. The measurements could not generally be made nearer than 0·001 inch, equal to 0″·5, and in most cases the *centre* of the wire was brought as nearly as possible to coincidence with the point to be measured; this minute correction consequently has not been applied.

The measurement of the photographs was effected in the following manner:—In the first place, the slide B was set at its central point 1·981 inch, and the slide A pulled out so far as was judged necessary to bring the periphery of the sun (or moon, as the case might be) under the centre of the cross. The photograph to be measured was then laid on the supports, due regard being paid to the position of the wires depicted thereon, so that they might coincide with the corresponding divisions on the circle C, that is, I. with 45°, II. with 135°, III. with 225°, and IV. with 315°. The photograph was next centered by means of the four screws a^2, and the requisite movement of the slide A by means of its screw a'', the slide B remaining at rest. When the periphery of the sun, notwithstanding its somewhat irregular outline, maintained, during the rotation of the circle C, a general coincidence with the centre of the microscope, the vernier a' was read off; this reading, minus the reading of the central position of A, namely, 2·0025, gave the measurement of the radius of the sun. The slide A was then unclamped and traversed right across, so as to bring the limb of the sun, after passing to the other side of the centre of the instrument, into coincidence with the cross when the circle C was rotated, and the vernier a' again read off. The difference between the first and second readings gave the diameter of the sun-picture. Thus, let

	in.
The first reading of photograph No. 20 be	3·907
Deduct the position of the centre on the scale a	2·0025
Radius of the sun=	1·9045

	in.	
The second reading of same photograph	0·094	
And the first reading having been	3·907	
Difference between first and second reading gives . .	2)3·813	sun's diameter.
	1·9065	radius.

These measurements, in the example cited, differ from each other 0·002 inch, which, as will be hereafter seen, is equal to about 1″ of arc. In Table I., columns 7 and 8, are given two series of measurements of the sun in these ways: in column 7 the numbers were obtained by the first, in column 8 by the second method.

The next operation was the rectification of the position of the photograph by means of the wires depicted on it. I must here call attention to the circumstance, that in the various phases of the eclipse the wires cannot be traced beyond the crescent of the sun; for the time occupied in taking the photograph is so extremely short, that the background of the sky in the immediate neighbourhood of the sun does not in the slightest degree depict itself on the collodion plate. The wires are visible on the sun's crescent, because they intercept his action, and produce a blank space corresponding to their shadows, but evidently the dark body of the moon and the adjacent sky could have no such effect. In photographs of the partial phases, therefore, the length of the wires depicted corresponds with the extent of the crescent unobscured, and is shorter the nearer the picture is to the totality. A wire which is visible in some cases is afterwards covered by the moon, so that the correct apposition of the picture could not be effected, in every instance, by means of the same wire. In the totality-pictures the whole four wires are visible, from the fact of one or other of them having intercepted either the light of the prominences or that of the corona, the latter even having depicted itself sufficiently to render the shadow of the wires quite perceptible. In the other phases, either wire IV. or wire II. was used for rectifying the picture; if wire II. were used, the circle was clamped to read 134° 54′ 10″, because this had been found, by measurement, to be the position which corresponded to 315° for wire IV.

Supposing wire IV. to have been visible, and the circle clamped so as to read exactly 315° after centering, it would generally happen that the picture of wire IV. was neither under the centre of the microscope, nor parallel with the wire corresponding to the slide A, because, in the first place, the picture may not have been taken exactly in the centre of the heliograph; and in the second place, in centering, the image of the wire may not have been brought exactly parallel with a normal diameter from 315° to 135° of the circle C. If such were the case, by means of the upper circle D the image of the wire was first made approximately parallel with a wire of the microscope, and then, by means of the screw b'' of the slide B, was brought exactly under this wire of the

microscope, so that the latter bisected longitudinally the broad image of the wire in the photograph; the slide A was now unclamped, and drawn along: if the microscope-cross continued to bisect, in the direction of its length, the wire of the photograph, the operation had been successful; if not, by again turning the upper circle D through a small arc, and moving the slide B sufficiently to cause bisection, this coincidence was finally brought about. Slide B was then screwed back to its original central position, namely 1·981 inch, and slide A to that position in which the periphery of the sun came exactly under the centre of the cross of the microscope, which in the example cited would be 3·907 inches.

The instrument was then in a position for measuring the position-angles of the cusps, which was effected by rotating the circle C so as to bring first one and then the other cusp under the microscope. Table III., columns 8 and 9, contains a series of such measurements, corrected, however, for the ascertained error of the position-wires of the heliograph, which is given in column 7, for the epoch of each photograph: the original numbers, before the correction was applied, were the means of three measurements for each cusp.

The position-angles of the cusps gave the means of finding the position-angle of the line joining the centres of the sun and moon, which is at right angles to the chord joining the cusps. For example, in photograph No. 20, the measurement of which has been quoted by way of illustration,

The position-angle of the northern cusp was found to be	13°	49′	30″*
That of the southern cusp	218	19	
Adding	360		
And dividing by 2	2)592	8	30
We obtain for the position-angle of the line joining the sun and moon's centres	296	4	15

The circle C was now fixed, so as to read the angle thus found to be that of the line joining the centres of the sun and moon, and the slides A and B were both unclamped, and so placed as to bring one of the cusps exactly to coincide with the centre of the microscope-cross. Slide A was then clamped, and slide B drawn along so as to bring the other cusp under the microscope. If it coincided with the centre of the cross, the operation was so far completed; but if not, by causing the circle C to move through a small arc, and adjusting A a little, the coincidence of both cusps was brought about. In the photograph cited, the circle had to be brought to read 296° 0′ 30″; and this, and other numbers obtained in a like manner, are the lines of centres given in Table III., column 11, corrected for the errors of the wires of the heliograph.

The coincidence of the cusps with the microscope-cross having been effected, slide B was moved so as to bring one cusp exactly under the centre of the microscope; and its position having been read off on the vernier b', the other cusp was made central, and the vernier b' again read off, the difference between the two readings giving the length

* These numbers are not corrected for the errors of the position-wires as in Table III. columns 8 and 9.

of the imaginary chord joining the cusps. Thus, in the photograph under consideration,

The first reading of vernier b' was . .	3·841
The second reading	0·124
Length of the chord = difference . . .	3·717
Length of the semichord or sine . . .	1·8585

Slide B was now brought to its normal central position, 1·981 inch, and clamped, and vernier a' of the slide A being read off, gave the position of the imaginary chord. The slide A was now moved sufficiently to bring, first, the moon's periphery and then the sun's periphery into coincidence with the centre of the microscope-cross, and the readings were recorded in each case. These readings gave the means of finding the versed sine of the moon, the versed sine of the sun, and the distance of the moon's periphery from the sun's periphery.

Thus the first reading of the vernier a' gave the position of the chord . .	2·407
The second reading the position of the moon's limb	1·154
Their difference the versed sine of the moon	1·253
Again, the first reading was	2·407
The third reading giving the position of the sun's limb	0·094
And their difference the versed sine of the sun	2·313
Position of the moon's periphery as above	1·154
Position of the sun's periphery as above	0·094
Their difference gave the distance of the sun and moon's peripheries . .	1·060

Table II., columns 2 to 10 inclusive, gives a series of readings of the verniers a' and b', and the resulting determinations of the chords and the versed sines of the cusps of the sun and the moon.

By making the vernier of the circle C read 360°, plus the correction for the error of the wires of the heliograph, the position of the cusps could be read off in differences of right ascension and declination from the sun's (or the moon's) centre. Thus, in the photograph quoted, No. 20, the circle was set at 0° 9′ 30″, and the northern cusp made to coincide with the centre of the cross of the microscope by moving both the slides; the same thing was done with respect to the southern cusp. The readings were as follows:—

Northern Cusp.

Slide A	3·853	Slide B	2·429
Centre	2·0025	Centre	1·981
Difference of declination . .	+1·8505	Difference of R.A. . .	+0·448

Southern Cusp.

Slide A	0·505	Slide B	0·803
Centre	2·0025	Centre	1·981
Difference of declination . .	−1·4975	Difference of R.A. . .	−1·178

Although the differences of declination and right ascension of the cusps from the centres were taken out in every case, they will not be recorded in this report, because the results have been worked out by other measurements. I cite the above merely to show the applicability of the instrument to the measuring of two such coordinates as declination and right ascension.

Measurements of the moon's radius could, I found, be obtained with great accuracy by centering all the photographs which had been taken so near totality as to give a large proportion of the lunar periphery. Experience in centering for the sun had previously proved that the operation could be performed with ease, even in those cases where a large portion of the sun's disk was shut off by the moon. The totality-negatives were also measured, and I had two positive copies on albuminized glass made by superposing the negatives; and these also were measured. In the case of the totality-pictures, besides the radius, the whole diameter was measured, by causing the slide A to carry the picture from one side to the other of the centre of the reading-microscope. These measurements are given in Table I. column 9.

The readings, it will be observed, are in inches and decimals of an inch, the values of which in *arc* are based entirely on an assumed diameter for the sun, in consequence of no steps having been taken in Spain to obtain data for an absolute scale. This was not attempted, because at that time it was not contemplated that such an extensive use of the photographs would afterwards be made. It might be practicable to ascertain the value in arc of an inch on the screen of the heliograph; but this would have no retrospective application to the pictures obtained in Spain, because it would not be possible to place the object-glass, the secondary lens, and the screen in relative positions absolutely the same*. By assuming for the semidiameter of the sun its tabular value, the arbitrary measurements were translated into measures of arc; but it will be seen that any error in the tabular semidiameter of the sun necessarily affects all the numbers based upon it.

Table I.

In Table I. are given the following quantities:—Column 1 contains the progressive numbers of the photographs. Where a break occurs in the consecutive order, it arises from the circumstance that the omitted plate was spoiled after the picture had been taken, or that the photograph could not be taken, for the reasons already given.

Column 2 contains the times noted as the epochs of the photograph, to the nearest half-second. In observing the time, the half-second beats of the chronometer were counted, from 1 to 120, a method which, I think, is less likely to lead to error than the

* Since this paper was handed in, the Kew heliograph has been removed to Cranford, and attempts have been made to obtain an absolute value in arc for the scale of the micrometer by procuring pictures of objects situated at the distance of about a mile. Hitherto these attempts have been attended with only partial success, on account of the want of definition of the resulting pictures, consequent on the feebleness of the light and the movements of the intervening atmosphere. I have it in contemplation to continue the experiments and to erect a scale of equal parts about 80 feet long, at a known distance, with the view of ascertaining the radial distortion of the image, and the value of each increment from centre to circumference.

counting whole seconds by two steps, when using a chronometer which beats half-seconds.

Column 3 contains the computed errors of the chronometer for the several epochs, and Column 4 the epochs of the photographs, corrected by deducting the numbers in column 3 from those of column 2.

The time-intervals in columns 5 and 6, the nature of which is sufficiently explained by their superscriptions, will be found useful in checking the numbers given in this and the subsequent Tables, they having been employed in the calculations.

Columns 7 and 8 give the results of the measurements of the sun's radius by the two methods already explained. On examining these numbers certain discrepancies will be apparent; and they may, on the whole perhaps, appear at first sight not so accordant as might have been expected. The greatest difference in column 7 is between No. 29, in which the sun's radius measured 1909·5 thousandths of an inch, and No. 16, in which it measured 1900·5 thousandths—the difference being $\frac{9}{1000}$ths of an inch, or about 4″·5. Some of this difference is due to errors in centering; for on taking the means of columns 7 and 8 for the photographs Nos. 8 and 16, the difference of radius was reduced to $\frac{7}{1000}$ths, =3″·5. Measures of the same photograph may vary in difficult cases, on account of the irregularity or faintness of the sun's border, from $\frac{1}{1000}$th to $\frac{8}{1000}$ths of an inch, but in most cases they are in complete agreement. There is, however, a real difference of photographic diameter in different pictures; for in disturbed states of the atmosphere the sun's diameter is enlarged by irradiation; and, on the other hand, when once the instantaneous apparatus has been so adjusted as to produce the best effect, any great diminution in the intensity of the light renders the picture more feeble, and the periphery of the sun consequently less distinctly defined; and it is just barely possible that the fainter portions of the limb are not depicted at all, whence would arise a diminution in the size of the picture.

The mean of the measurements given in column 7 is 1904·17, and of those in column 8, 1905·65,—the difference not being greater when converted into arc than 0″·7, while the mean of both sets of numbers is 1904·91. By assuming the radius of the sun to be 15′ 44″·8, as calculated by Mr. Hind from Leverrier's Tables, the value of $\frac{1}{1000}$th of an inch of my scale becomes 0″·4960, the logarithm of which is 9·6954654. This number has been employed in the reduction of the several measurements to their equivalents in arc.

Column 9 contains measurements of the moon, which are very accordant. The original negative of the second totality-picture presented greater difficulty in centering than the first totality-picture, in consequence of the triplication of the images of the protuberances; a positive albumen copy of it on glass was more easily centered. The greatest discordance in the measures is 4·5 thousandths, which are equivalent to 2″·2. The mean of all the measures gives 2002·25 thousandths =16′ 33″·1 as the radius of the moon, which agrees almost exactly with the number of Mr. Carrington, 16′ 33″, and that of Mr. Farley, 16′ 32″·9.

In column 10 are given the distances actually measured between the peripheries of the sun and moon in a direction at right angles to a line joining the cusps; these numbers differ in a few cases from those which were obtained for the same photographs by deducting the numbers in column 4 from those in column 3, Table II., but it has not been thought necessary to alter them in Table I. The cases in which discrepancies occur are the following:—

		inch.
No. 8.	in which the peripheral distances in Table II. differ from those in Table I. by	−0·003
No. 17.	„ „ „	+0·001
No. 19.	„ „ „	−0·001
No. 22.	„ „ „	+0·003
No. 29.	„ „ „	−0·001

The numbers for these particular photographs in Table II. result from a series of second measurements of several of the photographic plates, which it was found necessary to make again for Table II., in consequence of some minute errors in reading the vernier b' in the first series.

Column 11, Table I., gives the numbers of column 10 reduced to the adopted mean solar radius, namely, 1904·91 thousandths of an inch.

Column 12 gives the differences of the peripheral distances for two consecutive photographs, and, neglecting the augmentation of the moon's semidiameter, the approximate approach or retreat of the centres in the interval between their epochs. By dividing these numbers by those in column 6, were obtained the numbers in column 13, which are very nearly the rates of approach or retreat of the sun and moon's centres per minute.

Column 14 gives the approach and retreat per minute for the longer periods bracketed, and column 15 the same numbers reduced to seconds of arc. These rates of approach and retreat of the centres are affected by any errors in registering the time of the photographs, and also by all errors of measurement. The numbers do not run quite smoothly, and yet perhaps they are as good as could be expected. No account was taken of the augmentation of the moon's semidiameter, except for the middle of the eclipse; but the rates of approach or retreat for the beginning and end, even without this correction, admit of a comparison with the computed numbers, as the change of semidiameter was not great during the intervals.

	Approach of sun and moon's centres per minute at the commencement.	Relative motion of centres per minute at the middle.	Retreat of sun and moon's centres per minute at the end.
Measured . . .	25″·26	27″·84	30″·05
Carrington . .	24·87	27·27	29·85
Farley	25·14	27·40	29·61

TABLE I.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.
No. of photo-graph.	Observed time, Frodsham 3094.	Error of Frodsham 3094 at noon, 4·6 seconds fast; daily losing rate 2·1 seconds; de-duced error at epoch.	Epochs of photographs, corrected to Greenwich mean time.	Interval elapsed since the epoch of No. 6.	Interval elapsed between two con-secutive photo-graphs.	Measured radius of sun, in thousandths of an inch, By measuring distance of periphery from centre.	Measured radius of sun, in thousandths of an inch, By reversal and travers-ing slide A, to eliminate errors of centering.	Measured radius of the moon, in thousandths of an inch, by measuring distance of periphery from centre, &c.	Measured distance of sun and moon's peri-pheries at right angles to a line joining the cusps, in thousandths of an inch.	Distances reduced to the mean radius 1904·91, in thousandths of an inch.	Approach or retreat of peripheries in the interval, in thousandths of an inch.	Approach or retreat of the peripheries (or of the cen-tres nearly) per minute, in thousandths of an inch.	Approach or retreat of centres per minute, calculated for the intervals bracketed, in thousandths of an inch.	Numbers in column 14 reduced to seconds of arc.
	h m sec. s	sec.	h m sec.	min.	min.									"
6	1 47 96	+4·4	1 47 43·6	0·000	4·792	1906·5	1907·2	.....	3592	3585·2				
7	52 71	+4·4	52 31·1	4·792	3·533	1907·5	1909·5	.....	3403	3397·5	187·7	53·12	50·92	25·26
8	56 15	+4·4	56 3·1	8·325	6·533	1905·5	1906·5	.....	3074	3072·6	324·9	49·73		
9	2 2 79	+4·4	2 2 35·1	14·858	0·684	1905·5	1906·0	.....	3040	3038·2	34·4	50·29		
10	3 41	+4·4	3 16·1	15·542	3·833	1905·5	1906·5	.....	2840	2842·1	196·1	51·16	51·71	25·65
11	7 21	+4·4	7 6·1	19·375	4·083	1902·5	1904·5	.....	2628	2627·9	214·2	53·46		
12	11 31	+4·4	11 11·1	23·458	8·942	1904·5	1905·5	.....	2176	2176·1	451·8	50·53		
14	20 24	+4·4	20 7·6	32·400	2·608	1904·0	1905·7	.....	2040	2041·1	135·0	51·77	50·80	25·20
15	22 97	+4·4	22 44·1	35·008	5·050	1903·0	1904·7	.....	1776	1777·8	263·3	52·14		
16	27 103	+4·4	27 47·1	40·058	5·442	1900·5	1905·5	.....	1489	1487·1	290·7	53·42	52·19	25·89
17	33 36	+4·4	33 13·6	45·500	3·058	1901·5	1903·0	2001·5	1332	1333·9	153·2	50·10		
19	36 43	+4·4	36 17·1	48·558	5·100	1904·5	1906·5	2004·5	1060	1059·7	274·2	53·76		
20	41 55	+4·4	41 23·1	53·658	4·925	1904·0	1904·7	2002·5	792	792·2	267·5	54·31	53·87	26·72
21	46 46	+4·4	46 18·6	58·583	2·117	1904·5	1906·5	2001·5	680	679·8	112·4	53·09		
22	48 60	+4·4	48 25·6	60·700	2·943	.....	.....	.....	523	523·0	156·8	53·28	53·28	26·43
23 *a*	51 53	+4·3	51 22·2	63·643	4·292	.....	.....	.....	298	298·0	225·0	52·42		
24 *b*	55 88	+4·3	55 39·7	67·935		.....	.....	2001·5 *d*	.....	.....	.....	.....		
25	First totality.	.....	.....	.....	.....	.....	.....	2000·0 *e*	.....	.....	.....	.....	56·19	27·84 A
26	Second totality.	.....	.....	.....	.....	.....	.....	2004·5 *f*	.....	.....	.....	.....		
27 *c*	.....	.....	.....	.....	.....	.....	.....	2000·0 *g*	.....	.....	.....	.....		
28	3 13 116	+4·3	3 13 53·7	86·168	3·300	1902·5	1903·0	2001·5	20	20·0	.....	.....		
29	17 32	+4·3	17 11·7	89·468	2·934	1909·5	1911·0	2003·5	556	556·6	181·4	54·97		
30	20 24	+4·3	20 7·7	92·402	4·091	1900·5	1902·0	2003·5	740	738·0	171·8	58·55	56·71	28·13
31	24 35	+4·3	24 13·2	96·493	2·642	1904·5	1905·0	2002·5	908	909·8	232·3	56·78		
32	26 112	+4·3	26 51·7	99·135	7·167	1902·5	1904·0	.....	1142	1142·1	148·0	56·02	56·15	27·85
33	34 12	+4·3	34 1·7	106·302	3·300	1903·0	1904·7	.....	1259	1290·1	402·8	56·20		
35	37 48	+4·3	37 19·7	109·602	3·983	1902·5	1904·0	.....	1692	1692·9	194·7	59·00		
36	41 46	+4·3	41 18·7	113·585	2·650	1904·5	1907·0	.....	1886	1887·6	227·5	57·12	57·16	28·35
37	44 4	+4·3	43 57·7	116·235	7·933	1902·5	1904·5	.....	2116	2115·1	145·6	54·94		
39	51 116	+4·3	51 53·7	124·168	7·077	1903·5	1905·5	.....	2259	2260·7	459·9	57·97	57·97	28·75
41	59 5	+4·2	58 58·3	131·245	3·758	1902·5	1904·5	.....	2720	2720·6	416·7	58·88	58·88	29·20
42	4 2 96	+4·2	4 2 43·8	135·003	3·017	1904·5	1906·0	.....	3135	3137·3	229·1	60·96	60·59	30·05
43	5 98	+4·2	5 44·8	138·020	4·808	1902·5	1906·5	.....	3366	3366·4	181·4	60·13		
44	10 75	+4·2	10 33·3	142·828	5·709	1904·0	1904·0	.....	3547	3547·8				
45	16 40	+4·2	16 15·8	148·537		1906·0	1907·5	.....						
						1904·17	1905·65	2002·25						

a 23 is a solarized picture, and the contour not perfectly defined.
b 24 also is a solarized picture; and in its epoch there is reason to believe that there is some error.
c 27 is solarized; many impressions exist on the plate, in consequence of the telescope having been agitated by a gust of wind. Its epoch was not noted.
d 25. These measurements were made on the original negative.
e 25. These measurements were made on an albumen positive copy, obtained by printing from the negative by superposition (*i. e.* not by means of a camera).
f 26. From the original negative.
g 26. From the albumen positive.
h The augmentation of the radius of the moon, in the cases of Nos. 22 and 25 was taken account of in the computation of this quantity.

Sun's mean radius 1904·91 = log 3·2798745

Moon's mean radius = log 3·3015184

$\frac{2002·25}{1904·91} = 1·0511$ { ratio of lunar to solar radius.

97·3 = excess of moon's over sun's radius at the middle of eclipse.

Tabular radius of sun ..... = 944″·8 = log 2·9753399
Sun's measured radius in 1000ths of an inch ... = log 3·2798745
∴ Value of $\frac{1}{1000}$th of an inch in sec. of arc 0″·4960 = log 9·6954654
Moon's radius in 1000ths of an inch ..... = log 3·3015184
Value of $\frac{1}{1000}$th of an inch in sec. of arc ..... = log 9·6954654
∴ Moon's radius = 993″·1 = log 2·9969838
= 16′ 33″·1

Column 11.
The numbers in this column were obtained as follows:—
Nos. in Col. 7 = r
" 8 = r'
" 10 = d
" 11 = d'
$d' = \frac{d \cdot 1904·91 \times 2}{r + r'}$

Table II.

In Table II., columns 2, 3, and 4, is given a series of measurements from which are derived the versed sines of the sun and moon, and in columns 7 and 8 other measurements from which are obtained the lengths of the chord joining the cusps given in column 9. Columns 11 and 12 the resulting semidiameters of the moon and sun respectively, calculated upon these data. On account of the change in the apparent diameter of the moon during the eclipse, the measures in column 11 are not adapted for giving a mean result of the whole series; but the case is different for the sun, and hence the average of the measures of column 12 has been taken out for comparison with the mean semidiameter obtained by direct measurement: the mean semidiameter of the sun given by Table II. is 1903·1 thousandths of an inch, which differs by only $-1{\cdot}91$ thousandths $=-0''{\cdot}9$ from the value given in Table I. With respect to the moon's radius, it should be borne in mind that the photographs near the commencement and the end of the eclipse are not well adapted for such calculations, as a very minute error in measuring the chord or the versed sine introduces a great error in the resulting calculated semidiameter; and any rounding off or indistinctness of the cusps, especially near the epochs of commencement and end, militates greatly against exact determinations of the moon's radius by the method employed. For these reasons, it has been necessary to omit certain numbers of column 11, in deducing the averages for the moon's semidiameter, namely, Nos. 7, 10, 14, 39, 41, 42, and 43: by bringing together into three groups the remaining calculated semidiameters of the moon, we obtain for the mean epochs of these groups the following results, as compared with those deduced from Mr. Farley's numbers for the same epochs.

		Mean epoch, $2^h\ 17^m$.	Mean epoch, $3^h\ 1^m$.	Mean epoch, $3^h\ 34^m$.
Moon's radius	De La Rue .	$99''{\cdot}8$	$99''{\cdot}3$	$99''{\cdot}0$
	Farley . .	$99''{\cdot}1$	$99''{\cdot}9$	$99''{\cdot}9$
	Difference	$-\ 0{\cdot}3$	$+\ 0{\cdot}4$	$-\ 1{\cdot}0$

These numbers are remarkably near the computed numbers, and render manifest that even so minute a change as the decrease in the moon's semidiameter during the eclipse is traceable in the photographs. Taking the differences of semidiameter at the first and last epochs, the augmentation of the moon's radius becomes more apparent, and not far from the true numbers; thus

		Moon's semidiameter.	
		De La Rue.	Farley.
	h m		
At . .	2 17	$99''{\cdot}8$	$99''{\cdot}1$
At . .	3 34	$99''{\cdot}0{\cdot}9$	$99''{\cdot}9$
Difference . . .		$2{\cdot}9$	$2{\cdot}2$

TABLE II.—Calculations of the Radii of the Sun and Moon, from measurements of the Chords and Versed Sines.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	
	Slide A.					Slide B.						
No.	Position of a line joining the cusps.	Position of moon's peri-phery.	Position of sun's peri-phery.	Versed sine of the moon $=b$, in 1000ths of an inch.	Versed sine of the sun $=b'$, in 1000ths of an inch.	Position of the first cusp.	Position of the second cusp.	Length of chord, in 1000ths of an inch.	Sine, or semi-chord, $=a$, in 1000ths of an inch.	Calculated radius of the moon in 1000ths of an inch.	Calculated radius of the sun in 1000ths of an inch.	
7	3787	3683	91	104	3696	2643·0	1317·5	1325·5	662·7	2163·4	1907·4	A second measurement gave a similar result; the cusps are [rounded off, and difficult to measure.
8	3693	3496	96	197	3597	2848·0	1114·0	1734·0	867·0	2006·3	1903·0	
9	3525	3168	94	357	3431	3117·0	835·5	2281·5	1140·7	2000·9	1905·1	
10	3504	3133	93	371	3411	3148·5	807·5	2341·0	1170·5	2032·0	1906·3	Same result on remeasurement; the northern cusp not well adapted [for measurements.
11	3403	2936	96	467	3307	3267·0	694·0	2573·0	1286·5	2005·5	1903·7	
12	3294	2724	96	570	3198	3386·0	583·0	2804·0	1402·0	2009·2	1906·3	
14	3045	2271	95	774	2950	3575·0	393·0	3182·0	1591·0	2022·2	1904·0	This plate is fogged, and very difficult to measure.
15	2977	2136	96	841	2881	3615·0	353·0	3262·0	1631·0	2002·0	1902·2	
16	2835	1868	92	957	2735	3687·0	279·0	3415·0	1707·5	2001·8	1900·0	
17	2668	1583	93	1085	2575	3764·0	206·0	3558·0	1779·0	2000·9	1902·0	
19	2572	1428	97	1144	2475	3794·0	174·0	3620·0	1810·0	2003·9	1899·3	
20	2407	1154	94	1253	2313	3841·0	124·0	3717·0	1858·5	2004·8	1903·2	
21	2222	889	97	1333	2125	3870·0	93·0	3777·0	1888·5	2004·2	1901·6	[disturbance during exposure.
22	2133	774	91	1359	2042	3874·0	86·0	3788·0	1894·0	1999·3	1899·3	This plate difficult to measure, from the effects of atmospheric
23	.....	.....	...	.....	.....	.....	.....	.....	.....	.....	.....	Solarized, and too indistinct at the cusps.
24	.....	.....	...	.....	.....	.....	.....	.....	.....	.....	.....	Ditto ditto.
27	.....	.....	...	.....	.....	.....	.....	.....	.....	.....	.....	Ditto ditto.
28	3031	655	99	1376	1932	3880·0	82·0	3798·0	1899·0	1998·5	1899·3	
29	2186	831	92	1355	2094	3880·0	87·0	3793·0	1896·5	2004·7	1905·8	
30	2308	1007	99	1301	2209	3859·0	105·0	3754·0	1877·0	2004·5	1902·0	
31	2460	1239	97	1221	2363	3829·0	133·0	3696·0	1848·0	2009·0	1904·1	
32	2547	1386	97	1161	2450	3802·0	159·0	3643·0	1821·5	2009·4	1902·1	
33	2786	1788	96	998	2690	3715·0	250·0	3465·0	1732·5	2002·8	1902·9	
35	2899	1983	97	916	2802	3658·0	308·0	3350·0	1675·0	1989·4	1901·6	
36	3022	2209	93	813	2929	3588·0	376·0	3212·0	1606·0	1992·7	1904·7	
37	3099	2355	96	744	3003	3529·0	432·0	3097·0	1548·5	1983·4	1900·7	
39	3348	2815	95	533	3253	3325·0	635·0	2690·0	1345·0	1963·5	1904·6	The southern cusp rounded, and impression faint. [difficulty.
41	3539	3231	96	328	3463	3057·0	901·0	2156·0	1078·0	1935·5	1899·3	An extremely faint impression, and the cusps measured with great
42	3680	3461	95	219	3585	2880·0	1072·0	1808·0	904·0	1975·3	1906·5	A faint impression, and difficult to measure. [difficulty.
43	3773	3639	92	134	3681	2683·0	1285·0	1398·0	699·0	1890·3	1906·9	An extremely faint impression, and the cusps measured with great
										Mean.....	1903·1	

No.	Moon's S.D.	Sun's S. D.	No.	Moon's S.D.	Sun's S. D.	No.	Moon's S.D.	Sun's S. D.
8	2006·3	1903·0	20	2004·8	1903·2	31	2009·0	1904·1
9	2000·9	1905·1	21	2004·2	1901·6	32	2009·4	1902·1
11	2005·5	1903·7	22	1999·3	1899·3	33	2002·8	1902·9
12	2009·2	1906·3	28	1998·5	1899·3	35	1989·4	1901·6
15	2002·0	1902·2	29	2004·7	1905·8	36	1992·7	1904·7
16	2001·8	1900·0	30	2004·5	1902·0	37	1983·4	1900·7
17	2000·9	1902·0						
19	2003·9	1899·3						
Means.....	2003·8	1902·7		2002·7	1901·9		1997·8	1902·7
Value in arc.....	993″·8	943″·7		993″·3	943″·3		990″·9	943″·7
Calculated from Farley's numbers	994·1	944·8		992·9	944·8		991·9	944·8
De La Rue—Farley	−0·3	−1·1		+0·4	−1·5		−1·0	−1·1

Mean epoch 2h 17m 3h 1m 3h 34m

	Augmentation of moon's radius.	Farley's augmentation of moon's radius.
h m		
2 17	+″0·5	+″1·2
3 1	0·0	0·0
3 34	−2·4	−1·0
Between 2h 17m and 3h 34m	−2·9	
Ditto Farley's	−2·2	
Difference..	−0·7	

Radius of the moon, column 11 $= r$.
Radius of the sun, column 12 $= r'$.
Versed sine of moon, column 5 $= b$.
Versed sine of sun, column 6 $= b'$.
Chord of cusps, column 10 $= a$.

$$r = \frac{a^2}{2b} + \frac{b}{2} \qquad r' = \frac{a^2}{2b'} + \frac{b'}{2}.$$

Table III.

In Table III., column 2, the epoch of the photographs in Greenwich mean time is again given as in Table I.; in column 3 is given a series of numbers obtained by adding to the peripheral distances of the sun and moon (as shown in Table I., column 10) the number 97·3, which expresses in thousandths of an inch the excess of the mean measured radius of the moon over the mean measured radius of the sun, namely, 2002·2 − 1904·9 = 97·3.

Column 4 contains the numbers in column 3 reduced to seconds of arc, and column 6 the same numbers corrected by the quantities in column 5, which contains the corrections necessary on account of the augmentation of the moon's semidiameter from the mean diameter at the middle of the eclipse. The numbers in column 5 are derived by interpolation from Mr. Farley's calculations. The corrected numbers in column 6 show the distances of the sun and moon's centres at the epochs given in column 2.

Column 7 contains the errors of the wires of the heliograph from the assumed position of 45° for wire I., for the epochs of the several photographs. These numbers have been applied to the numbers in columns 8, 9, and 10, in which, respectively, are given the corrected angles of position of the cusps, and the line joining the sun and moon's centres.

Column 12 contains the measures of half the angles between the cusps, taken from the sun's centre, the numbers being half the differences between the position-angles of each pair of cusps, which are given in columns 8 and 9. The angles in this column were employed in the computation of Table IV.

TABLE III.

1. No.	2. Epoch of photograph, Greenwich mean time.	3. Distance of the sun and moon's perigeries increased by 97·3, the excess of the lunar radius at the middle of the eclipse in 1000ths of an inch.	4. The numbers in column 3 reduced to seconds of arc.	5. Correction to be applied for augmentation of moon's radius; interpolated from Mr. Farley's numbers.	6. Resulting distances of sun and moon's centres.	7. Error of position-wires of photoheliograph, and of their imprints on the photographs.	8. Position-angle of the cusps N. towards E. corrected by the quantities in column 7. Northern cusp.	9. Southern cusp.	10. Line of centres derived from the measurement of position-angles of cusps.	11. Line of centres ascertained by measuring the angle at right angles to an imaginary line joining the cusps.	12. Half the angle included between the cusps.
	h m sec.										
7	1 52 31·1	3682·5	1826·4	+1·4	1827·8	+ 5·1	317° 17′ 41″	278° 24′ 41″	296° 51′ 11″	° ′ ″ a	20 26 30
8	56 3·1	3494·8	1733·4	+1·4	1734·8	+ 4·0	324 11 55	269 42 30	296 57 12	297 5 00	27 14 42
9	2 2 35·1	3169·2	1572·2	+1·3	1573·5	+ 2·0	333 46 25	259 56 55	296 51 40	296 58 40	36 54 45
10	3 16·1	3135·5	1555·2	+1·3	1556·5	+ 1·8	334 52 32	258 52 27	296 59 29	56 52	38 0 2
11	7 6·1	2939·4	1457·9	+1·3	1459·2	+ 0·7	339 18 18	254 2 48	296 40 33	42 48	42 37 45
12	11 11·1	2725·2	1351·6	+1·2	1352·8	− 0·5	344 1 45	249 16 45	296 39 15	37 30	47 22 30
14	20 7·6	2273·4	1127·6	+1·0	1128·6	− 3·2	353 22 2	239 42 57	296 32 29	28 12	56 49 32
15	22 44·1	2138·4	1060·6	+1·0	1061·6	− 4·0	355 55 00	237 31 00	296 43 00	38 00	59 12 00
16	27 47·1	1875·1	930·0	+0·9	930·9	− 5·5	0 50 45	232 19 15	296 35 00	37 00	64 15 45
17	33 13·6	1584·4	785·8	+0·7	786·5	− 7·1	6 0 21	227 6 21	296 33 21	33 6	69 27 00
19	36 17·1	1431·2	709·9	+0·7	710·6	− 8·0	8 51 31	223 42 51	296 17 11	13 30	72 34 20
20	41 23·1	1157·0	573·8	+0·5	574·3	− 9·5	13 59 00	218 28 30	296 13 45	10 00	77 45 15
21	46 18·6	888·5	441·2	+0·4	441·6	−11·0	19 50 46	212 52 21	296 21 33	26 00	83 29 12
22	48 25·6	777·1	385·4	+0·3	385·7	−11·7	21 22 37	209 34 12	295 28 24	26 53	85 54 12
23	51 22·2	620·3	307·7	+0·2	307·9	−12·6	28 24 59	200 42 42	294 33 50	*b*	*f*
24	55 39·7	395·3	196·1	+0·1	196·2	−14·0	33 53 50	190 56 47	292 25 18	*c*	*g*
27	.....	117·3	58·2	−0·3	57·9	−18·0	359 41 00	260 47 00	130 14 00	*d*	*h*
28	3 13 53·7	653·9	324·3	−0·5	323·8	−19·4	30 1 19	208 45 24	119 23 21	119 24 24	89 22 2
29	17 11·7	835·3	414·3	−0·6	413·7	−20·3	34 6 18	203 18 53	118 43 35	118 41 28	84 36 17
30	20 7·7	1007·1	499·5	−0·7	498·8	−21·2	37 47 2	199 30 32	118 33 47	31 19	80 46 45
31	24 13·2	1239·4	614·7	−0·8	613·9	−22·5	42 3 30	194 17 00	118 10 15	9 50	76 6 45
32	26 51·7	1387·4	688·1	−0·9	687·2	−23·2	44 41 37	191 23 2	118 2 19	3 42	73 20 42
33	34 1·7	1790·2	887·9	−1·1	886·8	−25·4	52 14 14	183 30 49	117 52 31	117 49 24	65 38 17
35	37 19·7	1984·9	984·5	−1·2	983·3	−26·4	55 50 24	179 33 24	117 41 54	38 24	61 51 30
36	41 18·7	2213·4	1097·3	−1·3	1096·0	−27·6	60 4 36	175 16 21	117 40 28	37 36	57 35 52
37	43 57·7	2358·0	1169·5	−1·4	1168·1	−28·4	62 55 24	172 19 54	117 37 39	37 54	54 43 15
39	51 53·7	2817·9	1397·6	−1·6	1396·0	−29·8	72 30 18	162 30 3	117 30 10	32 48	44 59 52
41	58 53·3	3234·6	1604·3	−1·9	1602·4	−31·9	82 43 24	152 22 54	117 32 39	*e*	34 50 15
42	4 2 43·8	3463·7	1717·9	−2·0	1715·9	−33·0	88 59 00	145 33 25	117 16 12	23 30	28 17 12
43	5 44·8	3645·1	1807·9	−2·1	1805·8	−33·9	95 25 54	138 51 54	117 8 54	7 54	21 43 00

a Not measured.
b, *c*, *d* Not sufficiently distinct to ascertain the position of the line joining the cusps.
e Too faint to ascertain the position of the line joining the cusps with sufficient precision.
f, *g*, *h* The exact termination of the cusps could not be ascertained with precision.

No. 7.	296° 51′2		
No. 8.	296 57·2		
		h	m
Mean ...	296 54·2	Mean epoch	1 54·3
No. 9.	296 51·7		
No. 10.	296 52·5		
Mean ...	296 52·1	Mean epoch	2 2·9

Interval 8·6 } Increase of position-angle per minute = 0′·24

No. 37.	117° 32′6		
No. 39.	117 30·2		
		h	m
Mean ...	117 33·9	Mean epoch	3 47·9
No. 41.	117 32·6		
No. 42.	117 16·2		
No. 43.	117 8·9		
Mean ...	117 19·3	Mean epoch	4 2·5

Interval 14·6 } Decrease of position-angle per minute = 1′·0.

From Table III. it is possible to derive several elements of the eclipse; for example, the epochs of first and last contacts, and the duration of the eclipse.

The distance of the sun and moon's centres at the epoch of

No. 7 was . . .	1827″·8	No. 41 was . . .	1602″·4
No. 8 was . . .	1734·8	No. 42 was . . .	1715·9
No. 9 was . . .	1573·5	No. 43 was . . .	1805·8

whence is derived, as the mean motion per minute, in the interval between

7 and 9 . . . = 25″·263
41 and 43 . . . = 30·022

	At first contact.	Last contact.
The augmented moon's radius was	994″·5	990″·7
The sun's radius	944·8	944·8
	1939·3	1935·5

By deducting from these sums of the radii the corresponding distances of the centres in photographs 7, 8, and 9, and in 41, 42, and 43 respectively, and dividing the numbers so obtained by the corresponding rates of approach or retreat of the centres, the intervals are derived which have elapsed between the epochs of the first contact and the epochs of the photographs, on the one hand; and on the other, the intervals which must have elapsed between the epochs of the photographs and the last contact of the sun and moon. By subtracting from the epochs of Nos. 7, 8, and 9, and adding to those of Nos. 41, 42, and 43 their several intervals, the following periods result:—

	First contact. h min. sec.		Last contact. h min. sec.
No. 7 . .	1 48 6·4	No. 41 . .	4 10 4·0
No. 8 . .	1 47 57·4	No. 42 . .	4 10 2·7
No. 9 . .	1 48 6·4	No. 43 . .	4 10 4·0
Mean . .	1 48 3·4*	Mean . .	4 10 3·6

whence the duration of the eclipse is found to have been 2 h. 22 min. 0·2 sec.

The position-angle of the line joining the centres at 1 h. 54·3 min. was . . 296° 54′·2
and by adding the decrease in the position-angle since the period of first contact, as calculated from the numbers in Table III. . . . . 1·5
we obtain for period of first contact the position-angle 296 56

which agrees very nearly with the angle calculated by Mr. Carrington and Mr. Farley, namely 296° 54′. In the same manner, the position-angle of the line joining the centres at the epochs of last contact was found to be 117° 12′

Mr. Carrington's number is . . . 117 20
Mr. Farley's 117 18

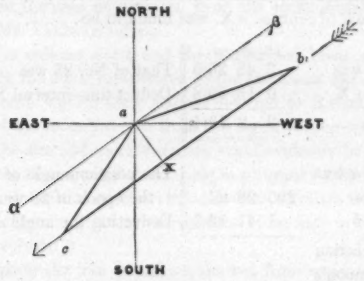
* The time of first contact observed with the achromatic was 1 h. 48 m. 6·6 sec. p. 354.

The beginning and end of the eclipse do not agree with the calculations; for example—

	First contact.			Last contact.			Duration.		
	h	min.	sec.	h	min.	sec.	h	min.	sec.
De La Rue . . .	1	48	3·4	4	10	3·6	2	22	0·2
Carrington . . .	1	47	56	4	10	15·2	2	22	19·2
Farley	1	47	57	4	10	15	2	22	18

But it is possible that the discrepancy may partly arise in consequence of the assumption of a greater angular measure for the diameters of the sun and moon than they in reality subtend; and this view is supported by my measures of the distances of the sun and moon's centres, which, as a whole, come out greater than the computed distances. It will be seen that, as the reduction of my arbitrary measures to their equivalents in arc is dependent on the tabular value for the sun's semidiameter, if this be in excess of the truth, my distances of the centres must come out greater than the real values; and this is actually the case, as will be hereafter seen. If the diameter either of the sun or of the moon, or of both, be less than the tabular numbers, the first contact must happen later, and the last contact sooner than the computed times: the epochs of these phenomena, derived from the measure of the peripheral distances of the sun and moon given above, tend to show that some correction is necessary to the tabular diameters.

The epoch of the middle of the eclipse, the direction of motion of the moon's centre, and the nearest approach of the centres of the sun and moon may also be derived from the photographs, by means of the distances of the centres and the epochs of two photographs, one before and one after totality.



In the above diagram, let a represent the position of the sun's centre, b the position of the moon's centre previous to totality, at the epoch of No. 22 photograph, and c the position of the moon's centre in photograph No. 28, after totality; $b\,c$ will represent the motion of the moon's centre across the solar disk during the interval, and the line $\beta\,\alpha$, parallel to $b\,c$ and passing through the sun's centre, the direction of motion referred to the sun's centre; $a\,X$ shows the nearest approach of the centres of the sun

and moon; X the position of the centre of the moon at the middle of the eclipse.

$a\,b$, the distance of the centres at the epoch of No. 22, was . . . . . 385″·7
$a\,c$, the distance of the centres at the epoch of No. 28, was . . . . . 323″·8

the position-angle of the line joining the centres at the epoch of No. 22 was 295° 28′ 24″
the position-angle of the line joining the centres at the epoch of No. 28 was 119 23 21
whence the angle $c\,a\,b$ . . . . . =176 5 3

	h	min.	sec.
The epoch of No. 22 was . . .	2	48	25·6
„ No. 28 was . . .	3	13	53·7

and the interval, in minutes and decimals of a minute, 25·468.

From these data the angle $X\,c\,a$, equal to the angle $\alpha\,a\,c$, was found to be 2° 7′ 43″·7
And the angle $X\,b\,a=\beta\,a\,b$ . . . . . 1 47 13·3

The side $b\,X$ was computed to be . . . . . 385″·512
The side $c\,X$ was computed to be . . . . . 323·576
And the line $b\,c$, which represents the space travelled over during the interval, was consequently . . . . . 709·088

	min.	sec.
The proportion of the interval of time occupied by the moon in travelling from b to X was computed to be	13	50·8
The proportion in travelling from X to c	11	37·3

The mean rate of motion per minute . . . . . 0 27″·84
The nearest approach of centres, $a\,X$, was found to be. . . 0 12·03

	h	m	sec.
The epoch of No. 22 was . .	2	48	25·6
Add time-interval b to X . .	0	13	50·8
Middle of the totality . . .	3	2	16·4

	h	m	sec.
That of No. 28 was . . .	3	13	53·7
Deduct time-interval X to c .	0	11	37·3
	3	2	16·4

	°	′	″
The position-angle of $a\,b$ at the epoch of 22 was . .	295	28	24
Adding the angle $\beta\,a\,b$ . .	1	47	13·3
We obtain as the direction of motion of the moon's centre during the totality from	297	15	37·3

	°	′	″
The position-angle of $a\,c$ at the epoch of 28 was . .	119	23	21
Deducting the angle $\alpha\,a\,c$.	2	7	43·7
to	117	15	37·3

By combining Nos. 22 and 29, and Nos. 23 and 28, similar numbers were computed, which, together with the preceding results, are given in the following summary:—

	Direction of motion of moon's centre.		Nearest approach of centres.	Relative motion of centres per minute.	Middle of totality.		
					h	m	sec.
Nos. 22 and 28 .	297° 15′ 37″·3	to 117° 15′ 37″·3	12″·03	27″·84	3	2	16·4
Nos. 22 and 29 .	297 8 53·6	to 117 8 53·6	11·27	27·78	3	2	18·4
Nos. 23 and 28 .	297 2 14·1	to 117 2 14·1	13·29	28·02	3	2	20·9
Mean . .	297 8 55	117 8 55	12·20	27·88	3	2	18·6

The following are the numbers computed for the same elements by

					h	m	sec.
Carrington . .	297° 17′ 0	to 117° 17′ 0	13″	27″·27	3	2	19·5
Farley . . .	. . . . .		12·7	27·35	3	2	20·0

Table IV.

In Table IV. column 1, are given the computed sines of half the angles of the opening of the cusps, which are set forth in column 12, Table III.; the sun's mean measured radius, namely, 1904·91, being employed in the computations. The resulting numbers correspond to the semichords given in Table II. column 10; but in most instances the calculated is greater than the measured semichord or sine.

In column 2 are given the cosines of the same angles referred to the sun. These numbers correspond to the distance of the sun's centre from the imaginary chord joining the cusps.

In column 3 is set forth the augmented semidiameter of the moon for the epoch of each photograph, the increase or decrease from the mean measured diameter 2002·2 being derived from Mr. Farley's values.

Calling the sines in column $1 = a$, and the augmented lunar semidiameter $= b$, the cosine of the moon for the angle represented by the same sine was derived by the formula $\sqrt{(b-a)(b+a)}$; column 4 gives these cosines referred to the lunar radius, and represents the distances of the moon's centre from the chord joining the cusps.

The distances of the sun and moon's centres could evidently be derived by taking out the sums or differences of such numbers as those in columns 2 and 4: in the cases actually under consideration, the distances of the moon and sun's centres result from the addition of these quantities; they are given in thousandths of an inch in column 5, and reduced to seconds of arc in column 6.

In column 7 are given the like quantities, derived from measurements of the distance of the peripheries, which are merely a repetition of the values given in Table III. column 6.

In column 8 are set forth the mean distances of the centres of the sun and moon, ascertained by taking the arithmetical mean between the numbers in columns 6 and 7.

In column 9 are the same distances computed by interpolation of the values calculated by Mr. Farley.

Column 10 gives the differences between the mean distances in column 8, and those in column 9, or De La Rue—Farley. The mean of the differences will be seen to be $+4''{\cdot}1$; that is, the distances of the centres of the sun and moon come out greater than the computed distances by $4''{\cdot}1$. This tends to show that the semidiameters of the sun and moon jointly, are less in reality by $4''$ than their tabular values. It is not intended to urge this as an absolute proof, but merely as supporting that view, which is further corroborated by the fact that the first contact occurred later, and the last contact sooner, than the predicted times. The distances of the sun and moon's centres, obtained by calculation from the angular opening of the cusps, will be presently employed to furnish data respecting the commencement and end of the eclipse, &c.; and it will be seen that the times thus obtained differ from those derived from the peripheral distances, and that they approach more nearly to the predicted times. The optical distortion of the sun's image would occur in the direction of a radius, and would not affect the numbers derived from measurements of the angular opening of the cusps, provided the picture were concentric with the optical axis of the instrument; while it would affect the numbers based on the measures of the distances of the sun and moon's peripheries, so that the quantity $4''$ is probably, from that cause, in excess of the true correction. The measurements of the angular openings of the cusps, and the measurements of the distances of the peripheries, both present peculiar difficulties. The difficulty of determining the precise termination of the cusp, especially when blunted by a lunar mountain, leads one to make the angular opening greater than it ought to be, and, consequently, the cosines and the distance of the centres less than they really are. On the other hand, the optical distortion of the image, combined with the irregularities of the peripheries of the sun and moon, tends to make the measurements of the distances of the peripheries, and consequently the distance of the centres, greater than they are in reality. These liabilities to error have, therefore, in the two cases, an opposite effect on the final results; hence a mean of the numbers obtained by the two methods will probably approach very nearly to the correction to be applied to the semidiameters of the sun and moon taken conjointly.

TABLE IV.

No.	1. Sun. Sine of ½ ∠ of opening of the cusps in 1000ths of an inch, = a.	2. Sun. Cosine of ½ ∠ of the opening of the cusps in 1000ths of an inch.	3. Mean radius of the moon, corrected for augmentation of S. D. in 1000ths of an inch, = b.	4. Moon. Cosine of ½ ∠ of the opening of cusps $\sqrt{(b-a)(b+a)}$, in 1000ths of an inch.	5. Distance of the centres of the sun and moon; col. 2 + col. 4 in 1000ths of an inch.	6. Distance of centres of sun and moon, reduced to seconds of arc.	7. Distance of centres brought from column 6, Table III.	8. Distance of centres. Mean of columns 6 and 7. Seconds of arc. *	9. Distance of centres computed for epochs of photographs, by interpolating Mr. Farley's numbers.	10. Measurements minus Farley's numbers.	11. Angle of position of the line joining centres, derived from column 10, Table III. *	12. Angle of position of line of centres N. towards E., computed from Mr. Farley's numbers.	13. Measured angle of position minus Farley's angle of position of line of centres.
7	666·7	1784·9	2005·0	1890·9	3675·8	1823″·1	1827″·8	1825″·4	1824″·2	+1″·2	296° 51′·2	296° 54′·0	− 2′·8
8	872·1	1693·6	2005·9	1805·4	3499·0	1735·4	1734·8	1735·1	1735·4	−0·3	296 57·2	296 53·8	+ 3·4
9	1144·1	1523·1	2004 8	1646·3	3169·4	1572·0	1573·5	1572·7	1570·0	+2·7	296 51·7	296 59·5	− 0·8
10	1172·8	1501·1	2004 8	1626·0	3127·1	1551·0	1556·5	1553·7	1552·7	+1·0	296 52·5	296 52·3	+ 0·2
11	1290·1	1401·1	2004·8	1534·6	2935·7	1456·0	1459·2	1457·6	1455·1	+2·5	296 40·5	296 50·9	− 10·4
12	1401·6	1290·0	2004·6	1433·2	2723·2	1350·6	1352·8	1351·7	1350·6	+1·1	296 39·2	296 48·9	− 9·7
14	1594·4	1042·3	2004·2	1214·4	2256·7	1119·3	1128·6	1123·9	1120·4	+3·5	296 32·5	296 42·4	− 9·9
15	1636·2	975·4	2004·2	1157·4	2132·8	1057·6	1061·6	1059·7	1053·0	+6·7	296 43·0	296 40·0	+ 3·0
16	1715·9	827·2	2004·0	1035·2	1862·4	923·7	930·9	927·3	921·4	+5·9	296 35 0	296 33·8	+ 1·2
17	1783·7	668·7	2003·6	912·6	1581·3	784·3	786·5	785·4	778·7	+6·7	296 33·3	296 25·7	+ 7·6
19	1817·5	570·5	2003·6	843·3	1413·8	701·2	710·6	705·9	698·2	+7·7	296 17·2	296 18·5	− 1·3
20	1861·6	404·0	2003·2	739·8	1143·8	567·3	574·3	570·8	563·0	+7·8	296 13·7	296 0·6	+13·1
21	1892·6	216·1	2002·0	655·8	871·9	432·4	441·6	437·0	432·1	+4·9	296 21·5	295 51·2	+10·3
22	1900·0	136·1	2002·8	633·4	769·5	381·7	385·7	383 7	375·4	+8·3	295 28·4	295 4·1	+24·3
23	.....	.....	.....	.....	.....	.....	307·9	307·9	296·6	.....	294 33·8	294 26·5	+ 7·3
24	.....	.....	.....	.....	.....	.....	196·2	196·2	181·2	.....	292 25·3	.....	.....
27	.....	.....	.....	.....	.....	.....	57·9	57·9	.....	.....	130 14·0	.....	.....
28	1904·8	21·0	2001·2	613·6	634·6	314·7	323 8	319·2	317·9	+1·3	119 33·3	120 10·7	−47·4
29	1896·5	179·1	2001·0	638·2	817·3	405·4	413·7	409·5	409·3	+0·2	118 42·6	119 7·3	−24·7
30	1880·3	305·2	2000·8	683·9	989·1	490·6	498·8	494·7	491·0	+3·7	118 33·8	118 50·3	−16·5
31	1849·2	457·2	2000·6	763·4	1220·6	605·4	613·9	609·6	605·0	+4·6	118 10·2	118 26·5	−16·3
32	1825·0	546·0	2000·4	819·1	1365·1	677·1	687·2	682·1	679·3	+2·8	118 2·3	118 17·3	−15·0
33	1735·3	765·8	2000·0	904·3	1760·1	882·9	886·8	884·8	881·6	+3·2	117 52·5	117 59·4	− 6·9
35	1679·7	898·5	1999·6	1084·9	1983·4	983·7	983·3	983·5	975·7	+7·8	117 41·9	117 53·7	−11·8
36	1608·3	1020·7	1999·4	1187·8	2208·5	1095·4	1096·0	1095·7	1089·7	+6·0	117 40·5	117 48·2	− 8·7
37	1544·7	1100·7	1999·2	1269·1	2369·8	1175·4	1168·1	1171·7	1165·5	+6·2	117 37·6	117 44·4	− 6·8
39	1346·9	1347·0	1999·0	1477·1	2824·1	1400·7	1396·0	1398·3	1395·0	+3·3	117 30·2	117 35·4	− 5·2
41	1088·2	1563·5	1998·4	1676·1	3239·6	1606·8	1602·4	1604·6	1601·8	+2·8	117 32·6	117 28·0	+ 4·6
42	902·7	1677·4	1998·2	1782·7	3460·1	1716·1	1715·9	1716·0	1712·3	+3·7	117 16·2	117 24·3	− 8·1
43	704·8	1769·7	1998·0	1869·6	3639·3	1805·0	1805·8	1805·4	1801·3	+4·1	117 8·9	117 21·4	−12·5
										Mean +4·1			

* Plate XVIII. exhibits a graphic representation of the path of the moon's centre, set off, in accordance with the numbers of columns 8 and 11, by means of an instrument constructed specially for such operations.

Proceeding as before, but with the distances of the centres given in Table IV., column 6, we obtain, as the rate of approach of the centres per minute between 7 and 9, 24″·945; and the rate of retreat of the centres per minute, between 41 and 43, 29″·255, whence we derive the following:

First contact.	h	m	sec.
No. 7 . .	1	47	51·6
No. 8 . .	1	47	52·7
No. 9 . .	1	47	51·6
Mean . .	1	47	52

Last contact.	h	m	sec.
No. 41 . .	4	10	12·4
No. 42 . .	4	10	13·8
No. 43 . .	4	10	12·4
	4	10	12·9

	h	m	sec.
Duration of the Eclipse . . .	2	22	20·9.

These numbers agree fairly with the calculations; for example—

	First contact. h	m	sec.	Last contact. h	m	sec.	Duration. h	m	sec.
De La Rue	1	47	52	4	10	12·9	2	22	20·9
Carrington	1	47	56	4	10	15·2	2	22	19·2
Farley	1	47	57	4	10	15·0	2	22	18

	Direction of motion of the moon's centre during totality.	Nearest approach of centres.	Relative motion of the centres per minute.	Middle of eclipse. h	m	sec.
Nos. 22 and 29	297° 8′ 25″ to 117° 8′ 25″	11″·10	27″·34	3	2	22·6
Nos. 22 and 28	297 14 34·1 to 117 14 34·1	11·79	27·33	3	2	23·2
Mean . .	297 11 29 to 117 11 29	11·44	27·34	3	2	22·9

The following are numbers computed for the same elements by

				h	m	sec.
Carrington . .	297° 17′ 0″ to 117° 17′ 0″	13″·0	27″·27	3	2	19·5
Farley . . .	. . . . .	12·7	27·35	3	2	20

By combining these results with the results of the peripheral measures already given, we obtain

First contact.	h	m	sec.
By peripheries . .	1	48	3·4
By cusps	1	47	52·0
Mean . . .	1	47	57·7

Last contact.	h	m	sec.
	4	10	3·6
	4	10	12·9
	4	10	8·2

	h	m	sec.
Duration of the Eclipse . . .	2	22	10·5.

	Direction of motion.	Nearest approach of centres.	Relative rate of motion of centres.	Middle of totality. h	m	sec.
Peripheries .	297° 8′·9 to 117° 8′·9	12″·20	27″·88	3	2	18·6
Cusps . . .	297 11·5 to 117 11·5	11·44	27·34	3	2	22·9
Mean .	297 10 to 117 10	11·8	27·61	3	2	20·7

These numbers agree very closely with the theoretical numbers, the chief difference being in the epoch of the end of the eclipse, which is earlier by 7 seconds than Mr. Carrington's computation, and by 6·8 seconds than that of Mr. Farley.

The following are the differences:—

	De La Rue—Carrington.	De La Rue—Farley.
	sec.	sec.
First contact	+1·7	+0·7
Last contact	−7·0	−6·8
Duration	−8·7	−7·5
Middle of eclipse	+1·2	+0·7
Nearest approach of centres	−1·2	−0·9
Direction of motion of the moon's centre during the totality	−7′	
Relative motion of the centres per minute	+0″·34	+0″·26
Position-angle of the line joining the centres at the first contact	+2′	+2′
Ditto at the last contact	−8′	−6′

The periods of first contact, and the middle of the eclipse, are accordant, but not so that of the end of the eclipse, the duration being less than the computed duration by 8 seconds. The half of this, or 4 seconds, would correspond to a distance moved through of 1″·9, by which quantity the radii of the moon and of the sun jointly would be smaller than the computed values. Without any desire to attach more importance to the results of the photographic measurements than they merit, I believe that I have made out satisfactorily that astronomical photography is capable of furnishing data on which great reliance can be placed, and which it would be difficult to collect in any other way. It possesses the advantage, in the case of sun-pictures, of instantaneous registration, and permits of measurements being made calmly and at leisure, and of their being repeated as often as may be considered desirable. Its employment in connexion with means of measurement will undoubtedly suggest future improvements; and although it is impossible at present to predict the destiny of astronomical photography, it appears likely that it will take a high rank among the methods of observation.

Solar Spots.

With the view of ascertaining whether any connexion exists between the luminous prominences and the faculæ, or the spots on the solar disk, photographs of the sun were obtained as soon as the heliograph could be got to work, and others would have been taken on each day previous to the eclipse if the weather had proved favourable. Several photographs were secured on the 14th, but only one on the 16th, which was not measured, having been overlooked; spots *a* and *b* were then well on the solar disk. Afterwards, until the 18th, none could be procured, but on the 19th and 20th several were obtained. With the exception of a spot *d*, which became visible on the 20th, with

a position-angle 86° 26′*, a group of spots, extending on the 18th from 114° 30′ to 121° 30′†, and a double spot X visible on the 14th in position-angle 243° 53′ to 245° 34′‡, there were none between which and the luminous prominences any connexion could be presumed to exist. The group 114° 30′ to 121° 30′ was surrounded by many faculæ; the spots in it underwent considerable changes on the 19th and 20th; the faculæ extended evidently beyond the visible portion of the sun's surface on the 18th; for a part which was not in sight on the 18th came into view on the 19th and 20th. Just in the neighbourhood of these faculæ there was visible in the telescope during the totality, a very brilliant sheet of light. On the 18th, besides the group of spots surrounded by faculæ just mentioned, and other small spots delineated in the index map, Plate XV., there were three conspicuous spots, which I have designated by the letters *a*, *b*, *c*. Spot *c* was visible on the 14th, but the two others had not yet come round; the three spots *a*, *b*, *c* continued to be visible on the 19th and 20th.

On the whole, however, no very intimate relation was discoverable between the prominences and the sun-spots; and recent photographic researches having convinced me that the formation of spots and their changes are among the least frequent of the great disturbances always occurring in the solar photosphere, I take this opportunity of stating my opinion that future investigations will rather tend to disprove any very close connexion between them.

On the 14th, at $4^h\ 12^m\ 6^s{\cdot}6$ Greenwich mean time, the spots visible on the sun's disk were the following:—

Spot.		Position-angles.	Distance from the centre in a decimal of the radius.
	c	45° 37′	·3825
Cluster .	α	112 21	·5548
	β	113 6	·5169
	γ	114 25	·4984
	δ	114 35	·5811
	ε	114 51	·6075
X first nucleus . . .		243 53	·8899
X second „ . . .		245 34	·9078

The spots α, β, γ, δ, and ε, somewhat changed, were still on the disk on the 18th, but I did not notice any spot which could have been brought by rotation into proximity with the western limb of the sun, with the exception of X, which was at some little distance in longitude on the hemisphere turned away from the earth.

Of the group of small spots surrounded by faculæ, visible on the eastern edge of the sun on the 18th, the following were selected and measured on photograph No. 6, whose epoch is $1^h\ 47^m\ 43^s{\cdot}6$.

* See in the index map, Plate XV., the prominences E and F.

† See in the index map, Plate XV., the prominences H and G.

‡ See in the index map, Plate XV., the prominence L, which, however, was at some distance from the position of X.

Spot.	Position-angle.	Distance from the centre in a decimal of the radius.
ζ	113° 33′	·9979
η	114 40	·9953
θ	116 31	·9643
ι	117 37	·9984

In consequence of the partial breaking up of the spots on the 19th and 20th, it was not easy to identify them, and the Greek letters may possibly not refer in all cases to the same spot. The following, selected from many other small spots surrounded by faculæ, were measured on a photograph taken on the 19th at $0^h\ 9^m\ 51^s$ Greenwich mean time:—

Spot.	Position-angle.	Distance from the centre in a decimal of the radius.
ζ	117° 26′	·9518
θ	119 45	·8872
ι	120 19	·9607
$\varkappa$	124 48	·9166

On the 20th, civil reckoning, or astronomical reckoning 19th day $23^h\ 46^m\ 45^s$, the photograph K was taken, when a fresh spot d had made its appearance on the eastern limb; the following are the results of the measurements of this spot, and of some others in the group surrounded by faculæ. All these spots had altered greatly since the previous day.

Spot.	Position-angle.	Distance from the centre in a decimal of the radius.
d	86° 26′	·9961
ζ	121 3	·8713
ι	123 53	·8896
θ	126 10	·7733
$\varkappa$	130 52	·8231

Table V.

Table V. contains the results of the measurements of the principal spots (a, b, c), the angles of position being corrected for the errors of the wires, and the distances given in a decimal of the radius of the sun. In the case of each spot, the particular edge measured is indicated on Plate XV. by a dotted line, and by means of Table V. my results may be reduced to those of other observers who may have measured a different part of the same spot.

Column 1 gives the number of the photograph; column 2 the date of the photograph; columns 3, 8, and 13 the distances of the spots a, b, c from the sun's centre in a decimal of the radius; columns 4, 9, 14 the averages of several measures; columns 5, 10, and 15 the position-angles; columns 6, 11, 16 the average position-angle for several photographs; and lastly, columns 7, 12, and 17 the mean epochs of the means of the measures.

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TABLE V.

1.	2.				3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.
No.	Greenwich mean time.				Spot *a*. Distance from the sun's centre in decimal of radius.		Spot *a*. Position-angle corrected for error of the wire.	Spot *a*. Mean.	Spot *a*. Mean epoch.	Spot *b*. Distance from the sun's centre in decimal of radius.		Spot *b*. Position-angle corrected for error of the wire.	Spot *b*. Mean.	Spot *b*. Mean epoch.	Spot *c*. Distance from the sun's centre in decimal of radius.		Spot *c*. Position-angle corrected for error of the wire.	Spot *c*. Mean.	Spot *c*. Mean epoch.
	d	h	m	sec.			° ′	° ′	h m			° ′	° ′	h m			$\overset{\circ}{4}\overset{\prime}{3}\overset{''}{7}$	° ′	h m
	14	4	12	6·6	Not on disk.	...	.....	...	.....	.....	...	.....	.....	.....	·3825	...			
6	18	1	47	43·6	·6873		74 35·8			·6773		129 1·5			·6064		301 5·5		
7			52	31·1	·6877		75 6·5			·6798		129 16·8			·6070		300 54·3		
8			56	3·1	Defective.		74 48·3			·6764		129 16·0			·6019	·6053	301 3·0	301 0	1 54·72
9		2	2	35·1	Defective.	·6867	.....	75 00	1 57·33	·6759	·6780	129 5·0	129 12	1 58·21	·6061		300 58·0		
10			3	16·1	·6868		75 8·2			·6784		129 14·3							
11			7	6·1	·6853		75 21·3			·6801		129 18·3							
12			11	11·1	·6832		75 21·0			·6764		129 23·5							
14			20	7·6	·6837	·6852	75 23·2	75 16	2 20·46	·6748	·6758	129 25·2	129 23	2 20·46					
15			22	44·1	·6867		75 3·0			·6762		129 29·0							
16			27	47·1	·6871		75 17·5			·6759		129 14·5							
17			33	13·6	·6809		75 20·6			·6752		129 41·1							
19			36	17·1	·6817		75 16·6			·6748		129 42·6							
20			41	23·1	·6825	·6821	75 0·0	75 13	2 39·31	·6730	·6737	129 31·5	129 34	2 41·13					
21			46	18·6	·6833		75 16·6			·6723		129 33·6							
22			48	25·6	.....	...	.....	...	.....	·6730		129 22·7							
29		3	17	11·7	.....	...	.....	...	.....	.....	...	.....	.....	.....	·6172		300 48·3		
30			20	7·7	.....	...	.....	...	.....	.....	...	.....	.....	.....	·6170	·6175	300 22·7	300 34	3 22·10
31			24	13·2	.....	...	.....	...	.....	.....	...	.....	.....	.....	·6179		300 28·6		
32			26	51·7	.....	...	.....	...	.....	.....	...	.....	.....	.....	·6179		300 35·8		
33			34	1·7	.....	...	.....	...	.....	.....	...	.....	.....	.....	·6198		300 25·4		
35			37	19·7	.....	...	.....	...	.....	.....	...	.....	.....	.....	·6163		300 28·4		
36			41	18·7	.....	...	.....	...	.....	.....	...	.....	.....	.....	·6187	·6189	300 30·6	300 29	3 39·16
37			43	57·7	.....	...	.....	...	.....	.....	...	.....	.....	.....	·6207		300 31·4		
39			51	53·7	.....	...	.....	...	.....	.....	...	.....	.....	.....	·6188		300 23·8		
41			58	58·3	Too faint.	...	.....	...	.....	·6611		130 4·9	.....	.....	·6260	·6219	300 24·9	300 29	3 57·87
42		4	2	43·8	·6735		75 6·0			·6676		130 2·0			·6209		300 39·0		
43			5	44·8	·6734	·6729	74 56·0	74 53	4 8·82	·6676	·6635	130 10·8	130 10	4 6·85	·6261		300 33·9		
44			10	33·3	·6743		74 41·3			·6619		130 6·3			·6241	·6238	300 53·3	300 33	4 10·86
45			16	15·8	·6684		74 49·5			·6595		130 27·0			·6212		300 13·0		
46	19	0	9	51	·5312	...	70 48·8	...	.....	·5454	...	140 18·0	.....	.....	·7483		296 53·0		
K		23	46	40	·3581	...	59 31·0	...	.....	·4294	...	161 26·0	.....	.....	·8713		295 41·0		

July 18, $2^h\ 2^m\ 35^s$ Greenwich mean time, the moon had occulted spot *c*, at a distance of ·6061 of the sun's radius from the sun's centre, and between the position-angles 300° 58′ and 302° 33′, reckoning from the respective edges of the spot.

At $3^h\ 11^m\ 11^s{\cdot}7$ the spot *c* was partly uncovered by the moon's edge at a distance of ·6172, and between the angles 300° 48′ and 302° 45′.

At $3^h\ 58^m\ 58^s{\cdot}3$ the moon's limb had partially passed off spot *b*, at a distance of ·6611, between the angles 128° 3′ and 130° 5′.

I am not aware that any practical use has ever been made of the occultation of a sun-spot by the moon during an eclipse; but as it is probable that during the eclipse under discussion such phenomena were recorded by a great number of observers, as on the occasion of previous eclipses, I have thought it desirable to make the measurements given in Table V., which will, I believe, afford better means than have been before available for turning such observations to account.

Photographs of the Totality.

Copies of the two totality-pictures which accompanied this paper were produced in the following way: the original negatives were placed in the focus of an enlarging-camera, and positive collodion copies on glass procured, on which the lunar disk was enlarged to 9 inches in diameter. These positive copies were then placed in the focus of the camera, and a number of negatives made, to print any impressions that might be required, in which the lunar disk was reduced to the sizes of the several engravings accompanying the paper. The photographic copies therefore are two removes from the original, and, something being lost at each operation, they do not present all the details visible in prints taken direct from the original negatives. The corona, for example, which is depicted on the original negatives, is to a great extent lost in the copies, because in bringing clearly out the details of the prominences, the corona in most cases becomes over-printed.

A few positive 9-inch copies on glass have been presented to Observatories and public Societies; but it was not possible to do this very extensively, in consequence of the extreme difficulty of copying, occasioned by the density of the original negatives. They could only be procured on days when the sun was perfectly unobscured by haze or cloud, and ultimately the injury to the second original negative prevented my continuing the work sufficiently long to obtain a supply as great as I desired. Secondary copies can, however, still be procured.

In Plate IX. are given mezzotint fac-similes of the two totality-pictures the size of the originals; although they will serve to give a general idea of the photographs, and to illustrate what has to be said respecting them, they are, after all, but imperfect substitutes for the photographs themselves. Copies of the photographs would have been inserted in this memoir, had past experience of the permanence of such pictures warranted the Council in doing so*.

* In the Author's copies, Plate IX. *a*, photographic copies from the originals are given. They are two removes from the originals.

In order that the phenomena presented by the photographs may be clearly understood, I propose to give an explanation of certain appearances in them, which might otherwise occasion some difficulty. First, with regard to No. 25 photograph (the first totality-picture) (Plate IX. fig. 1). The sensitive plate was in the heliograph and the slide which covered it removed, a minute or so before totality, a temporary screen being first held just before the object-glass, so as to stop off all light. Thus every thing was in readiness, and it was only necessary to remove the temporary screen, to expose the plate at the proper moment. The very instant of the disappearance of the sun, I gave the signal for its removal, which was immediately done, and Mr. BECKLEY, who was watching the chronometer, gave the signal for covering the object-glass exactly one minute after it had been uncovered. I had given instructions that no attempt was to be made to note the precise epoch of total obscuration; for each operator had too much to occupy his attention to admit of any work being done which was not absolutely essential to the photographic operations. In order to regulate the time of exposure, the precise position of the second-hand of the chronometer was noticed when the plate was first exposed, and the signal was given for replacing the screen when the second-hand had completed a revolution.

The telescope followed the motion of the sun so well that the prominences retained a perfectly fixed position on the sensitive plate; and from the results on the second plate, presently to be spoken of, it is known that they must have depicted themselves to some extent, though very faintly, in a second. On the other hand, the comparatively feeble corona would have required even a longer period to thoroughly imprint itself than the whole time allotted for the first picture. Consequently, as the moon moved from the west to the east, she kept shutting off the prominences and corona on the east, thus stopping further action; while, on the west, she permitted fresh portions of the corona to commence a new action. The luminous prominences, when once they had produced their effect, could not be obliterated, although they might be subsequently covered by the moon; for it is well known in photography that latent images remain on the plate for a long time, and become apparent on applying the developing fluid. In the case of the corona, the full effect not having been produced even at the end of the operation, it will be evident that its picture was necessarily the most intense on the eastern side, just at that part where the moon's periphery had arrived at the close of the work; while it is clear that on the western side the action would continuously follow up the moon's progress, and that therefore an impression gradually becoming fainter towards the lunar disk, would indicate the point which the moon had reached when the image of the eclipse was shut off. With this explanation, the appearance of the moon's edge beyond the prominences on the eastern side in the first totality-picture can present no difficulty. If, during the exposure of the plate, fresh prominences had become uncovered on the western side, they would have imprinted themselves; and if the plate had remained in the heliograph during the entire period of totality, the whole of the prominences would necessarily have recorded themselves on a

single plate, although only a part had been visible at one time, and the plate would have shown very nearly the position of the moon at the conclusion of the operation. Unless an instantaneous picture of the phenomena of totality could be procured*, as in the case of the other phases, no photograph would show the precise state of matters at any one moment; consequently, if it be desired to know what was the condition of things at any one instant of the period during which the plate was in the heliograph, for example, at the commencement of the totality, and a minute afterwards, recourse must be had to the expedient of completing the circle of the lunar disk for the position she occupied at these two epochs respectively. The photograph itself affords the necessary data for effecting this; for it will be found that the disk of the moon, as depicted on the photograph 25, represented in Plate IX. fig. 1, is not quite a complete circle, and that the longest diameter is in a direction at right angles to prominence A (see Plate XV.).

By measuring the diameter in the direction of prominence A with a divided beam compass, and taking the half of the quantity as a radius, it was a matter of no great difficulty to find the centre of the picture for the two epochs in question, namely, near the commencement and near the end of the first minute, as indicated by the photograph, and to draw in the lunar disk from either centre. In this way two photographs, α and β, 7 inches in diameter, were corrected, and served as originals for Plates X. and XI., which show the state of the phenomena as accurately as if two instantaneous pictures had been taken.

Plate X., which is the copy of α, represents the appearance of the phenomena of totality at the commencement, and Plate XI. the copy of β nearly at the end of the first minute. A line drawn to the centres, laid down for the two epochs, was found to correspond absolutely with the direction ascertained independently to be that of the motion of the moon's centre, and measured $23''$. Allowing a period of five seconds for the production of a picture sufficiently intense to show itself clearly on the plate, both at the commencement and at the end of the exposure, the period traceable would be fifty seconds, and $\frac{23'' \times 60}{50} = 27''{\cdot}6$ would be the motion of the moon's centre during a minute, a result not differing by more than a few tenths of a second of arc from the mean derived from the measures of the other phases of the eclipse.

In Plate IX. fig. 1, representing the untouched photograph No. 25, a portion of the prominence R is visible; but it came into view after the commencement of totality, and was therefore painted out in completing the lunar disk in the touched photograph α, represented in Plate X.; at the epoch shown in photograph β, represented in Plate XI., however, the lunar disk had revealed so much of prominence R as is seen in the original picture.

The dark lines in the original photograph, represented in Plate IX. fig. 1, situated respectively above the floating cloud C, and across the broad part of G, are the images in shadow of the position-wires. In the original negatives and positive copies taken

* The possibility of doing this during future total eclipses will be presently pointed out.

carefully from them, the continuations of the images of these wires are depicted in positions diametrically opposite. The wires are not central with the picture of the lunar disk, which was adjusted by means of the finder to be as much on the right of the plate as possible, in order that the details on the eastern limb might not be lost. It will be perceived that the light was inflected sufficiently during the taking of the picture to complete the image of the protuberance G, notwithstanding the intervening wire.

About eighty seconds were required for covering and taking out photograph 25, placing plate No. 26 in the heliograph, drawing back the slide which covered it, allowing time for the vibrations imparted to the instrument to cease, and removing the temporary cover from the telescope; so that the exposure of plate No. 26 commenced about two minutes and twenty seconds after the commencement of total obscuration, and continued until within a second of the reappearance of the sun, having been, like plate No. 25, exposed as nearly as possible one minute to the actinic influence of the prominences. The action of the prominences was, however, on account of the accidental disturbance of the heliograph, not allowed to continue on one part of the plate during the whole time; and hence their impressions are not so strongly depicted as those of the prominences on photograph No. 25, represented in Plate IX. fig. 2.

On this account the appearance of photograph No. 26 was not easy at first to comprehend; and it gave me considerable difficulty for some time to make out with precision the true nature of the result obtained. A gust of wind had arisen close upon the time this picture was being taken, and I was induced to imagine that the telescope had so been caused to vibrate; but further examination of the picture showed that this could not have been the case, for three very distinct images were imprinted of the prominence E (the boomerang), proving that after each disturbance the instrument followed the sun's movement correctly. This afforded a clue to what had actually occurred; and I found on inquiry that two of my assistants had looked at the eclipse through the finder of the heliograph, and had, as it appeared, inadvertently disturbed it in right ascension, which the wear of the worm-wheel and tangent-screw permitted them to do. Fortunately a firm radius-bar, which I had had made previous to leaving England, held the telescope so firmly in declination that it could not be readily moved in that direction; if it had been so moved, the resulting picture might have defied interpretation, and have rendered the photograph useless for exact measurements.

The clue once obtained, it was easy to make out three impressions of the wheatsheaf A (Plate XV.), three of the floating cloud D, three of each of the three points h', h'', h''' of the fallen tree H, and less easily three of the mountain-peak R because they overlapped. There were produced only two impressions of prominence Q on the western side, because at the epoch of the first action, when the point I was as yet partly visible, the moon still covered Q. The next impression was formed just when a part of Q had become visible, and when also part of the spade L had been revealed. The prominence L appears to have been sufficiently brilliant to imprint itself continuously while its image was traversing the plate. In the last impression on the plate, the prominences and corona continued

their action undisturbed during the remainder of the time of exposure: this picture is the one which includes those images of the several prominences depicted furthest to the right in each case, Plate IX. fig. 2. During the exposure of the plate in its third period there was a slight irregularity in the motion of the driving-apparatus, which to a very small extent enlarged the prominences in the direction of right ascension.

In making a representation of the state of the phenomena at the end of totality, it was only necessary to paint out the two impressions of each of the several prominences belonging to the two other periods depicted on the photograph, and to correct the slight exaggeration caused by the irregularity of the driving-apparatus. In this way was produced the touched photograph γ, represented in Plate XII., which faithfully shows the state of matters about a second, or less, before the reappearance of the sun. It has been possible to make out, from photograph 26, three corrected pictures, showing the appearance of the prominences at three different epochs of that period of totality during which it was in the heliograph; but only one of the resulting pictures has been engraved, namely, that shown in Plate XII.

The irregularity of many of the protuberances on the concave side adjacent to the lunar disk is very striking, and appeared to me, while observing the eclipse with the achromatic, to be greater than could be attributable to the indentation which would be caused by any amount of irregularity on the lunar periphery. The extent of this irregularity could be readily estimated during the other phases of the eclipse with the telescope, and is also depicted clearly on the several photographs, which afford a permanent record of the moon's profile. In Plate XVI., which was produced by etching an enlarged positive copy of photograph 22 and electrotyping from it, is shown the profile of the moon's limb between the position-angles 44°·5 and 191°; in Plate XVII., produced in a like manner from photograph 28, the moon's profile is depicted between the position-angles 228° and 9°; altogether the plates exhibit 287° of the moon's outline, with which the concave edge of the luminous prominences shown in Plates XIII. and XIV. may be compared. As the moon moved onwards, the great amount of indentation of the concave side of the protuberances appeared to me to become less on the eastern side and greater on the western side. Some of the irregularity on the concave side is undoubtedly due to the periphery of the lunar disk, but all of it cannot be so accounted for. We may assume that some prominences are not in absolute contact with the sun's photosphere, but, on the contrary, are supported at a distance from it, as in the case of the floating cloud D. Notably, in that part of the prominence G between the position-angles 112° and 124° the irregularity of the concave boundary cannot be accounted for by the form of the moon's limb; on the other hand, in support of the position that in certain cases the irregularity must be due to the profile of the moon's disk, we have good evidence in the second totality-photograph, Plate IX. fig. 2, where part of the luminous prominence Q is depicted at two epochs, namely, just as it became visible, and at the end of totality. It will be seen that the part q'—q'' has the same amount of indentation at both epochs. In most cases the irregularity of the contour of the prominences

appears to be much greater than that portion of the moon's limb corresponding in position-angle with it.

On comparing the results of the expedition of 1860 with those obtained in 1851, one cannot fail to be impressed with the general similarity in the aspect of the prominences at the two epochs: on both occasions were seen luminous masses of vast extent, perfectly detached from the sun, and far beyond the lunar disk; the same irregularity of outline on the convex side, running out into points; the same apparent outpouring of faint vapours, falling as it were towards the sun (as in the faint portions of prominence A); lastly, although not seen with the eye in 1860, there is recorded by the photographic retina a similar hooked projection to that seen in 1851, and named by the Astronomer Royal "the boomerang," from its approximation in form to the Australian weapon bearing that name.

Reference to Plate XV. affords better means of judging of the dimensions of the protuberances depicted on the two photographs than the photographs themselves, on account of the latter merely showing their distance from the moon's periphery, whereas in the index map, Plate XV., the distances from the sun's disk are shown. We thus see that while the two photographs give a fair representation of the height above the sun's periphery of the prominences D, E, F, G, M, N, O, P, Q, R, they do not do so in respect of the remainder. Notably, the prominence A, if it extended inwards to the sun's periphery, and was not, like the floating cloud D, supported at some distance from it, must have been of twice the height revealed to us; and again, the prominence K must, as regards the brighter part of it, have been nearly four times the height seen above the moon's limb. The brighter part of the prominence K, at the epoch of the second totality-picture, was covered by the moon; and although the fainter hooked part projected beyond the lunar disk, it was too faint to imprint itself in the second picture, on account of the disturbance of the telescope, which did not allow it to remain sufficiently long on one part of the plate. In the first picture it depicted itself clearly, and, if it had done so in the second, would have admitted, in connexion with A, of a chord being drawn which would have afforded a capital basis for measurements. The actinic power of "the boomerang" is quite remarkable; for it imprinted itself, distinctly, three times on photograph 26, although it was invisible to the eye. Not only, therefore, is photography of value in recording phenomena visible to the human eye, but it is also able to render evident bodies which emit only those rays which belong to the invisible part of the spectrum.

The question whether the luminous prominences would appear as bright or dark markings on the sun's disk admits of a probable solution by means of photography, which furnishes data as to the degree of luminosity of the prominences relatively to that of the sun's photosphere. It will be recollected that I have stated that one of the prominences (A) was visible some minutes previous to totality, and that it continued visible to other observers several minutes after the reappearance of the sun. It so happened that photograph No. 28 was taken about twenty seconds after the reappearance of the sun,

when the distance between the moon and the sun's peripheries was only 9″·6. This photograph was obtained with the full aperture (3·4 inches) of the heliograph, and the plate was exposed by removing a temporary cover which had been placed before the object-glass, and replacing it as quickly as possible. The time of exposure would certainly not exceed a second, yet the image is completely solarized (bleached) from over-exposure; moreover, the wind, which rose suddenly at that period, violently shook the heliograph in the direction of right ascension, by successive impulsions against the object-end, which projected beyond the walls of the observatory; and many impressions of the solar crescent are consequently depicted on the plate, on which, however, not the slightest trace of prominence A could be made out. With the aperture of the object-glass reduced to 2 inches in diameter, and using the instantaneous apparatus, a picture of partial phase could under similar circumstances have been procured in $\frac{1}{10}$th of a second, and therefore in less than $\frac{1}{5}$th of a second with the full aperture of the telescope. Moreover, as in the second totality-picture (No. 26) the prominences were depicted three times, in consequence of two disturbances of the telescope in right ascension, during the minute the plate was exposed in the heliograph, we know that on the average twenty seconds are about sufficient to bring out the picture of the luminous prominences strongly. These triplicate images are not, however, of equal intensity, one being very faint; and therefore, assigning to this latter (what its appearance warrants) an exposure of half the time of the other two, we have twelve seconds as the time required to depict the most luminous of the prominences fairly. It results, therefore, that the light of the luminous prominences is fully $58 \times 12 = 696$ times less bright than that of the photosphere of the sun.

On August 12, 1862, I succeeded, as I have already stated in the foot-note, page 334, in obtaining an *extremely* faint impression of the moon with the Kew heliograph in three minutes. The full aperture of the object-glass was employed, and the chemicals used were in the highest degree of sensitiveness. An impression of the luminous prominences of equal intensity would, according to data furnished by the second totality-picture No. 26, have been produced in a second. It may therefore be safely estimated that the image of the luminous prominences (for equal areas) is 180 times more brilliant than that of the moon. Assuming for the ratio of the light of the sun in comparison with the light of the moon 200,000 to 1, it would follow that the image of the luminous prominences is $\frac{200,000}{180} = 1111$ times less brilliant than that of the sun; taking the mean of the two estimates it would be 900 times less brilliant than that of the sun.

Although in all probability the prominences are less bright than the dark nuclei of the solar spots, it does not follow that they would appear as very dark markings on the sun's disk, for to a great extent they may permit of the transmission of the light emitted by the photosphere; and, besides, it is by no means probable that there is any intimate connexion between the solar spots and the prominences, for the vast extent of the sun's limb which is surrounded by the prominences precludes such an idea, and leads to the

conviction that they are far more generally distributed on the solar disk, and of proportions greatly exceeding any which the spots ever attain to.

Since the prominences would appear to be scattered so widely over the sun's surface, the question has arisen whether it would be possible to render these wonderful appendages apparent at other periods than those of total eclipses of the sun. For the purpose of solving this problem, Mr. James Nasmyth devised an apparatus which, in part, consisted of a cylindrical box, blackened inside, having in one end an aperture of such dimensions that it exactly permitted of the passage of the sun's image when projected by a telescope, whilst the surface surrounding the aperture was sufficiently large to receive the images of all objects situated beyond the solar periphery.

The Astronomer Royal also has made experiments with the same view, using, in part, the Nasmyth apparatus; but the existence of the luminous prominences could not be detected by its means, in all probability on account of the great amount of illumination of that part of our atmosphere which is in apparent contiguity with the sun. On the occasion of Professor Piazzi Smyth's experimental visit to the Peak of Teneriffe he took out with him this apparatus, because it was thought that the more attenuated stratum of atmosphere at that elevation would interfere less with the success of the experiment. Only negative results were, however, obtained, and the problem remains to be solved.

Is it probable that photography may lead to a solution of the difficulty? I am inclined to think that it may possibly do so. It would, however, be quite futile to attempt to delineate the luminous prominences, when beyond the sun's periphery, by means of photography, after the experience afforded by the experiments before cited; for most unquestionably they would not produce an image so intense as that of our own atmosphere in apparent contiguity with the sun's disk and illuminated by his rays. My hope is that their forms may be depicted on the brighter solar disk itself, and their existence rendered evident by means of the stereoscope, which has already enabled me to make out the real nature of the radiating lines on the lunar surface.

During the year 1861, by means of my 13-inch equatorial reflector, I succeeded in procuring photographs of the sun's surface, on a scale of 3 feet for the sun's diameter. These colossal photographs were obtained by enlarging the focal image by means of a secondary magnifier, constructed especially to ensure a flat field and the coincidence of the visual and chemical foci. They show, in a remarkably striking manner, the mottling of the sun's photosphere, which appears to be entirely composed of an undulating mass of waves, like the surface of the sea agitated by wind.

Two pictures of the same sun-spot, taken at an interval sufficiently great to admit of the sun's rotation causing the necessary angular shift of its position, evidently possess the stereoscopic relation. By placing them in the stereoscope in such a way that the positions of the two pictures, relatively to each other, shall be reversed, that last taken being placed on the left, that first taken to the right (supposing the image to be erect), I have obtained a stereoscopic picture of a sun-spot, and some surrounding faculæ, which represented the various parts of the picture in their true relative positions in

regard to altitude, and in other respects. I have ascertained in this way that the faculæ occupy the highest positions of the sun's photosphere, the spots appearing like holes in the penumbræ, which appeared lower than the brighter regions surrounding them; in one case parts of the faculæ were discovered to be sailing over a spot, apparently at some considerable height above it.

My hope of rendering evident the luminous prominences is dependent upon an extension of this experiment. I believe that, with a careful adjustment of the time of exposure of the sensitive plate, I shall succeed in obtaining the outline of the luminous prominences (the so-called red flames) as very delicate dark markings on the more brilliant mottled background of the photosphere. These delineations, except with the aid of the stereoscope, would be confounded with the other markings of the sun's surface, but they would assume their true aspect, and stand out from the rest, as soon as two suitable pictures were viewed by the aid of that instrument.

The difficulties in the way of doing this are, however, of a special kind, as will readily be seen from the following considerations: when the aperture of the instantaneous slide is at a maximum, and the rapidity of motion at a minimum, a picture of the sun will result which will be of one uniform maximum density, without the slightest trace of any marking even of a dark sun-spot. As the aperture is reduced and the velocity of the slide augmented the spots will become depicted, but no trace of the penumbræ will be seen; then we shall get the penumbræ, and subsequently traces of the faculæ and of the general mottling of the sun's disk; lastly, by still further reducing the aperture, the faculæ and the mottling will be well brought out, but especially the latter. The photographic process, it will be recollected, is one of progressive action, and even the faintest parts of the picture may, by a long exposure, produce as much intensity as the bright parts do by a shorter action; and evidently, with sufficient time, all distinctions of bright and less bright must cease to exist on the photographic plate, if all parts have produced the maximum density of effect which the plate is capable of affording. It rarely (it might be said never) happens that all parts of the picture are portrayed with the best effect; and in heliography the apparatus has to be variously adjusted, according as the spots or the mottlings of the sun's surface are required to be especially well shown.

Measurements of the Totality-Photographs.

The main object of the observations of the total eclipse of 1860 was to ascertain whether the luminous prominences are objective phenomena belonging to the sun, or whether they are merely subsidiary phenomena, produced by some action of the moon's edge on light emanating originally from the sun. If the luminous prominences are attached to the sun, it is evident that they would continually change their positions with respect to the moon's centre as the moon moved across the solar disk. For example, a prominence situated in the direction of the moon's path would maintain its position-angle in reference to the moon's centre unchanged, but it would be gradually and at

length entirely covered by the moon. On the other hand, a prominence situated at right angles to the moon's path would change its position-angle in respect of the moon's centre, but would remain uncovered to almost the same extent at the end of totality as at the commencement. Moreover, prominences situated in positions intermediate between these two would change their respective position-angles, referred to the moon's centre, less, but would be covered more, in the proportion of their relative degrees of proximity to the line of the moon's path; and *vice versâ*, the change in position-angle of the several prominences would be greater in proportion as they were situated nearer to a line at right angles to the moon's path*. No stronger evidence that the prominences belong to the sun can be adduced, than that of a change in the angular position of a prominence in reference to the moon's centre, because it is not probable that *different* parts of the moon's periphery would produce precisely the same effect on light emanating from the sun. The relative changes of position may be calculated for any given locality; for which purpose it is necessary to know either its geographical position, or to determine the relative motions of the sun and moon during an eclipse by other means. In a former part of this paper I have stated that the geographical position was ascertained, also that the exact path of the moon's centre across the solar disk was made out by certain measurements of the photographs; and in Plate XVIII. I have given a graphic representation of the moon's path in reference to the sun's centre during the eclipse. Mr. CARRINGTON and Mr. FARLEY'S elements of the eclipse, founded on the geographical position, have already been cited, and will be presently made use of for computing the changes of position of the several luminous prominences which should occur if they belong to the sun.

If the prominences belong to the sun, photographic images of the same protuberances taken at any one locality, at different epochs of totality, ought to coincide exactly when the photographs are superposed one over the other†; and measurements of their positions with respect to the moon's centre ought to correspond with their computed positions. Moreover, photographs taken at different places (sufficiently near in longitude) ought to agree in their details,—although, for very distant localities, it is possible that they might not do so; for it is conceivable that during long intervals a change might occur in the luminous prominences, or fresh prominences might be brought into view by the sun's rotation.

Furthermore, on consideration it will be evident that to two observers differently situated the protuberances would not have precisely the same dimensions; that is, they would appear to project more or less beyond the moon's limb. Within the zone of

* An inspection of Plate XV. will render this evident, and it will be seen that the angle γ gradually diminishes as the prominence is more distant from A.

† That this was the case was shown, on the occasion of this paper being read, by sliding the first totality-picture over the second, and projecting their images on a screen by means of an electric lamp. It was thus seen that the pictures of the several prominences correspond exactly in form and position, each to each, in the two photographs.

totality, indeed, some prominences might be visible at one station and not at another, in consequence of the parallactic shift of the moon with respect to the sun.

Photographs offer great advantages over eye observations in determining changes of position in the prominences; but nevertheless there are some difficulties even in photographic measurements.

For example, with objects which are terminated by a softened outline, there is some difficulty in determining, absolutely, where the boundary ceases to exist: this was found to be especially the case with the luminous prominences depicted on the original negative, and also with the prominences as shown in positive photographic copies taken from it by superposition on an albumenized plate. Some doubt also existed in regard to the centering of No. 26 photograph; but this was found not to affect the results so much as the uncertainty of the precise termination of the prominences, in the photographs both negative and positive.

Table VI.

Table VI. gives the results of a series of angular measures of the luminous prominences, with reference to the moon's centre, both on the original negatives and on the albumen positive copies. Columns 4, 5, 6, 7 refer to the first totality-picture; columns 8, 9, 10, and 11 to the second totality-picture. Columns 7 and 11 are the measured positions, corrected for the errors of the wires, for the epochs of the two photographs; column 12 the difference in the position-angle of certain prominences at the commencement and at the end of totality. A mere inspection of column 12 renders it evident that the nearer a prominence is situated to 27° 10′, in reference to the sun's centre (which is the case with prominence A), the greater is its angular shift in reference to the moon's centre; and the nearer a prominence is to the line of motion of the moon's centre, the less is the angular change, as, for example, the bead h''' of the protuberance H.

Table VI.—Angular Position of the Prominences referred to the Moon's centre.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
			No. 25 photograph, 1st of totality, measured with the new micrometer.				No. 26 photograph, 2nd of totality, measured with the new micrometer.				
Letter designating protuberance on diagram.	Synonym.	The part measured.	Original negative.	Albumen positive (copy).	Mean.	Mean corrected for error of wires, by adding 15″·5.	Original negative.	Albumen positive (copy).	Mean.	Mean corrected for error of wires, by adding 16″·5.	Angular motion between the epochs of the two photographs.
A.	Cauliflower, Wheatsheaf	a	28 16	27 47·5	27 56·7	28 14	28 15	28 37	28 26	28 45·5	5 32
		a'	32 1·5	31 48	31 54·7	32 10	.....	26 8	26 8	26 24	
B.	Small spot	...	46 30	45 18	45 54	46 9					
C.	Detached cloud	c	58 14·5	58 2·5	58 8·5	58 24	53 23	53 45	53 34	53 50	4 34
D.	Base of Boomerang	d	72 16·5	72 2	72 9·2	72 25	67 57	68 16	68 6·5	68 23	4 2
E.	,,	e									
F.	.....	f	85 17·5	85 18	85 17·7	85 33					
H.	Part of trunk of fallen tree	h'	.....	.....	.....	.....	105 16·5	105 19	105 17·7	105 34	
		h''	.....	.....	.....	.....	106 28	106 35·5	106 31·7	106 48	
		h'''	109 30	.....	109 30	109 45	108 33	108 42	108 37·5	108 53	0 52
G.	Part of long eastern protuberance	g'	134 9	134 12	134 10·5	134 26					
I.	Mitre	i'	155 1	155 3	155 2	155 17					
K.	Hook	k''	195 9	195 17	195 13	195 28					
L.	Spade	l	.....	.....	.....	.....	261 33	261 58	261 45·5	262 2	
		p	.....	.....	.....	.....	263 24	263 18	263 21	263 37	
M.	Bead	m	.....	.....	.....	.....	275 58	275 52	275 55	276 11	
Q.	.....	q	.....	.....	.....	.....	298 13	298 9	298 11	298 27	
		q''	.....	.....	.....	.....	309 35·5	309 26	309 30·7	309 47	
		q'	.....	.....	.....	.....	340 59·5	341 26	341 12·7	341 29	
R.	Mountain, Ship	r'	348 26·5	348 12	348 19·2	348 35	344 26	344 51	344 38·5	344 55	3 45
	,,	r''	.....	.....	.....	.....	351 3	351 22	354 12·5	351 39	

By Mr. Carrington's calculations the relative motion of the centres during totality was $22''{\cdot}8 = \beta\alpha$, Plate XV.

The moon's radius at totality by measurements of the photographs was $993''{\cdot}1 = \alpha\beta$, Plate XV.

The position-angle of the moon's path $\beta\alpha$ prolonged in the direction of α was ascertained by measurements of photographs to be ..... $117^{\circ}\ 16'$

The position-angle of prominence A at the point a was ascertained by measurements of photographs to be ..... $28\ 14$

Whence the angle $\alpha\beta\alpha$, Plate XV. ..... $= 88\ 56$

With these elements the angle $\gamma =$ the angular shift of prominence A during totality was found to be $5^{\circ}\ 21'$.

In a like manner, taking into account the distance of each prominence from the moon's centre, the angular shift of the other prominences was calculated, whence

Angular shift of prominence A = 5 21
,, ,, ,, ,, C = 4 35
,, ,, ,, ,, E = 3 46
,, ,, ,, ,, H = 0 41
,, ,, ,, ,, R = 3 56

The edge a of the prominence A (the cauliflower or wheatsheaf) is not far from a line at right angles to the path of the moon's centre, and is well situated, therefore, for ascertaining whether the change of position-angle accords with the demands of that hypothesis which assumes that the luminous prominences belong to the sun.

Disregarding, for the moment, the errors in the assumed places of the wires, we have the following results.

Position of the prominence A, measured on the edge a:—

	First totality-picture.	Second totality-picture.	Angular shift.
Original negative . .	$28^{\circ}\ 10'$	$22^{\circ}\ 15'$	$5^{\circ}\ 55'$
Albumen positive . .	27 47·5	22 37	5 10·5
Difference . . .	− 22·5	+ 22	

Both these values of the angular shift have to be diminished by $1'$, in consequence of the alteration of the position-wires of the instrument in the interval between the two epochs, and they become respectively $5^{\circ}\ 54'$ and $5^{\circ}\ 9'{\cdot}5$.

Considering the difficulties experienced, from the causes before mentioned, in making measurements of position-angle, it is quite justifiable to take the mean of the above results; for, besides the uncertainty in determining the exact boundary of a prominence in the photograph, there is superadded, also, the difficulty of placing the photograph, quite correctly, in its proper angular position on the measuring-instrument, on account of the wires being very faintly imprinted on the western side of the picture in the first, and on the eastern side in the second, totality-picture.

The mean of the two measures gives $5^{\circ}\ 32'$ as the angular shift in the position of prominence A. Assuming, in accordance with theory, the motion of the moon's centre to have been $92''{\cdot}8$ during totality, we have $5^{\circ}\ 21'$ as the theoretical change of position-angle, which, deducted from the mean $5^{\circ}\ 32'$, gives the difference of $0^{\circ}\ 11'$.

On taking an average of the measurements of all the prominences, the difference between the measured and the computed angular shift is only half this quantity, as will presently be seen.

Table VII.

Table VII. contains the results of measurements of positive photographs 9 inches in diameter. Columns 4, 5, 6, 7, and 8 relate to No. 1 totality-picture; and columns 9, 10, 11, 12, and 13, to No. 2 totality picture. Columns 7 and 12 show, respectively, the differences in the determinations of the position-angles of the several prominences in No. 1 and No. 2 totality-pictures, on the 9-inch photograph and the original negatives and positive-albumen copies by superposition. I give preference to the measures by means of the instrument of the original negatives and the direct albumen copies, so far as regards the position-angles; but for measuring the amount of motion of the moon's centre during the totality, I am, for the reason already assigned, dependent on the 9-inch photographs. The whole difficulty, with respect to the latter, consisted in exactly ascertaining the position of the centre of the moon in the two pictures, and in measuring correctly a picture of the sun, enlarged to the same scale, with a divided

beam compass, in order to obtain a value in arc for the measurements taken in inches and decimals of an inch. This accomplished, it was only necessary to measure the distance from the moon's centre or periphery of certain parts of the prominences in both totality-pictures, and to compare the results of the measures obtained on one picture with those on the other, for the purpose of ascertaining the amount of motion of the lunar disk in reference to the several prominences, and to reduce the resulting numbers to their value in arc.

The amount of motion is given in column 14 for those prominences which are visible in both photographs.

Thus, in a direction nearly at right angles to the path of the moon's centre, the apparent motion of the periphery was 1″, while at 83° from that point, or 7° from the line of motion of the lunar centre, it was 93″.

The motion of the moon's periphery, in respect of the several prominences situated at an angle θ would be $92''.8 . \sin(\theta - 27° 10')$, 27° 10′ being the position of a line at right angles to the motion of the lunar centre, as deduced from the photographic measures already given. For the following calculations, the angle θ was obtained from Plate XV., in which the prominences are referred to the sun's centre.

	Part measured.	Angle θ.		Measured motion.	Computed motion.	Measure − computation.
		°	′	″	″	″
Prominence A	a	26	9	1	1·6	−0·6
C	c	56	34	44	45·6	−1·6
C	c'	61	39	51	52·5	−1·5
E	e	67	4	65	59·5	+5·5
H	h'''	110	9	93	92·1	+0·9
R	r	346	14	60	60·9	−0·9
						+0·3 Mean.

From Table VI. are derived the following numbers:—

	Part measured.	Angular shift in respect of the moon's centre during totality.				
		Measured.		Calculated.		Measure − computation.
		°	′	°	′	′
Prominence A	a	5	32	5	21	+11·0
C	c	4	34	4	35	− 1·0
E	e'	4	2	3	46·5	+15·5
H	h'''	0	52	0	41·5	+10·5
R	r'	3	45	3	56·5	−11·5
						+ 4·9 Mean.

=1″·4 motion of the moon's centre in excess of the computed quantity.

It would be extremely difficult to obtain more convincing proofs that the luminous prominences belong to the sun than the foregoing numbers offer; but I have still one more to bring forward.

* The relative motion of the centres during totality as calculated by Mr. Carrington.

Table VII.

Distances of Prominences from the Moon's Periphery and their Position-angles referred to the Moon's centre.														Distance and position-angles referred to the sun's centre.	
			No. 25 photograph. First totality-picture.					No. 26 photograph. Second totality-picture.						Both totality-pictures engraved round No. 6 photograph of sun. Plate XV.	
Designating letter on Plate XV.	Name of protuberance.	Part measured.	Measures on the enlarged positive copy, the original of Plate XIII.	Numbers in column 4, corrected for error of the zero by subtracting 8′.	Mean of the measures of the original negative, and albumen print therefrom, Table VI. column 7.	Column 5 minus column 6.	Measured height of the prominences above the moon's periphery.	Measures on the enlarged positive copy, the original of Plate XIV.	Numbers in column 9 corrected for error of zero by subtracting 7′.	Mean of the measures of the original negative and the albumen print therefrom, Table VI. column 11.	Column 10 minus column 11.	Measured height of the prominences above the moon's periphery, corrected for error of centering.	Motion of the moon in covering and uncovering prominences.	Position-angle on sun's centre by graphic process.	Height of prominences above the sun's periphery, corrected for error of centering in those cases in which the positions are derived from the copy of No. 26.
1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.
			° ′ ″	° ′ ″	° ′ ″	′	″	° ′ ″	° ′ ″	° ′ ″	″	″	″	° ′	″
A.	Cauliflower	a.	28 40	28 32	28 14	+18	0 40	22 30	22 23	22 42·5	−19·5	0 39	0 1	28 9	1 18
		a′.	32 24	32 16	32 10	+ 6		26 35	26 28	26 24	+ 4			29 39	
C.	Detached cloud	c.	58 30	58 22	58 24	− 2	0 44	54 14	54 7	53 50	+17	0 00	0 44	56 34	0 54
		c′.	.....	.....	.....	.....	1 45	.....	.....	.....	.....	0 54	0 51	61 39	1 53
D.		d.	72 20	72 12	72 25	−13	0 40	68 5	67 58	68 23	−25	.....	.....	.....	.....
E.	Boomerang	e.	68 50	.....	.....	.....	2 31	.....	.....	.....	.....	1 36	1 5	67 4	2 40
		e′.	72 20	72 12	72 25	−13	.....	.....	.....	.....	.....	.....	.....	.....	.....
F.		f.	85 40	85 32	85 33	− 1	.....	.....	.....	.....	.....	.....	.....	85 4	.....
		f′.	.....	.....	.....	.....	0 38	.....	.....	.....	.....	.....	.....	.....	0 39
		k′.	.....	.....	.....	.....	1 50	105 30	105 23	105 34	−11	0 12	1 38	.....	.....
H.	Fallen tree	h″.	.....	.....	.....	.....	1 38	107 5	106 58	106 48	+10	0 3	1 35	.....	.....
		h‴.	110 0	109 52	109 45	+ 7	1 33	108 55	108 48	108 53	− 5	0 00	1 33	110 9	.....
G.	Long protuberance	g.	.....	.....	.....	.....	1 3	.....	.....	.....	.....	.....	.....	.....	1 2
		g′.	134 30	134 22	134 26	− 4	1 5	.....	.....	.....	.....	.....	.....	136 14	1 14
I.	Mitre	i.	.....	.....	.....	.....	1 6	.....	.....	.....	.....	.....	.....	.....	1 28
		i′.	155 20	155 12	155 17	− 5	.....	.....	.....	.....	.....	.....	.....	157 29	.....
K.	Hook	k.	.....	.....	.....	.....	0 44	.....	.....	.....	.....	.....	.....	.....	1 38
		k″.	195 50	195 42	195 28	+14	.....	.....	.....	.....	.....	.....	.....	196 19	0 54
L.	Spade	l.	.....	.....	.....	.....	.....	262 20	262 13	262 2	+11	0 21	.....	260 4	0 48
		l′.	.....	.....	.....	.....	.....	264 5	263 58	263 37	+21	.....	.....	.....	.....
M.	Bead	m.	.....	.....	.....	.....	.....	276 40	276 33	276 11	+22	0 16	.....	274 44	0 24
		m′.	.....	.....	.....	.....	.....	298 40	298 33	298 27	+ 6	0 27	.....	297 54	0 37
Q.		q.	.....	.....	.....	.....	.....	309 40	309 33	309 47	−14	0 31	.....	309 49	0 31
		q″.	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
R.	Mountain	r.	348 50	348 42	348 25	+ 7	0 00	344 45	344 38	344 55	−17	1 00	1 00	346 14	1 16
						Mean +1′·3					Mean +0′·0				

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It is known that Señor Aguilar, Professor Montserrat, and Father Secchi obtained photographs of the eclipse at Desierto de las Palmas, south of the central line, whilst my own were taken on the north of it. The latitude of my position was 42° 42′ N., that of Señor Aguilar 40° 4′ 4″ N.; my position, measured on a meridian, is about 18′ north of the central line of the eclipse, Señor Aguilar's about 4′ to the south of the central line.

Señor Aguilar was so good as to send me paper copies of the four photographs taken at Desierto de las Palmas; and I have been able, in return, to send him copies of mine on glass, 9 inches in diameter. I have enlarged the copies presented to me to exactly the same dimensions (9 inches for the moon's diameter). One amongst the number, namely, the first taken, is sufficiently good, when magnified, to admit of measures being made on it with a fair amount of accuracy, although there is evidence, in the woolliness of the photograph, of the telescope not having followed the sun very well. Notwithstanding a want of precision in the details, I was able to ascertain with certainty that the distances between the point r' of R and a of A, between a of A and c of C, between c of C and g of G, between g of G and i of I, and between i of I and k'' of K, in Señor Aguilar's photographs, correspond exactly with the same points in mine. I have already mentioned that when my first totality negative was superposed over the second the several prominences exactly coincided in their relative positions, and that the distance between any given points of two prominences on my first totality-photograph is absolutely the same in the second totality-photograph. Here we have the evidence carried still a step further; for the distances between given points in two prominences in Señor Aguilar's photographs accord entirely with the distances between the same points on my own.

I may mention that the prominence G in Señor Aguilar's photograph, from the commencement of the broad part on towards h of the fallen tree H, is much confused; H is not seen, because it is mixed up with G, which in consequence is as broad from h to g' as in its broadest part. With this explanation, however, there will be experienced no difficulty in comparing the photographs. The boomerang is not depicted on Señor Aguilar's photographs. As my position was north of the central line, and the moon's centre was, as we have seen, shifted, by parallax, about 12″ below the sun's, it follows that I ought to have seen more of the prominence A, and less of I and K, than could be seen at Desierto de las Palmas; this is fully borne out by the photographs taken at the respective localities. The height of A above the moon's periphery is 40″ in my first totality-picture; in the corresponding picture taken at Desierto de las Palmas it is 32″, the difference being 8″. Prominence K in my first totality-picture, measured at k', is 44″, in Señor Aguilar's 60″, difference 16″. The mean of the two measures $\frac{8+16}{2}=12''$, the relative parallactic displacement of the moon's disk at the two stations of Rivabellosa and Desierto de las Palmas, ascertained as nearly as the want of definition in the photograph obtained at the latter station permitted. It is probably less by about 2″ than the true displacement.

In conclusion, the two totality-pictures No. 25 and No. 26, when reduced to a suitable size and placed in the stereoscope, No. 25 on the left, and No. 26 on the right, afford a very beautiful view of the phenomena of totality, and one which could not be enjoyed by mortal eyes in looking at the real eclipse. Not only does the stereoscope render evident the fact of the moon being an object intervening between the observer and the sun, but it also shows it as a sphere. The triplication of the prominences must be corrected in No. 26; but any attempt to complete the lunar disk by painting on a positive copy, as α, β, and γ photographs, the originals of Plates X., XI., and XII., is immediately detected, and the corrected lunar disk appears perfectly flat. In placing the photographs in the stereoscope, the prominence A must be placed upwards, and at right angles to the line joining the centres of the photographs.

Appendix.

Having brought to a successful issue the photographic record of a total eclipse, it may not be out of place to point out for the guidance of others what steps I would recommend should hereafter be adopted.

In the foot-note to p. 334, I have mentioned that with my 13-inch reflector intense photographs of the moon were obtained in four seconds, and that, under precisely similar atmospheric circumstances, it required three minutes to obtain a feeble impression of the moon with the Kew heliograph, which, for the present, is mounted on an outrigger attached to the declination axis of my reflector. It will be remembered that for the totality-pictures obtained at Rivabellosa, under exceptionally favourable conditions in respect of the sun's altitude and the state of the atmosphere, the sensitive plate was exposed exactly one minute, the resulting photograph being remarkably dense, even to a fault. A picture of the moon, of greater intensity than the feeble image given by the Kew heliograph, could be obtained with my reflector in a second, so that it would produce pictures of the prominences in $\frac{1}{180}$th part of the time required by the heliograph, or in $\frac{60}{180}=\frac{1}{3}$ of a second. Making sufficient allowance for the difficulties in determining the exact ratio of actinic intensity in the foci of the two instruments, and also for a condition of the atmosphere less favourable than that under which the 'Himalaya' photographers worked, it may be safely estimated that, with a 13-inch reflector, perfect pictures of the prominences could be procured in two seconds. A 13-inch reflector would, however, be a cumbrous instrument to transport and erect at a distance from home; but a 9-inch reflector—or its equivalent, a 6-inch refractor, specially corrected for the actinic rays, is within the compass of such an expedition. These telescopes might be mounted with clockwork drivers on rigid equatorial stands, which must, however, be so designed as to admit of an adjustment of the polar axis to suit various latitudes. For each instrument an observatory should be constructed to take to pieces. Each observatory would require not less than four plate-holders, and about six baths to contain nitrate of silver,

which must have been carefully fused*. The collodion employed should be iodized a month before use with the cadmium iodizer, and, before starting, carefully decanted into clean vessels, tied over with bladder to prevent evaporation.

I would recommend that three instruments, having a focal length of about 10 feet, should be prepared and kept, with the portable observatory, in readiness to be placed at the disposal of any such expedition as that organized by the Astronomer Royal. These instruments would give pictures of the luminous prominences in four seconds at the outside; and under favourable atmospheric conditions, in less than a second.

Observers intending to take charge of an instrument should practise with it in taking lunar photographs, previous to starting, so as to familiarize themselves completely with it. Not fewer than four persons should accompany each telescope, and two of them ought to be accomplished photographers.

Pictures of the partial phases would be best obtained with such an instrument as the Kew heliograph.

Lastly, I have deemed it to be desirable that positive copies of the eclipse-pictures should be placed in some Institution readily accessible to the public, and I have therefore presented to the South Kensington Museum a series of enlarged positive copies on glass, 9 inches in diameter, which are at present exhibited in the International Exhibition, Class XIII.

* "Report on Celestial Photography," by the author, in the Report of the British Association for 1859.

XIX. *On a New Method of Approximation applicable to Elliptic and Ultra-elliptic Functions.*—Second Memoir*. *By* CHARLES W. MERRIFIELD. *Communicated by* W. SPOTTISWOODE, *Esq.*, *F.R.S.*

Received March 20,—Read April 3, 1862.

SINCE my first memoir on this subject was read before the Society, Mr. SYLVESTER has published a method, more general than mine, of applying rational approximation to facilitate the computation of the integrals of irrational functions. This method, at which he had arrived independently, included, *à majori*, the one which was the subject of my memoir. Aided by his papers, my subsequent studies have enabled me to view the method with more generality, as well as with more precision and completeness of detail, and I am now able to present it in a sufficiently finished and practical form for the immediate use of the computer. I have also computed auxiliary Tables, to render its application easier in certain cases.

Any rational formula, which gives approximately the value of a function to be integrated, may be integrated in lieu of it, and the result will in general be an approximate value of the integral sought. But for such a process to be of any practical utility, the convergence of the formula must be excessive, for the complexity of the integral forms is so great that the labour would be enormous, unless the terms were very few in number. In the discovery of formulæ sufficiently convergent for the purpose, lies the success of the method.

We are by no means restricted to functions under a square root, or even to pure radical forms at all. The principle applies with equal generality to functions which are given implicitly as roots of equations, and thus to a class of differential equations; and Mr. SYLVESTER has well remarked that these formulæ not only afford facilities for computation, as by a method of quadratures, but also enable us to assign superior and inferior limits to an integral, without losing its generality of form.

I shall begin with the approximation to the square root, giving it in its general form, and explaining its exact analytical signification. I shall then show its application to Elliptic Functions, and how, in the ordinary cases, certain simple reductions can be effected, which greatly lessen the labour of computation; and I shall give these reductions for the cases more commonly occurring, with some examples and working formulæ. I shall then add a short account of the extension of the method.

The paragraphs in the first two sections of this paper bear a consecutive number for convenience of reference.

* For the First Memoir, see the Philosophical Transactions for 1860, p. 223.

SECTION I.—*Approximants to the Square Root.*

1. Mr. SYLVESTER gives, for the approximants to the square root, the following statement:—

"Let r be an approximate value of $\sqrt{N}$; then by that mode of application of NEWTON's method of approximation to the equation $x^2=N$, which is equivalent to the use of continued fractions, we may easily establish the following theorem, viz., that

$$\frac{r^2+N}{2r},\ \frac{r^3+3rN}{3r^2+N},\ \frac{r^4+6r^2N+N^2}{4r^3+4rN},\ \frac{r^5+10r^3N+5rN^2}{5r^4+10r^2N+N^2},$$

will be successive approximations to $\sqrt{N}$."

2. Their general form is

$$y=\frac{(r+\sqrt{N})^i+(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i}\sqrt{N}, \quad \ldots\ldots\ldots\ldots \quad (1.)$$

which is always rational. In this form the approximation to $\sqrt{N}$ as i increases is obvious. The method of my previous memoir is simply the particular case of $i=2^k$.

3. If we wish to approximate to $N^{-\frac{1}{2}}$, we may take the reciprocal of (1.), or, what is simpler, we may divide (1.) by N, thus obtaining

$$z=\frac{(r+\sqrt{N})^i+(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i}\frac{1}{\sqrt{N}}. \quad \ldots\ldots\ldots\ldots \quad (2.)$$

Before we can integrate these formulæ, we must reduce them by means of the method of rational fractions; the simplest and most general way is as follows:—

4. Let ϱ be an ith root of unity; then, obviously,

$$\log(1-x^i)=\log(1-\varrho x)+\log(1-\varrho^3 x)+\ldots\ldots+\log(1-\varrho^i x).$$

Multiplying the differential coefficient of this by $(-x)$, we obtain

$$\frac{ix^i}{1-x^i}=\frac{\varrho x}{1-\varrho x}+\frac{\varrho^2 x}{1-\varrho^2 x}+\frac{\varrho^3 x}{1-\varrho^3 x}+\ldots\ldots+\frac{\varrho^i x}{1-\varrho^i x};$$

and since $\frac{1}{1-x^i}=1+\frac{x^i}{1-x^i}$ and $\frac{1+x^i}{1-x^i}=1+\frac{2x^i}{1-x^i}$,

$$\frac{i}{1-x^i}=\frac{1}{1-\varrho x}+\frac{1}{1-\varrho^2 x}+\frac{1}{1-\varrho^3 x}+\ldots\ldots+\frac{1}{1-\varrho^i x},$$

$$i\frac{1+x^i}{1-x^i}=\frac{1+\varrho x}{1-\varrho x}+\frac{1+\varrho^2 x}{1-\varrho^2 x}+\frac{1+\varrho^3 x}{1-\varrho^3 x}+\ldots\ldots+\frac{1+\varrho^i x}{1-\varrho^i x}$$

Making $x=\frac{r-\sqrt{N}}{r+\sqrt{N}}$, we may thus divide $N^{\pm\frac{1}{2}}$ into i fractions, each of the form

$$\frac{1}{i}\frac{(r+\sqrt{N})+\varrho^k(r-\sqrt{N})}{(r+\sqrt{N})-\varrho^k(r-\sqrt{N})}N^{\pm\frac{1}{2}},$$

k being any integer not exceeding i.

5. If we add the pairs k and $i-k$, we obtain for the sum of the pair,

$$\frac{2(r+\sqrt{N})^2-2(r-\sqrt{N})^2}{(r+\sqrt{N})^2+(r-\sqrt{N})^2-(\varrho^k+\varrho^{i-k})(r^2-N)}N^{\pm\frac{1}{2}}$$

$$=\frac{8rN}{2(r^2+N)-(\varrho^k+\varrho^{i-k})(r^2-N)}\text{ or }\frac{8r}{\text{same denominator}},$$

according to whether the upper or lower sign be taken. Now, because ϱ is an ith root of unity, $\varrho^k+\varrho^{i-k}=2\cos\frac{2k\pi}{i}$, and the sum of the pair reduces itself, for $\sqrt{N}$, to

$$\left.\begin{aligned} y_k&=\frac{1}{i}\frac{4rN}{(r^2+N)-\cos\frac{2k\pi}{i}(r^2-N)}=\frac{1}{i}\frac{2rN}{r^2\sin^2\frac{k\pi}{i}+N\cos^2\frac{k\pi}{i}}\\ &=\frac{1}{i}\frac{2rN}{N+\sin^2\frac{k\pi}{i}(r^2-N)}=\frac{1}{i}\frac{2rN}{r^2-\cos^2\frac{k\pi}{i}(r^2-N)}. \end{aligned}\right\}\quad\ldots\ldots\quad(3.)$$

For $N^{-\frac{1}{2}}$ we have the simpler forms,

$$\left.\begin{aligned} z_k&=\frac{1}{i}\frac{4r}{(r^2+N)-\cos\frac{2k\pi}{i}(r^2-N)}=\frac{1}{i}\frac{2r}{r^2\sin^2\frac{k\pi}{i}+N\cos^2\frac{k\pi}{i}}\\ &=\frac{1}{i}\frac{2r}{N+\sin^2\frac{k\pi}{i}(r^2-N)}=\frac{1}{i}\frac{2r}{r^2-\cos^2\frac{k\pi}{i}(r^2-N)}. \end{aligned}\right\}\quad\ldots\ldots\quad(4.)$$

All that remains is to integrate these terms, and sum them.

6. Our grouping the terms in pairs has limited the value of k to range from i to $\frac{1}{2}(i-1)$ when i is odd. There is an odd term which, however, presents no difficulty, being simply $\frac{r}{i}$ in the case of $\sqrt{N}$, and $\frac{r}{iN}$ in the case of $N^{-\frac{1}{2}}$. When i is even, k is limited to range from 1 to $\frac{1}{2}i-1$, and the odd term becomes $\frac{r^2+N}{ir}$ in the case of $\sqrt{N}$, and $\frac{r^2+N}{iNr}$ in the case of $N^{-\frac{1}{2}}$. It is important to bear in mind that the term just mentioned is an odd term, and therefore not affected with the coefficient 2, which appears in the terms composed of pairs corresponding to imaginary roots.

7. The value of i, which I consider to be most useful for general purposes, is $i=8$: in this case the odd term becomes $\frac{r^2+N}{8r}$ or $\frac{r^2+N}{8Nr}$, and the other values of $\frac{k\pi}{i}$ are three in number, viz. 22° 30′, 45°, 67° 30′. With proper precautions $i=8$ will almost always give seven or more figures correct.

8. If we now give infinite values to k and i and pass from the summation to the definite integral, we have $\left(\text{putting } \lambda=\frac{k}{i}\right)$

$$z=N^{-\frac{1}{2}}=\int_0^{\frac{1}{2}}\frac{2rd\lambda}{r^2+(N-r^2)\cos^2\lambda\pi}=\frac{2}{\pi}\frac{r}{N}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{1+\frac{r^2-N}{N}\sin^2\phi};$$

and since

$$\int_0^{\frac{1}{2}\pi}\frac{d\phi}{1+p\cdot\sin^2\phi}=\frac{\frac{1}{2}\pi}{\sqrt{1+p}},$$

this is an identical equation, as it ought to be.

9. This use of approximants, therefore, is simply the application of the method of

quadratures to a definite integral, which we substitute for the surd proposed for evaluation.

10. It would appear at first sight that a full application of the method of quadratures in the ordinary way, with the help of differences, would give better results than the mere summation of the ordinates. But this is not the case; for the differences diverge immediately. If we use differential coefficients for the quadrature, instead of differences, we have an opposite anomaly, namely that the correction of the summation appears to be absolutely *nil*, inasmuch as the differential coefficients which appear in the series are all of odd order, and the numerator of each of them contains the factor $\sin\phi\cos\phi$, which vanishes at both the limits 0 and $\frac{1}{2}\pi$. LEGENDRE has discussed this point. See the Appendix to the second volume of his 'Fonctions Elliptiques,' p. 578.

11. The application of the method to integrations, then, lies in the substitution for

$$\int_0^t \frac{M}{\sqrt{N}}\,dt \text{ of } \int_0^t\int_0^\lambda \frac{2Mr\,.\,d\lambda\,.\,dt}{r^2+(N-r^2)\cos^2\lambda\pi},$$

in which, since λ and t are perfectly independent of each other, we may change the order of integration, thus obtaining

$$\int_0^\lambda \left\{\int_0^t \frac{2Mr\,.\,dt}{r^2+(N-r^2)\cos^2\lambda\pi}\right\} d\lambda\,;$$

and the rest of the operation depends upon our being able to perform the integration in } generally, and then to determine the integral in λ by quadratures. The great advantage of the method turns upon the easy application of the method of quadratures, in consequence of our not requiring to *difference* the ordinates.

12. One way of exhibiting generally the degree of convergence is as follows: $N^{\pm\frac{1}{2}}$ always lies between

$$N^{\pm\frac{1}{2}}\frac{(r+\sqrt{N})^i+(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i} \text{ and } N^{\pm\frac{1}{2}}\frac{(r+\sqrt{N})^i-(r-\sqrt{N})^i}{(r+\sqrt{N})^i+(r-\sqrt{N})^i},$$

and the error of either is therefore always less than their difference,

$$N^{\pm\frac{1}{2}}\frac{4(r^2-N)^i}{(r+\sqrt{N})^{2i}-(r-\sqrt{N})^{2i}}.$$

13. There is another mode, by which, in any given case, we may see how far it is necessary to carry our work in order to obtain a given number of decimals correctly in the result. Let θ_m be determined by the equation

$$\int_0^{\theta_m}\frac{d\theta_m}{\cos\theta_m}=m\int_0^{\theta_1}\frac{d\theta_1}{\cos\theta_1},$$

and let $\sin\theta_1=\frac{\sqrt{N}}{r}$ or $\frac{r}{\sqrt{N}}$, whichever may be less than unity; then the mth approximant will be $\frac{\sqrt{N}}{\sin\theta_m}$. This is easily seen from the general term of the approximant, since

$$\int\frac{d\theta}{\cos\theta}=\frac{1}{2}\log_e\left(\frac{1+\sin\theta}{1-\sin\theta}\right).$$

14. A table of meridional parts, such as is given in the books on Navigation, if carried far enough, would solve this equation. I have calculated an auxiliary Table for the purpose, as follows:—

Let $\operatorname{cosec}\varphi - 1 = z$, $\log \tan\left(\frac{1}{4}\pi + \frac{1}{2}\varphi\right) = y$, then

$$y = \frac{1}{2}\log \frac{2+z}{z} = \frac{1}{2}\log 2 - \frac{1}{2}\log z + \frac{1}{2}\log\left(1+\frac{1}{2}z\right)$$

$$= \frac{1}{2}\log 2 - \frac{1}{2}\log z + \frac{z}{4} - \frac{z^2}{16} + \frac{z^3}{48} - \frac{z^4}{128} + \ldots\ldots\ldots$$

To bring this formula to the same unit as the common Table of meridional parts, we must multiply it by the number of minutes in the arc equal to unity, or by L = 3437·74677 07849 4, whence we have $\frac{1}{2}$ L log, 2 = 1191·43224 08243 2, and $\frac{1}{2}$ L log, 10 = 3958·85223 39129 100. These data give the following Table, the argument being the common logarithm of z with its sign changed; that is, the number of places which are correct:

$-\log z$.	y.
1	5234·14859
2	9117·70966
3	13068·84816
4	17026·92712
5	20985·70200

$-\log z$.	y.
6	24944·54650
7	28903·39796
8	32862·25012
9	36821·10235
10	40779·95458

$-\log z$.	y.
11	44738·80681
12	48697·65905
13	52656·51128
14	56615·36352
15	60574·21575

15. As a simple example, let N = 3, $r = 2$; $\therefore \frac{\sqrt{N}}{r} = \sin 60°$: the meridional parts for $60° = 4527$; and in order that the error may not exceed unity in the tenth place of figures, we must have m or $i = \frac{40780}{4527} = 9$; so that we must make $i = 9$ at least, for the 10th figure to be correct.

16. These methods of course only exhibit the degree of approximation on the surd itself. The proportionate approximation is generally greater on the integral than on the simple surd, because the first approximant is usually so chosen as to be identical with the surd at one of the limits, and it is only near the other limit that the discrepancy tells.

Section II.—*Details of Reduction and Computation.*

17. The chief assistance, which can be provided *à priori* for the computer, consists in the exhibition and discussion, for the ordinary forms, of the integral $\int_0^t \frac{2Mr\,dt}{r^2 + (N - r^2)\cos^2 \lambda\pi}$ and of the auxiliary functions which present themselves in its reduction.

18. In applying these methods to elliptic integrals, the radical and the first approximant r must both be of a simple form, and it is advisable that $r^2 - N$ or $N - r^2$ should be

of a square form. For the common form of the elliptic radical $\sqrt{(1-\sin^2\theta.\sin^2\varphi)}$, our choice is practically limited to

(1) $r=1$, (2) $r=\cos\varphi$, (3) $r=\sin\theta.\cos\varphi$,
(4) $r=\cos\theta$, (5) $r=\cos\theta.\sin\varphi$.

And on these suppositions I now proceed to the integration of the general form of the reduced approximant for $\int_0^\varphi(1-\sin^2\theta.\sin^2\varphi)^{-\frac{1}{2}}d\varphi=\int z_k d\varphi$. I omit mention of the constants of integration, because very slight changes in the function may alter them. The first of our three cases require, as they stand, no constant, and these are the most useful cases.

(1) $r=1$, $r^2-\mathrm{N}=\sin^2\theta.\sin^2\varphi$,

$$z_k=\frac{2}{1-\sin^2\theta.\cos^2\frac{k\pi}{i}.\sin^2\varphi},$$

$$\int z_k d\varphi=2\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\tan^{-1}\left\{\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}\tan\varphi\right\}.$$

(2) $r=\cos\varphi$, $r^2-\mathrm{N}=-\cos^2\theta.\sin^2\varphi$,

$$z_k=\frac{2\cos\varphi}{\cos^2\varphi+\cos^2\theta.\cos^2\frac{k\pi}{i}.\sin^4\varphi}=\frac{2\cos\varphi}{1-\left(1-\cos^2\theta.\cos^2\frac{k\pi}{i}\right).\sin^2\varphi},$$

$$\int z_k d\varphi=\left(1-\cos^2\theta.\cos^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\log_e\left\{\frac{1+\sin\varphi\left(1-\cos^2\theta.\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}}{1-\sin\varphi\left(1-\cos^2\theta.\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}}\right\}.$$

(3) $r=\sin\theta.\cos\varphi$, $r^2-\mathrm{N}=-\cos^2\theta$,

$$z_k=\frac{2\sin\theta.\cos\varphi}{\left(1-\cos^2\theta.\sin^2\frac{k\pi}{i}\right)-\sin^2\theta.\sin^2\varphi},$$

$$\int z_k d\varphi=\left(1-\cos^2\theta.\sin^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\log\left\{\frac{\left(1-\cos^2\theta.\sin^2\frac{k\pi}{i}\right)^{\frac{1}{2}}+\sin\theta.\sin\varphi}{\left(1-\cos^2\theta.\sin^2\frac{k\pi}{i}\right)^{\frac{1}{2}}-\sin\theta.\sin\varphi}\right\}.$$

(4) $r=\cos\theta$, $r^2-\mathrm{N}=-\sin^2\theta\cos^2\varphi$,

$$z_k=\frac{2}{\cos\theta}\frac{1}{1+\tan^2\theta.\cos^2\frac{k\pi}{i}.\cos^2\varphi},$$

$$\int z_k d\varphi=2\left(1-\sin^2\theta.\sin^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\tan^{-1}\left\{\cos\theta.\tan\varphi\left(1-\sin^2\theta.\sin^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\right\}.$$

(5) $r=\cos\theta\sin\varphi$, $r^2-\mathrm{N}=-\cos^2\varphi$,

$$z_k=\frac{2\cos\theta.\sin\varphi}{\cos^2\theta.\sin^2\varphi+\cos^2\frac{k\pi}{i}.\cos^2\varphi},$$

$$\int z_1 d\varphi = 2\left(\cos^2\frac{k\pi}{i} - \cos^2\theta\right)^{-\frac{1}{2}} \tan^{-1}\left\{\frac{\cos\theta}{\cos\varphi}\left(\cos^2\frac{k\pi}{i} - \cos^2\theta\right)^{-\frac{1}{2}}\right\}$$

$$= \left(\cos^2\theta - \cos^2\theta\frac{k\pi}{i}\right)^{-\frac{1}{2}} \log_e \left\{\frac{1 - \frac{\cos\varphi}{\cos\theta}\left(\cos^2\theta - \cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}}{1 + \frac{\cos\varphi}{\cos\theta}\left(\cos^2\theta - \cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}}\right\}.$$

(6) If we make $t = \tan\frac{1}{2}\varphi$, we obtain

$$(1 - \sin^2\theta \,.\, \sin^2\varphi)^{-\frac{1}{2}} d\varphi = 2(1 - 2\cos 2\theta \,.\, t^2 + t^4)^{-\frac{1}{2}} dt.$$

Taking $r = 1 - t^2$, the terms which we have to integrate are of the form

$$\int \frac{4(1-t^2)\,dt}{(1-t^2)^2 + 4\cos^2\theta \,.\, \cos^2\frac{k\pi}{i} \,.\, t^2}.$$

Putting $q^2 = 1 - \cos^2\theta \,.\, \cos^2\frac{k\pi}{i}$, we have

$$\int z_1 d\varphi = \frac{1}{q} \log_e \left(\frac{t^2 + 2qt + 1}{t^2 - 2qt + 1}\right).$$

The same expression serves for the integral

$$\int \frac{2dt}{\sqrt{(1 + 2\cos 2\theta \,.\, t^2 + t^4)}},$$

if we put $q^2 = 1 - \sin^2\theta \,.\, \cos^2\frac{k\pi}{i}$.

19. It will be observed that the first four cases, and the sixth, depend upon a radical of the form $\sqrt{(1 - \sin^2 \mathrm{A} \,.\, \sin^2\omega)}$, where ω is restricted to the selected values of $\frac{k\pi}{i}$. Assuming the modulus sin A not to vary, it would therefore in general be better to begin by computing the radical for the selected values. I have computed, and I append to this paper, a Table of this radical, the selected values of $\frac{k\pi}{i}$ being 22° 30′, 45°, and 67° 30′, while A ranges by whole degrees from 1° to 90° inclusive. Every entry but the last in the 2nd, 3rd, and 4th columns of the Table was computed by myself in duplicate with VEGA's ten-figure logarithms, by the help of two or more of the following formulæ, some of which are from LEGENDRE.

20. Putting Δ for $\sqrt{(1 - \sin^2 \mathrm{A} \,.\, \sin^2\omega)}$,

(1) Make $\sin \mathrm{A} \,.\, \sin\omega = \sin \mathrm{M}$; then $\Delta = \cos \mathrm{M}$,
$\log \sin \mathrm{M} = \log \sin \mathrm{A} + \log \sin\omega$, $\log \Delta = \log \cos \mathrm{M}$; or else

(2) Make $\tan \mathrm{A} \,.\, \cos\omega = \tan \mathrm{M}$; then $\Delta = \cos \mathrm{A} \,.\, \sec \mathrm{M}$,
$\log \tan \mathrm{M} = \log \tan \mathrm{A} + \log \cos\omega$, $\log \Delta = \log \cos \mathrm{A} +$ ar. co. $\log \cos \mathrm{M}$.

Moreover, let L be the tabular angle nearest to the angle M: it is not necessary to obtain the value of M: so that we have simultaneously,

$$\log \sin \mathrm{M} = \log \sin \mathrm{L} \pm s,$$
$$\log \tan \mathrm{M} = \log \tan \mathrm{L} \pm t,$$
$$\log \cos \mathrm{M} = \log \cos \mathrm{L} \mp c;$$

then we shall also have, and with great approximation,

$$\begin{aligned}\log s &= \log(t \cdot \cos^2 \mathrm{L}) \mp (t - t \cdot \cos^2 \mathrm{L}) \\ &= \log(c \cdot \cot^2 \mathrm{L}) \mp (c + c \cdot \cot^2 \mathrm{L}), \\ \log c &= \log(s \cdot \tan^2 \mathrm{L}) \pm (s + s \cdot \tan^2 \mathrm{L}) \\ &= \log(t \cdot \sin^2 \mathrm{L}) \pm (t - t \cdot \sin^2 \mathrm{L}), \\ \log t &= \log(s \cdot \sec^2 \mathrm{L}) \mp (s - s \cdot \sec^2 \mathrm{L}) \\ &= \log(c \cdot \operatorname{cosec}^2 \mathrm{L}) \mp (c - c \operatorname{cosec}^2 \mathrm{L}).\end{aligned}$$

I have given the whole set of six, but my Table was computed with the pair for log c. By way of example, I add a specimen copy of one of my working sheets. The use of so many as ten figures is not altogether unnecessary, because otherwise, when Δ is nearly equal to unity, the value of $\log(1-\Delta)$ or of $\log\frac{1-\Delta}{1+\Delta}$ cannot be had with exactness.

21. The following formulæ will also be found in many cases preferable, both for exactness and facility, to the ordinary use of logarithmic tables by means of differences. These formulæ, as well as those of the previous paragraph, are but applications of TAYLOR'S theorem, reduced to a shape fit for the computer. Even where only seven figures are required their application is frequently much easier, and gives more exact results, than interpolation by differences. In what follows, x is supposed to be the nearest tabular entry.

22. *To find* $\log y$ *from* $\log\tan y$.—Let us assume simultaneously

$$\log y = \log x \pm l, \quad \log\tan y = \log\tan x \pm t.$$

Putting $u = \log x$, $z = \log\tan x$, we have

$$\frac{du}{dz} = \frac{\sin 2x}{2x} \text{ and } \frac{d^2u}{dz^2} = \mathrm{M}\frac{\sin 2x}{2x}\left(\cos 2x - \frac{\sin 2x}{2x}\right),$$

M being the modulus of the logarithms.

Hence, by TAYLOR'S theorem,

$$l = t\,\frac{\sin 2x}{2x}\left\{1 \mp \mathrm{M}t\left(\frac{\sin 2x}{2x} - \cos 2x\right)\right\} \text{ nearly.}$$

Taking the logarithm, this becomes

$$\begin{aligned}\log l &= \log\left(t\,\frac{\sin 2x}{2x}\right) \mp t\left(\frac{\sin 2x}{2x} - \cos 2x\right) \\ &= \log\left(\frac{t \cdot \sin x \cdot \cos x}{x}\right) \mp \frac{t \cdot \sin x \cdot \cos x}{x} \pm t \mp 2t\sin^2 x.\end{aligned}$$

The latter is the better shape for a working formula, because $\log\sin x$ and $\log\cos x$ are found in the same page and line as $\log\tan x$, while $\log\sin 2x$ must be looked for elsewhere. The first term alone is sufficient when x is small; but when x much exceeds $45°$, $\cos 2x$ changes its sign, and even the entire formula is insufficient. The maximum value of the coefficient of t in the second term is 1·0631, corresponding to $x = 78°\,33'\,26''{\cdot}5$. In many cases, where the first term alone is insufficient, a rough interpolation, made at

sight from the following Table, will answer the purpose; it is a Table of the value of $\left(\frac{\sin 2x}{2x}-\cos 2x\right)$ and of its logarithm, from $x=45°$ to $x=90°$.

45°	0·63662	9·80387
50	0·73816	9·86815
55	0·83149	9·91986
60	0·91349	9·96070
65	0·98041	9·99141

70°	1·02910	0·01246
75	1·05658	0·02390
80	1·06216	0·02619
85	1·04334	0·01843
90	1·	0·

The Table shows that, past 45°, the formula

$$\log l=\log\left(\frac{t\sin x\cos x}{x}\right)\mp t$$

is a better approximation than when the $\mp t$ is omitted. It is to be remarked that t is at its minimum for $x=45°$, and increases both towards $x=0$ and $x=90°$. Near the latter limit, where great accuracy is required, we must proceed as follows.

Find the correction for the logarithm of the complement of the arc by the above process, and then find $\log(\frac{1}{2}\pi-y)$ from $\log y$. For this purpose, I observe that $\log y=\log x\pm l$ is equivalent to $y=x.10^{\pm l}$, hence

$$\tfrac{1}{2}\pi-y=\tfrac{1}{2}\pi-x.10^{\pm l}=(\tfrac{1}{2}\pi-x)-x(10^{\pm l}-1).$$

Now, let $\pm A=10^{\pm l}-1$, whence

$$\log(\pm mA)=\log(\pm l)\pm\tfrac{1}{2}l-\frac{1}{12}Ml^2 \text{ nearly, and also}$$

$$\log(\tfrac{1}{2}\pi-y)=\log(\tfrac{1}{2}\pi-x)-\left(\frac{\pm mAx}{\frac{1}{2}\pi-x}\right)-\tfrac{1}{2}M\left(\frac{\pm mAx}{\frac{1}{2}\pi-x}\right)^2-\ldots.$$

It is not often that the third term of either formula will be required.

I have gone into all this detail, because the inverse tangent is continually presenting itself in all these integrations, and because no book that I know shows the proper way of handling it.

23. The following constants are needed for these and similar formulæ:—

$10+\log m=9·63778\ 43113\ 00537,$	$\log M=0·36221\ 56886\ 99463,$
$10+\log 1°=8·24187\ 73675\ 90828,$	$-\log 1°=1·75812\ 26324\ 09172,$
$10+\log 1'=6·46372\ 61172\ 07184,$	$-\log 1'=3·53627\ 38827\ 92816,$
$10+\log 1''=4·68557\ 48668\ 23541,$	$-\log 1''=5·31442\ 51331\ 76459.$

24. As an example of finding the inverse tangent, let it be required to find $\log y$ and $\log(\frac{1}{2}\pi-y)$ from

$$\log\tan y=9·02313\ 50437.$$

Here we must take

$$\log\tan x=9·02303\ 57359$$
$$+t=\quad 9\ 93078;\qquad \therefore\ x=6°\,1'\,10''=21670''$$
$$\tfrac{1}{2}\pi-x=302330''$$

$\log \sin x = 9{\cdot}0206346$	$\log \sin^3 x = 8{\cdot}0413$	$\log 21670 = 4{\cdot}33585\ 89113$
$\log \cos x = 9{\cdot}9975988$	$\log 2 = 0{\cdot}3010$	$\log 1'' = 4{\cdot}68557\ 48668$
$\log t = 5{\cdot}9969834$	$\log t = 5{\cdot}9970$	$\log x = 9{\cdot}02143\ 37781$
ar. co. $\log x = 0{\cdot}9785662$	$\log (2\sin^3 x) = 4{\cdot}3393$	$l = +9\ 85786$
$\log \frac{t \sin 2x}{2x} = 5{\cdot}9937830$	$2\sin^3 x = 21{,}800$	$\log y = 9{\cdot}02153\ 23567$
2nd correction -14	$\frac{t \sin 2x}{2x} = 985{,}8$	
$\log l = 5{\cdot}9937816$	$1007{,}6$	
$\frac{1}{3}l \quad +493$	$-t = -998{,}1$	$\log 302330 = 5{\cdot}48048\ 12441$
$\log m\mathrm{A} = 5{\cdot}9938309$	2nd correction $= 14{,}5$	$\log 1'' = 4{\cdot}68557\ 48668$
$\log x = 9{\cdot}0214338$	The comma cuts off the eighth decimal.	$\log (\frac{1}{2}\pi - x) = 0{\cdot}16605\ 61109$
ar. co. $\log (\frac{1}{2}\pi - x) = 9{\cdot}8339439$		correction $= -70666$
log correction $= 4{\cdot}8492086$		$\log (\frac{1}{2}\pi - y) = 0{\cdot}16604\ 90443$

Verification.—The numbers corresponding to these logarithms of y and of $\frac{1}{2}\pi - y$ are 0·10508 29743 and 1·46571 33525, the sum of which, to the very last figure, is exactly $\frac{1}{2}\pi$.

25. *To find* $\log \frac{y+1}{y-1}$ *from* $\log y$.—Let $\log \frac{y+1}{y-1} = \log \frac{x+1}{x-1} \mp p$, and $\log y = \log x \pm q$; then $\log p = \log \left(\frac{2qx}{x^2-1}\right) \mp \frac{1}{2}q \mp \frac{q}{x^2-1}$, nearly. This formula obviously fails where y is near unity; in this case $\log \frac{y+1}{y-1}$ cannot be had with great accuracy, unless y itself be given absolutely. All the cases of $\int \frac{dy}{y^2-a^2}$ may be included in the above formula by giving proper signs to p and q. It may save trouble to remark that x must not always be taken to the extreme limit of the Table, because $\log (x+1)$ and $\log (x-1)$ have also to be taken out. As an example, let

$\log y = 0{\cdot}36290\ 63835$		
$\log x = 0{\cdot}36285\ 93030$	$x = 2{\cdot}306$,	$x+1 = 3{\cdot}306$, $x-1 = 1{\cdot}306$
$q = +4\ 70805$		$\log (x+1) = 0{\cdot}51930\ 28492$
$\log q = 5{\cdot}6728411$		$\log (x-1) = 0{\cdot}11594\ 31769$
ar. co. $\log (x^2-1) = 9{\cdot}3647540$		sum $= 0{\cdot}63524\ 60261$
$5{\cdot}0375951$	$\frac{q}{x^2-1} = 0{\cdot}0001090$	difference $= 0{\cdot}40335\ 96723$
$\log x = 0{\cdot}3628593$	$\frac{1}{2}q = 135$	$-p = -5\ 02761$
$\log 2 = 0{\cdot}3010300$	2nd corr^n $= 0{\cdot}0001225$	$\log \frac{y+1}{y-1} = 0{\cdot}40330\ 93962$
$5{\cdot}7014844$		
2nd correction $= -1225$		
$\log p = 5{\cdot}7013619$		

This example has been so chosen as to admit of easy verification. In fact $y=2{\cdot}30625$, and $\log\frac{y+1}{y-1}=\log\frac{529}{209}=0{\cdot}40330\ 93959\ 24$. The error is therefore only of three units in the tenth decimal place, where there was no reason to expect accuracy.

26. The only other formulæ which I shall give are the following, for finding the logarithm of a number, and *vice versâ*. They are indispensable where more than seven figures are required.

Let $\log(x\pm h)=\log x\pm k$, then

$$\log k=\log\left(\frac{mh}{x}\right)\mp\frac{1}{2}\frac{mh}{x}\text{ nearly,}$$

$$\log h=\log(\mathrm{M}xk)\pm\frac{1}{2}k\text{ nearly.}$$

The values of $\log m$ and $\log \mathrm{M}$ have been given in paragraph 23.

27. As an example of the application of the method to the evaluation of elliptic integrals of the third class, let us take the integral

$$\int_0^\varphi\frac{d\varphi}{(1-\sin^2\alpha.\sin^2\varphi)\,(1-\sin^2\theta.\sin^2\varphi)^{\frac{1}{2}}}$$

for the values $\alpha=45°$, $\theta=30°$, $\varphi=60°$.

I have selected these values because they can be obtained without reduction or interpolation from the Table of $\Delta(\theta,\varphi)$ which I have given, and also because $\sin^2\alpha=\sin\theta$, and therefore the integral can be reduced to one of the first class, *plus* an inverse tangent, thus admitting of easy verification. For this case

$$z_k=\frac{2}{(1-\sin^2\alpha.\sin^2\varphi)\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}.\sin^2\varphi\right)},$$

$$\int z_k\,d\varphi=\frac{2\sin^2\alpha}{\sin^2\alpha-\sin^2\theta.\cos^2\frac{k\pi}{i}}.\frac{1}{\cos\alpha}\tan^{-1}(\cos\alpha.\tan\varphi)$$

$$-\frac{2\sin^2\theta\cos^2\frac{k\pi}{i}}{\sin^2\alpha-\sin^2\theta.\cos^2\frac{k\pi}{i}}\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\tan^{-1}\left\{\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}\tan\varphi\right\}.$$

Making $\frac{k\pi}{i}$ successively 22° 30′, 45°, 67° 30′, and, for the odd term, 90°, we find, after a few obvious reductions, that eight times the value of the integral is

$$\left\{\frac{17}{3}+\frac{2}{\Delta^2(45°,67\frac{1}{2})}+\frac{2}{\Delta^2(45°,22\frac{1}{2})}\right\}\frac{1}{\cos 45°}\tan^{-1}\{\cos 45°.\tan 60°\}$$

$$-\frac{1}{\cos 30°}\tan^{-1}\{\cos 30°.\tan 60°\}-\frac{2}{3}\frac{1}{\Delta(30°,45°)}\tan^{-1}\{\Delta(30°,45°).\tan 60°\}$$

$$-\frac{\tan^{-1}\{\Delta(30°,67\frac{1}{2}).\tan 60°\}.\cos^2 22\frac{1}{2}}{\Delta^2(45°,67\frac{1}{2}).\Delta(30°,67\frac{1}{2})}-\frac{\tan^{-1}\{\Delta(30°,22\frac{1}{2}).\tan 60°\}.\cos^2 67\frac{1}{2}}{\Delta^2(45°,22\frac{1}{2}).\Delta(30°,22\frac{1}{2})}.$$

As these inverse tangents range generally from 45° to 60°, I computed them by the shortened formula of paragraph 22, namely $\log\left(\frac{t.\sin x.\cos x}{x}\right)\pm t$; this being sufficient to give eight figures of decimals accurately. I found

$$\log\tan^{-1}\{\cos 45°\tan 60°\} = 9{\cdot}94747\ 15296,$$
$$\log\tan^{-1}\{\cos 30°\tan 60°\} = 9{\cdot}99246\ 23739,$$
$$\log\tan^{-1}\{\Delta(30°, 45°)\tan 60°\} = 0{\cdot}00766\ 92607,$$
$$\log\tan^{-1}\{\Delta(30°, 67\tfrac{1}{2}).\tan 60°\} = 9{\cdot}99727\ 33807,$$
$$\log\tan^{-1}\{\Delta(30°, 22\tfrac{1}{2}).\tan 60°\} = 0{\cdot}01665\ 09657.$$

I hence obtained the following values:—

For the positive terms.	For the negative terms.
7·10091 3039	1·13483 2441
4·37212 6152	0·72539 4027
2·70421 6251	1·66839 9478
14·17725 5442	0·16728 4032
3·69590 9978	3·69590 9978
8)10·48134 5464	
1·31016 8183 value required	

A more exact value of the integral, otherwise obtained, is

$$\tfrac{1}{3}\mathrm{F}(30°, 60°)+\tan^{-1}\left(\frac{4\sin 60°}{\sqrt{13}}\right)=1{\cdot}31016\ 8161,$$

which differs from the previous value by 2 units in the eighth decimal place.

28. In order to find how many places ought to have been accurately obtained, I observe that the method followed gives $\mathrm{N}=\frac{13}{16}$, $r=1$, whence

$$\log\left(\frac{1}{r}\sqrt{\mathrm{N}}\right)=9{\cdot}95491=\log\sin 64°\ 20'\ 30''.$$

The corresponding meridional parts are 5086·5, which must be multiplied by $i=8$, giving 40692·0. Referring to the Table in paragraph 14, I find that this nearly corresponds to *ten* places correct, and therefore that the integral ought to be correct to at least that extent. That it is not so, is due to my having curtailed the formula for finding the logarithms of the inverse tangents. But my object was only to give seven decimals correct, and my going beyond that was simply because, with a ten-figure Table, putting down the additional figures gave me less trouble (once I had to use more than seven) than abbreviation would have done. This remark may at first sight seem strange to any one who has not had some practice in using large Tables. But the logarithmic corrections are given in the shape of arithmetical complements: with reference

to the 10th figure, therefore, considered as an integer, the index is right as it stands, and we need not bestow thought on the proper placing of the correction, as we must if we use any other number of figures.

29. If we had been content with five decimals, the calculation would have been very easy, for in that case we might have used six-figure logarithms, and have made $i=4$, thus omitting the terms containing $22°\frac{1}{2}$ and $67°\frac{1}{2}$. We should get

7·10091 3039	1·13483 2441
1·86022 6468	0·72539 4027
4)5·24068 6571	1·86022 6468
1·31017 1643 value required.	

30. It is worth while to notice a case which will sometimes occur, namely (using the notation of the last example), that the values may be so selected as to give, for one of the values of k, $\sin\alpha=\sin\theta.\cos\frac{k\pi}{i}$, and thus each of the terms into which $\int z_1 d\phi$ was divided would become infinite. Of course the difficulty is only apparent; for in this case the proper value is $\int z_1 d\phi=\int\frac{2d\phi}{(1-\sin^2\alpha.\sin^2\phi)^{\frac{1}{2}}}$, of which the integral may be at once found by differentiating the expression $\frac{\sin\phi.\cos\phi}{1-\sin^2\alpha.\sin^2\phi}$.

Section III.—*Extension of the Method.*

In respect of rapid approximation and precision of limit, the foregoing processes leave nothing to be desired, as far as concerns the radical of the square root; but they do not go beyond that. Mr. Sylvester has given an elegant extension of the method to radicals of a higher index, by means of symmetric functions*.

The more general problem before us is that of approximating to the integrals of irrational functions by means of rational substitutions.

Let φ and ψ be functional symbols, and y a function of z; then, that $\varphi(z).y_m$ and $\varphi(z):y_m$ should both be approximations to $\varphi(z)$, depends upon y_m approaching unity as m increases. Assuming that y_m and y_1 are connected by the equation $y_m=\psi(m, y_1)$, our problem is to choose ψ so that, in the first place, the approximation shall be exceedingly rapid, and, in the next place, that $\varphi(z).y_m$ and $\varphi(z):y_m$ shall both (or at least one of them) be thoroughly manageable, and easily integrable. In the case of the approximants already given, the equation $y_m=\psi(m,y_1)$ has been made $\int\frac{dy_m}{1-y_m^2}=m\int\frac{dy_1}{1-y_1^2}$.

I am acquainted with three general methods which effect the object more or less. The first is the obvious one afforded by the Newtonian approximation to the roots of an equation; viz., let a be a first approximate solution, obtained by trial, of the equation $fx=0$, and call $f'x$ the differential coefficient of fx; then a second approximation is

* See the Philosophical Magazine for December 1860, Supplementary Number, vol. xx. p. 525, note A.

$a-\frac{fa}{f'a}=b$; a third approximation will evidently be $b-\frac{fb}{f'b}=c$, and so forth. If we apply this method to the pure equation $x^n=p$, the convergent terms which we obtain are as follows:—

$$b=\frac{(n-1)a^n+p}{na^{n-1}},$$

$$c=\frac{(n-1)\left\{(n-1)a^n+p\right\}^n+n^n.p.a^{n(n-1)}}{n^3a^{n-1}\left\{(n-1)a^n+p\right\}^{n-1}}, \text{ \&c.}$$

The second method is that of the reversion of series; it is sufficiently discussed by ARBOGAST*.

The third method was suggested to me by Mr. CAYLEY's remark that Mr. SYLVESTER's third approximation is a particular case, for $n=2$, of the common form (of the books on the binomial theorem) $\sqrt[n]{N}=\frac{(n+1)N+(n-1)a^n}{(n-1)N+(n+1)a^n}.a$, approximately, a being a first approximation. In order to gain generality, and thereby symmetry, I shall pass from the particular form $\sqrt[n]{N}$ to the more general $\varphi^{-1}N$ by the following Lemma:—

Let $$N=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

and let $x_1, x_2, x_3, x_4 \ldots$ be determined by the system of equations,

$$\left.\begin{aligned} N&=a_0\left(1+\frac{a_1}{a_0}x_1\right)\\ &=a_0\left(1+\frac{a_1}{a_0}x_2\left(1+\frac{a_2}{a_1}x_1\right)\right)\\ &=a_0\left(1+\frac{a_1}{a_0}x_3\left(1+\frac{a_2}{a_1}x_2\left(1+\frac{a_3}{a_2}x_1\right)\right)\right), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

and so forth; also let $N-a_0=\mu$, and

$$1=(\mu-a_1x-a_2x^2-a_3x^3-\ldots.)(\lambda_0+\lambda_1x+\lambda_2x^2+\lambda_3x^3+\ldots.), \quad . \quad . \quad (3.)$$

then $$x_1=\frac{\lambda_0}{\lambda_1},\ x_2=\frac{\lambda_1}{\lambda_2},\ x_3=\frac{\lambda_2}{\lambda_3}\ldots.x_n=\frac{\lambda_{n-1}}{\lambda_n}.$$

For, if we substitute these values in the equations (2.) after placing them in the following form,

$$\left.\begin{aligned} \mu=N-a_0&=a_1x_1\\ &=a_1x_2+a_2x_2x_1\\ &=a_1x_3+a_2x_2x_3+a_3x_3x_2x_1, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

and so forth, we obtain

$$\mu=\frac{a_1\lambda_0}{\lambda_1}=\frac{a_1\lambda_1+a_2\lambda_0}{\lambda_2}=\frac{a_1\lambda_2+a_2\lambda_1+a_3\lambda_0}{\lambda_3}=\text{\&c.}, \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

which are the same equations as we should get by multiplying the two series in (3.) and equating to zero the coefficients of x and of its powers. The coefficient $\lambda_0=\frac{1}{\mu}$, obviously.

* Calcul des Dérivations, pp. 288–296.

The coefficients λ may now be found in a variety of ways; by solving equations (4.) or (5.), by simple division, or by ARBOGAST's processes *. The object of the preceding lemma is to connect the quantities x_n with the coefficients of division and of recurring series. Our results in any way are,

$$x_1=\frac{\mu}{a_1},\ x_2=\frac{a_1\mu}{a_2\mu+a_1^2},$$

$$x_3=\frac{a_2\mu^2+a_1^3\mu}{a_3\mu^3+2a_2a_1\mu+a_1^3},$$

$$x_4=\frac{a_3\mu^3+2a_2a_1\mu^2+a_1^3\mu}{a_4\mu^3+(2a_3a_1+a_2^2)\mu^2+3a_2a_1^2\mu+a_1^4}.$$

If for a_2, a_3, &c. we substitute the coefficients of the binomial theorem, so as to make $N=(a+x)^n$, we obtain

$$a+x_1=\frac{(n-1)a^n+N}{na^n}\cdot a,$$

$$a+x_2=\frac{(n-1)a^n+(n+1)N}{(n+1)a^n+(n-1)N}\cdot a,$$

$$a+x_3=\frac{(n^2-1)a^{2n}+(4n^2+2)a^nN+(n^2-1)N^2}{(n+1)(n+2)a^{2n}+4(n^2-1)a^nN+(n-1)(n-2)N^2}\cdot a.$$

Making $n=2$, we obtain Mr. SYLVESTER's approximants to the square root, and λ_n is then the coefficient of x^n in the development by ascending powers of

$$\frac{1}{(N-a^2)-2ax-x^2};$$

and so far the method agrees with the Newtonian approximation by continued fractions; but from this point the two methods diverge. For $n=3$, λ_n is the coefficient of x^n in the development of

$$\frac{1}{(N-a^3)-3a^2x-3ax^2-x^3};$$

and the successive approximants are

$$\frac{2a^3+N}{3a^3}\cdot a,\ \frac{a^3+2N}{2a^3+N}\cdot a,\ \frac{4a^6+19a^3N+4N^2}{10a^6+16a^3N+N^2}\cdot a,\ \frac{5a^9+45a^6N+30a^3N^2+N^3}{15a^9+51a^6N+15a^3N^2}\cdot a,\ \&c.;$$

while the second approximant obtained by successive substitution is

$$\frac{16a^9+51a^6N+12a^3N^2+2N^3}{36a^9+36a^6N+9a^3N^2}\cdot a.$$

What these methods all effect is simply a rational approximation to the value of y in the equation $\varphi(y,z)=0$. Then, making $y=\frac{du}{dz}$, we have only to integrate in order to find the value of u. They thus constitute a means of approximately solving, in respect of u, differential equations of the form $\varphi\left(z,\frac{du}{dz}\right)=0$; but they do not effect the solu-

* See his 'Calcul des Dérivations,' pp. 26, 29; or DE MORGAN, 'Diff. Calc.' p. 331.

tion of this equation in respect of z, and still less do they solve the more general form $\phi\left(u, z, \frac{du}{dz}\right)=0$.

It may suggest processes of reduction in some cases to remark, that there are many other functions of y_m and $\phi(z)$, which will approximate to $\phi(z)$ as m increases, besides the simple product or quotient of $\phi(z)$ by y_m.

There is one point about these higher approximants, of which a solution, even if accompanied with considerable restrictions, would be extremely desirable,—I mean the resolution of the denominators into factors. I do not suppose that the problem, in its perfectly general form, admits of a compact solution; but any class of cases, of even moderate generality, for which it could be elegantly solved, would probably have very useful applications. The criterion of convergence and the measure of approximation would also have their interest.

Specimen sheet of work for the Table. Arc of 60°.

log sin $22\frac{1}{2}$ =	9·58283	96605	83*	log tan² L =	9·0913166	log cos L =	9·97473	27132
log sin A =	9·93753	06316	96	log s =	5·3160144	c =	+	25545
log sin M =	9·52037	02922	79		4·4073310	log Δ =	9·97473	52677
log sin L =	9·52039	09944		s+				
s =	−2	07021		s tan² L =	−233			
s tan² L =		25546		log c =	4·4073077			

log sin 45° =	9·84948	50021	68*	log tan² L =	9·7781471	log cos L =	9·89794	07883
log sin A =	9·93753	06316	96	log s =	4·1137763	c =	−	7797
log sin M =	9·78701	56338	64		3·8919234	log Δ =	9·89794	00086
log sin L =	9·78701	43344		s+				
s =	+	12995		s tan² L =	+ 21			
s tan² L =		7821		log c =	3·8919255			

log sin $67\frac{1}{2}$ =	9·96561	53459	21*	log tan² L =	10·2501550	log cos L =	9·77806	24352
log sin A =	9·93753	06316	96	log s =	4·7816118	c =	− 1	07593
log sin M =	9·90314	59776	17		5·0317668	log Δ =	9·77805	16759
log sin L =	9·90313	99296		s+				
s =	+	60480		s tan² L =	+168			
s tan² L =	1	07589		log c =	5·0317836			

log tan $22\frac{1}{2}$ =	9·61722	43146	62*	log sin² L =	8·6140815	log cos $22\frac{1}{2}$ =	9·96561	53459 21*
log cos A =	9·69897	00043	36	log t =	5·5451559	log sec L =	0·00911	84784
log tan M =	9·31619	43190	08		4·1591374		9·97473	38243
log tan L =	9·31615	92213		t−		c =	+	14430
t =	+ 3	50877		t sin² L =	+327	log Δ =	9·97473	52673
t sin² L =		14018		log c =	4·1591701			

log tan 45 =	10·00000	00000	00*	log sin² L =	9·3009948	log cos 45 =	9·84948	50021 68*
log cos A =				log t =	5·3429218	log sec L =	0·04845	06017
log tan M =	9·69897	00043	36		4·6439166		9·89793	56039
log tan L =	9·69894	79790		t−		c =	+	44049
t =	+2	20253		t sin² L =	+176	log Δ =	9·89794	00082
t sin² L =		44047		log c =	4·6439342			

log tan $67\frac{1}{2}$ =	10·38277	56853	38*	log sin² L =	9·7730722	log cos $67\frac{1}{2}$ =	9·58283	96605 83*
log cos A =	9·69897	00043	36	log t =	4·7719327	log sec L =	0·19521	55228
log tan M =	10·08174	56896	74		4·5450049		9·77805	51834
log tan L =	10·08175	16044		t−		c =	−	35075
t =	−	59147		t sin² L =	− 24	log Δ =	9·77805	16759
t sin² L =		35075		log c =	4·5450025			

Note.—The entries marked *, and the whole of the letter-press, were printed on the sheets. The letter A on this page corresponds to θ in the Table.

Table of the value of the function $\log \Delta(\theta, \omega)$ or $\log \sqrt{(1-\sin^2\theta \cdot \sin^2\omega)}$ for four values of ω, viz. 22° 30′, 45°, 67° 30′, and 90°.

θ.	$\log \Delta(\theta, 22° 30')$.	$\log \Delta(\theta, 45°)$.	$\log \Delta(\theta, 67° 30')$.	$\log \cos \theta$.	θ.
1	9·99999 03138	9·99996 69274	9·99994 35386	9·99993 38497	1
2	9·99996 12643	9·99986 77197	9·99977 41349	9·99973 53589	2
3	9·99991 28792	9·99970 24074	9·99949 17311	9·99940 44063	3
4	9·99984 52048	9·99947 10407	9·99909 62308	9·99894 07898	4
5	9·99975 83051	9·99917 36910	9·99858 74990	9·99834 42260	5
6	9·99965 22633	9·99881 04507	9·99796 53618	9·99761 43489	6
7	9·99952 71805	9·99838 14325	9·99722 96070	9·99675 07098	7
8	9·99938 31764	9·99788 67716	9·99637 99831	9·99575 27754	8
9	9·99922 03891	9·99732 66254	9·99541 62004	9·99461 99270	9
10	9·99903 89748	9·99670 11740	9·99433 79300	9·99335 14589	10
11	9·99883 91084	9·99601 06211	9·99313 48042	9·99194 65764	11
12	9·99862 09825	9·99525 51957	9·99183 64168	9·99040 43940	12
13	9·99838 48090	9·99443 51515	9·99041 23213	9·98872 39328	13
14	9·99813 08170	9·99355 07689	9·98887 20337	9·98690 41185	14
15	9·99785 92542	9·99260 23560	9·98721 50300	9·98494 37781	15
16	9·99757 03868	9·99159 02501	9·98544 07479	9·98284 16370	16
17	9·99726 44986	9·99051 48183	9·98354 85857	9·98059 63156	17
18	9·99694 18915	9·98937 64594	9·98153 79027	9·97820 63255	18
19	9·99660 28857	9 98817 56066	9·97940 80200	9·97567 00654	19
20	9·99624 78189	9·98691 27246	9·97715 82198	9·97298 58164	20
21	9·99587 70469	9·98558 83197	9·97478 77462	9·97015 17377	21
22	9·99549 09429	9·98420 29331	9·97229 58056	9·96716 58605	22
23	9·99508 98979	9·98275 71478	9·96968 15661	9·96402 60827	23
24	9·99467 43204	9·98125 15899	9·96694 41603	9·96073 01625	24
25	9·99424 46358	9·97968 69297	9·96408 26837	9·95727 57115	25
26	9·99380 12870	9·97806 38852	9·96109 61968	9·95366 01869	26
27	9·99334 47337	9·97638 32245	9·95798 37258	9·94988 08840	27
28	9·99287 54524	9·97464 57677	9·95474 42643	9·94593 49269	28
29	9·99239 39363	9·97285 23905	9·95137 67741	9·94181 92587	29
30	9·99190 06948	9·97100 40265	9·94788 01877	9·93753 06317	30
31	9·99139 62526	9·96910 16705	9·94425 34100	9·93306 55951	31
32	9·99088 11517	9·96714 63813	9·94049 53211	9·92842 04835	32
33	9·99035 59484	9·96513 92852	9·93660 47788	9·92359 14023	33
34	9·98982 12144	9·96308 15797	9·93258 06231	9·91857 42135	34
35	9·98927 75363	9·96097 45359	9·92842 16784	9·91336 45194	35
36	9·98872 55150	9·95881 95031	9·92412 67607	9·90795 76446	36
37	9·98816 57654	9·95661 79121	9·91969 46797	9·90234 86165	37
38	9·98759 89157	9·95437 12781	9·91512 42488	9·89653 21441	38
39	9·98702 56072	9·95208 12065	9·91041 42888	9·89050 25944	39
40	9·98644 64943	9·94974 93945	9·90556 36388	9·88425 39665	40
41	9·98586 22425	9·94737 76364	9·90057 11640	9·87777 98629	41
42	9·98527 35294	9·94496 78273	9·89543 57674	9·87107 34581	42
43	9·98468 10429	9·94252 19663	9·89015 64024	9·86412 74638	43
44	9·98408 54812	9·94004 21611	9·88473 20869	9·85693 40901	44
45	9·98348 75524	9·93753 06317	9·87916 19193	9·84948 50022	45

TABLE

(*continued*).

ϕ.	log Δ (ϕ, 22° 30′).	log Δ (ϕ, 45°).	log Δ (ϕ, 67° 30′).	log cos ϕ.	ϕ.
46°	9·98288 79722	9·93498 97136	9·87344 50980	9·84177 18731	46°
47	9·98228 74657	9·93242 18620	9·86758 09423	9·83378 33303	47
48	9·98168 67640	9·92982 96543	9·86156 89166	9·82551 08951	48
49	9·98108 66053	9·92721 57944	9·85540 86587	9·81694 29168	49
50	9·98048 77326	9·92458 31150	9·84910 00105	9·80806 74967	50
51	9·97989 08943	9·92194 45794	9·84264 30543	9·79887 18039	51
52	9·97929 68415	9·91927 32846	9·83603 81532	9·78934 19787	52
53	9·97870 63285	9·91660 24627	9·82928 59976	9·77946 30249	53
54	9·97812 01111	9·91392 54820	9·82238 76555	9·76921 86852	54
55	9·97753 89457	9·91124 58470	9·81534 46333	9·75859 13013	55
56	9·97696 35878	9·90856 71996	9·80815 89399	9·74756 16513	56
57	9·97639 47918	9·90589 33160	9·80083 31625	9·73610 87645	57
58	9·97583 33090	9·90322 81075	9·79337 05490	9·72420 97077	58
59	9·97527 98869	9·90057 56154	9·78577 51006	9·71183 93361	59
60	9·97473 52675	9·89794 00084	9·77805 16759	9·69897 00043	60
61	9·97420 01874	9·89532 55788	9·77020 61055	9·68557 12291	61
62	9·97367 53742	9·89273 67330	9·76224 53190	9·67160 92909	62
63	9·97316 15478	9·89017 79885	9·75417 74828	9·65704 67649	63
64	9·97265 94174	9·88765 39622	9·74601 21530	9·64184 19615	64
65	9·97216 96810	9·88516 93618	9·73776 04387	9·62594 82593	65
66	9·97169 30239	9·88272 89739	9·72943 51756	9·60931 32999	66
67	9·97123 01175	9·88033 76506	9·72105 11125	9·59187 80116	67
68	9·97078 16179	9·87800 02961	9·71262 51046	9·57357 54170	68
69	9·97034 81644	9·87572 18497	9·70417 63081	9·55432 91617	69
70	9·96993 03790	9·87350 72689	9·69572 63771	9·53405 16846	70
71	9·96952 88643	9·87136 15099	9·68729 96519	9·51264 19176	71
72	9·96914 42028	9·86928 95088	9·67892 33280	9·48998 23640	72
73	9·96877 69551	9·86729 61579	9·67062 76041	9·46593 53400	73
74	9·96842 76594	9·86538 62846	9·66244 57824	9·44033 80750	74
75	9·96809 68303	9·86356 46269	9·65441 43168	9·41299 62305	75
76	9·96778 49569	9·86183 58088	9·64657 27859	9·38367 51767	76
77	9·96749 25025	9·86020 43157	9·63896 37732	9·35208 80330	77
78	9·96721 99032	9·85867 44678	9·63163 26324	9·31787 89102	78
79	9·96696 75670	9·85725 03958	9·62462 71226	9·28059 88450	79
80	9·96673 58731	9·85593 60134	9·61799 68925	9·23967 02300	80
81	9·96652 51705	9·85473 49940	9·61179 28070	9·19433 24413	81
82	9·96633 57778	9·85365 07456	9·60606 61106	9·14355 53039	82
83	9·96616 79819	9·85268 63874	9·60086 74357	9·08589 44712	83
84	9·96602 20377	9·85184 47281	9·59624 56795	9·01923 45656	84
85	9·96589 81672	9·85112 82461	9·59224 67793	8·94029 60083	85
86	9·96579 65594	9·85053 90708	9·58891 24439	8·84358 45184	86
87	9·96571 73697	9·85007 89667	9·58627 88989	8·71880 01636	87
88	9·96566 07187	9·84974 93212	9·58437 57166	8·54281 91639	88
89	9·96562 66934	9·84955 11322	9·58322 48116	8·24185 53184	89
90	9·96561 53459	9·84948 50022	9·58283 96696	—log. infin.	90

XXX. *On Simultaneous Differential Equations of the First Order in which the Number of the Variables exceeds by more than one the Number of the Equations.* By George Boole, *F.R.S., Professor of Mathematics in Queen's College, Cork.*

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It is a fundamental proposition of analysis that a system of n differential equations of the first order containing $n+1$ variables admits of n integrals, each of which is expressed by a function of the variables equated to an arbitrary constant.

But when a system of n differential equations of the first order connects $n+r$ variables, r being greater than unity, no existing theory assigns in a general manner the number of theoretically possible integrals of the above species, or shows us how to discover them. Yet such cases are of great importance.

I wish to develope here the theory of a method for the solution of the above classes of equations, which was published by me in the 'Proceedings of the Royal Society' for March 6th of the present year, and which enables us to assign the number of theoretically possible integrals, and to reduce their discovery to the solution of a system of simultaneous differential equations equal in number to the number of integrals, and expressible as exact differential equations.

The solution of the problem as thus reduced may be effected by known methods, but I have thought it desirable to discuss this part of the subject also in direct sequence to the other, and in conformity with its method.

Of the Connexion between ordinary and partial Differential Equations.

It has been found convenient, in researches bearing upon the general theory of differential equations, to use the term 'integral' in two distinct senses, viz. to denote, as above, a relation satisfying the differential equation or system of equations, and expressed by the equating of a function of the variables to a constant, and to denote the function itself. The particular sense intended will always be shown by the connexion.

With this convention two systems of differential equations will be said to be equivalent when they have in either of the above senses (and the one implies the other) the same system of integrals. This will explain the meaning of the following proposition.

Proposition I.—*A system of n ordinary differential equations of the first order connecting $n+r$ variables may be converted into an equivalent system of r linear partial differential equations of the first order.*

Let $x_1, x_2 \ldots x_{n+r}$ be the variables, then the supposed given system of differential equa-

tions may, by algebraic solution with respect to $dx_1, dx_2, \ldots dx_n$, be reduced to the form

$$\left.\begin{aligned} dx_1 &= A_{11}dx_{n+1} + A_{12}dx_{n+2} \ldots + A_{1r}dx_{n+r} \\ dx_2 &= A_{21}dx_{n+1} + A_{22}dx_{n+2} \ldots + A_{2r}dx_{n+r} \\ &\ldots\ldots\ldots\ldots\ldots \\ dx_n &= A_{n1}dx_{n+1} + A_{n2}dx_{n+2} \ldots + A_{nr}dx_{n+r} \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(I.)}$$

the coefficients A_{11}, A_{12}, &c. being functions of the variables.

Let $P=c$ be an integral of the system. Then

$$\frac{dP}{dx_1}dx_1 + \frac{dP}{dx_2}dx_2 \ldots + \frac{dP}{dx_{n+r}}dx_{n+r} = 0.$$

Substituting in this equation the values of $dx_1, dx_2, \ldots dx_n$ given by (I.), we have

$$\begin{aligned} &\left(\frac{dP}{dx_{n+1}} + A_{11}\frac{dP}{dx_1} + A_{21}\frac{dP}{dx_2} \ldots + A_{n1}\frac{dP}{dx_n}\right)dx_{n+1} \\ &+\left(\frac{dP}{dx_{n+2}} + A_{12}\frac{dP}{dx_1} + A_{22}\frac{dP}{dx_2} \ldots + A_{n2}\frac{dP}{dx_n}\right)dx_{n+2} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+\left(\frac{dP}{dx_{n+r}} + A_{1r}\frac{dP}{dx_1} + A_{2r}\frac{dP}{dx_2} \ldots + A_{nr}\frac{dP}{dx_n}\right)dx_{n+r} = 0. \end{aligned}$$

As the differentials $dx_{n+1}, dx_{n+2}, \ldots dx_{n+r}$ are now independent, we have, on equating their coefficients separately to 0,

$$\left.\begin{aligned} \frac{dP}{dx_{n+1}} + A_{11}\frac{dP}{dx_1} + A_{21}\frac{dP}{dx_2} \ldots + A_{n1}\frac{dP}{dx_n} &= 0, \\ \frac{dP}{dx_{n+2}} + A_{12}\frac{dP}{dx_1} + A_{22}\frac{dP}{dx_2} \ldots + A_{n2}\frac{dP}{dx_n} &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \frac{dP}{dx_{n+r}} + A_{1r}\frac{dP}{dx_1} + A_{2r}\frac{dP}{dx_2} \ldots + A_{nr}\frac{dP}{dx_n} &= 0, \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(II.)}$$

a system of r linear partial differential equations, the common integrals of which will be the integrals of the system (I.). We say 'the *common* integrals of which,' because in fact these equations express the conditions to all of which P must be subject in order that $P=c$ may be an integral of the system (I.).

The formal connexion between the systems (I.) and (II.) deserves to be carefully noticed. The several partial differential equations of the system (II.) may be formed by inspection from the columns in the right-hand member of the system (I.) by the following rule. For the differential dx_{n+i} in any column write the differential coefficient $\frac{dP}{dx_{n+i}}$, to this add the series of differential coefficients $\frac{dP}{dx_1}, \frac{dP}{dx_2}, \ldots \frac{dP}{dx_n}$ multiplied in succession by the descending coefficients of the column, and equate the final result to 0.

The symmetrical form of the equation

$$\frac{dP}{dx_1}dx_1 + \frac{dP}{dx_2}dx_2 \ldots + \frac{dP}{dx_{n+r}}dx_{n+r} = 0$$

shows that by an exactly similar rule any system of n partial differential equations of the first order, the terms of which consist of the differential coefficients of P multiplied by functions of the independent variables $x_1, x_2, \dots x_{n+r}$ may be converted into an equivalent system of r common differential equations of the first order.

For the proposed system of partial differential equations is by algebraic reduction expressible in the form

$$\left.\begin{aligned}
\frac{dP}{dx_1} &= A_{11}\frac{dP}{dx_{n+1}} + A_{12}\frac{dP}{dx_{n+2}} \dots + A_{1r}\frac{dP}{dx_{n+r}},\\
\frac{dP}{dx_2} &= A_{21}\frac{dP}{dx_{n+1}} + A_{22}\frac{dP}{dx_{n+2}} \dots + A_{2r}\frac{dP}{dx_{n+r}},\\
\frac{dP}{dx_n} &= A_{n1}\frac{dP}{dx_{n+1}} + A_{n2}\frac{dP}{dx_{n+2}} \dots + A_{nr}\frac{dP}{dx_{n+r}}.
\end{aligned}\right\} \quad \dots\dots\dots \text{(III.)}$$

If the values of $\frac{dP}{dx_1}, \frac{dP}{dx_2}, \dots \frac{dP}{dx_n}$ in this system be substituted in the previous general equation, and the coefficients of the differential coefficients $\frac{dP}{dx_{n+1}}, \frac{dP}{dx_{n+2}}, \dots \frac{dP}{dx_{n+r}}$ in the result be separately equated to 0, we shall have

$$\left.\begin{aligned}
dx_{n+1} + A_{11}\,dx_1 + A_{21}\,dx_2 \dots + A_{n1}\,dx_n &= 0,\\
dx_{n+2} + A_{12}\,dx_1 + A_{22}\,dx_2 \dots + A_{n2}\,dx_n &= 0,\\
dx_{n+r} + A_{1r}\,dx_1 + A_{2r}\,dx_2 \dots + A_{nr}\,dx_n &= 0.
\end{aligned}\right\} \quad \dots\dots\dots \text{(IV.)}$$

These equations may in like manner be formed by inspection from the columns of the second member of (III.), by writing for $\frac{dP}{dx_{n+i}}$ in any column dx_{n+b} adding to this $dx_1, dx_2, \dots dx_n$ multiplied by the descending coefficients of the column, and equating the final sum to 0. The rule for the one case differs from that for the other only in that differentials take the place of differential coefficients.

As an objection may be felt to the legitimacy of that step of the above process in which, the differential coefficients $\frac{dP}{dx_1}, \frac{dP}{dx_2}, \dots \frac{dP}{dx_n}$ being eliminated, the coefficients of the remaining ones are separately equated to 0, I will point out another mode of procedure which leads to the same result, and which is founded upon LAGRANGE'S method of solution. Let the equations of the system (III.) be added together after having been multiplied respectively by $\lambda_1, \lambda_2, \dots \lambda_n$, which are to be regarded as indeterminate functions of the variables $x_1, x_2, \dots x_{n+r}$. The result will be a linear partial differential equation of the first order, of which the Lagrangian auxiliary system of ordinary differential equations will be

$$\frac{dx_1}{\lambda_1} = \frac{dx_2}{\lambda_2} \dots = \frac{dx_n}{\lambda_n} = \frac{-dx_{n+1}}{A_{11}\lambda_1 \dots + A_{n1}\lambda_n} \dots = \frac{-dx_{n+r}}{A_{1r}\lambda_1 \dots + A_{nr}\lambda_n}.$$

Hence, eliminating $\lambda_1, \lambda_2, \dots \lambda_n$, or more strictly speaking the ratios

$$\frac{\lambda_1}{\lambda_n}, \frac{\lambda_2}{\lambda_n}, \dots \frac{\lambda_{n-1}}{\lambda_n},$$

we have

$$A_{11}\,dx_1 \ldots + A_{n1}\,dx_n = -dx_{n+1},$$
$$\ldots\ldots\ldots\ldots$$
$$A_{1r}\,dx_1 \ldots + A_{nr}\,dx_n = -dx_{n+r},$$

which agrees with the system (IV.).

Of the Determination of the Number of Integrals of a system of Differential Equations of the First Order and Degree.

We still suppose the given system of differential equations to be expressed under the general form (I.), and reduced by Prop. I. to the equivalent partial differential system (II.).

Now if for the expression of that system we introduce a series of symbols $\Delta_1, \Delta_2, \ldots \Delta_r$ defined as follows, viz.

$$\Delta_i = \frac{d}{dx_{n+i}} + A_{1i}\frac{d}{dx_1} + A_{2i}\frac{d}{dx_2} \cdots + A_{ni}\frac{d}{dx_n}, \quad \ldots\ldots\ldots \quad (1.)$$

the system will assume the form

$$\Delta_1 P = 0,\ \Delta_2 P = 0, \ldots \Delta_r P = 0, \quad \ldots\ldots\ldots\ldots \quad (2.)$$

and we shall now establish the following proposition

Proposition II.—*If* $\Delta_i P = 0$, $\Delta_j P = 0$ *represent any two linear partial differential equations of the system* (II.), *then will the equation of which the symbolical expression is*

$$(\Delta_i \Delta_j - \Delta_j \Delta_i) P = 0 \quad \ldots\ldots\ldots\ldots \quad (3.)$$

also be a linear partial differential equation of the first order, and it will be satisfied by all the common integrals of the two equations from which it is formed.

For, representing any one of the quantities $x_1, x_2, \ldots x_n$ by x, and any function of those quantities by X, Δ_i consists of a series of terms of the form $X\frac{d}{dx}$. Again, representing any one of the same series of quantities $x_1, x_2, \ldots x_n$ by y, and any function of them by Y, Δ_j will consist of terms of the form $Y\frac{d}{dy}$. Hence $(\Delta_i\Delta_j - \Delta_j\Delta_i)P$ will consist of terms of the form

$$\left(X\frac{d}{dx}Y\frac{d}{dy} - Y\frac{d}{dy}X\frac{d}{dx}\right)P.$$

Effecting the differentiations, this term becomes

$$X\frac{dY}{dx}\frac{dP}{dy} + XY\frac{d^2P}{dxdy} - Y\frac{dX}{dy}\frac{dP}{dx} - YX\frac{d^2P}{dxdy},$$

or

$$X\frac{dY}{dx}\frac{dP}{dy} - Y\frac{dX}{dy}\frac{dP}{dx},$$

which involves only the first differential coefficients of P. Hence (3.) will be a linear partial differential equation of the first order.

Hence also the ultimate form of (3.) will be the same as if Δ_i, when operating on $\Delta_j P$, operated only on the coefficients $A_{ij} \ldots A_{nj}$ involved in Δ_j, and *vice versâ*. Thus the

ultimate form of (3.) will be

$$(\Delta_i A_{ij}-\Delta_j A_{ii})\frac{dP}{dx_1}+(\Delta_i A_{2j}-\Delta_j A_{2i})\frac{dP}{dx_2}\ldots\ldots+(\Delta_i A_{nj}-\Delta_j A_{ni})\frac{dP}{dx_n}=0. \quad . \quad . \quad . \quad (4.)$$

Secondly, the equation $(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$ will be satisfied by any common integral of $\Delta_i P=0$ and $\Delta_j P=0$.

For let $\varphi=c$ be a common integral of the latter equations. Then, identically,

$$\Delta_i\varphi=0, \quad \Delta_j\varphi=0;$$

therefore, since Δ_j and Δ_i involve only operations of differentiation together with algebraic ones,

$$\Delta_j\Delta_i\varphi=0, \quad \Delta_i\Delta_j\varphi=0;$$
$$\therefore \quad \Delta_i\Delta_j\varphi-\Delta_j\Delta_i\varphi=0;$$

whence (3.) is also identically satisfied.

PROPOSITION III.—*If, by the successive application of Prop.* II., *and by permitted processes of algebraic elimination, we derive from the system of partial differential equations*

$$\Delta_1 P=0, \ \Delta_2 P=0, \ldots\ldots \Delta_r P=0,$$

into which the given system of differential equations has been converted, a final system of partial differential equations which, while including the above system, shall be such that the application of Prop. II. *to any pair of the equations contained shall lead only to an identity, then the number of integrals of the given system of differential equations will be equal to the number of variables they contain, diminished by the number of partial differential equations of the above final system.*

The developed form of the system

$$\Delta_1 P=0, \ \Delta_2 P=0, \ \ldots \ \Delta_r P=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

is the following:—

$$\left.\begin{aligned}
&\frac{dP}{dx_{n+1}}+A_{11}\frac{dP}{dx_1}\ldots+A_{n1}\frac{dP}{dx_n}=0,\\
&\frac{dP}{dx_{n+2}}+A_{12}\frac{dP}{dx_1}\ldots+A_{n2}\frac{dP}{dx_n}=0,\\
&\quad\ldots\ldots\ldots\ldots\\
&\frac{dP}{dx_{n+r}}+A_{1r}\frac{dP}{dx_1}\ldots+A_{nr}\frac{dP}{dx_n}=0.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Comparing these with (4.), Prop. II., which is the developed form of the equation $(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$, we see that the latter equation is necessarily algebraically independent of the above system; for no equation derived from that system by algebraic processes could be free, as (4.) is, from all the differential coefficients $\frac{dP}{dx_{n+1}}\ldots\frac{dP}{dx_{n+r}}$.

Again, as $(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$ is satisfied by all the common integrals of $\Delta_i P=0$ and $\Delta_j P=0$, it follows that the system of $r+1$ equations,

$$\Delta_1 P=0, \ldots \Delta_r P=0, \ (\Delta_i\Delta_j-\Delta_j\Delta_i)P=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

will be satisfied by all the common integrals of the system (1.). To this system of $r+1$

equations we can also give a form analogous to the developed form of the system (2.). It will be noticed that the differential coefficients $\frac{dP}{dx_{n+1}} \ldots \frac{dP}{dx_{n+r}}$ appear there, each in only one equation, and each with the coefficient unity. Now let the last equation of (3.) in its developed form be divided by the coefficient of $\frac{dP}{dx_n}$, and let also the value of $\frac{dP}{dx_n}$ which it gives be substituted in the other equations of (3.); then we shall have in the whole a system of $n+1$ equations possessing the same general character as the system (2.). To this new system the same procedure may be applied, viz. the genesis of a new equation by means of Prop. II., and the transference of another differential coefficient $\frac{dP}{dx_{n-1}}$ to the list of those which form the respective *first* terms of the equations of the system. We will suppose this procedure to have been repeated until a system composed of m partial differential equations *such that the further application of Prop.* II. *leads only to identities* has been formed. If $n+r-m=p$, that system will be of the form

$$\left.\begin{aligned} &\frac{dP}{dx_{p+1}}+H_{11}\frac{dP}{dx_1}\ldots+H_{p1}\frac{dP}{dx_p}=0,\\ &\frac{dP}{dx_{p+2}}+H_{12}\frac{dP}{dx_1}\ldots+H_{p2}\frac{dP}{dx_p}=0,\\ &\qquad\ldots\ldots\ldots\ldots\\ &\frac{dP}{dx_{p+m}}+H_{1m}\frac{dP}{dx_1}\ldots+H_{pm}\frac{dP}{dx_p}=0. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4.)$$

And if, in analogy with former notation, we write

$$\frac{d}{dx_{p+i}}+H_{1i}\frac{d}{dx_1}+H_{2i}\frac{d}{dx_2}\ldots+H_{pi}\frac{d}{dx_p}=\Delta_i,$$

it will take the symbolical form

$$\ldots\ldots \Delta_1 P=0,\ \Delta_2 P=0,\ \ldots\ \Delta_m P=0;$$

but it will differ from all former systems of equations in that all the conditions represented by

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0 \quad \ldots\ldots\ldots\ldots \quad (5.)$$

will be identically satisfied,—satisfied, in consequence not of any ascertained peculiarity of the integral P, but of the constitution of the system of symbols $\Delta_1, \Delta_2, \ldots \Delta_m$.

The course of argument has shown that the common integrals of the system (4.) will be identical with those of the parent system (2.). Now we shall show that the existence of the condition (5.) renders the integration theoretically possible; that the system of p ordinary differential equations into which, by Prop. II., the final system of partial differential equations (4.) is resolvable admits of exactly p integrals. As $p=n+r-m$, this is to say that the number of integrals is equal to the number of original variables diminished by the number of final partial differential equations.

The proof of this will consist of two parts:—1st. It will be shown that, if a system of p integrals exists, the conditions represented by (8.) will be identically satisfied.

2ndly. It will be shown that, when the conditions represented by (5.) are identically satisfied, the solution either of the final system of partial differential equations (4.), or of the corresponding system of ordinary differential equations, by a system of p integrals is theoretically possible.

It will follow from these conjoined, that the number of actually existing integrals is exactly p.

1st. The system of ordinary differential equations corresponding to (4.) may be expressed in the form

$$\left.\begin{aligned} dx_1 &= H_{11}dx_{p+1} + H_{12}dx_{p+2} \ldots + H_{1m}dx_{p+m}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ dx_p &= H_{p1}dx_{p+1} + H_{p2}dx_{p+2} \ldots + H_{pm}dx_{p+m}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6.)$$

Now suppose this system to have p integrals. Then, by means of these, $x_1, x_2, \ldots x_p$ can be eliminated from the coefficients H_{11}, &c. in the second members, which will thus become exact differentials of those functions of the variables $x_{p+1} \ldots x_{p+m}$ which express the values of $x_1, \ldots x_p$. Hence we shall have the system of conditions

$$\frac{d}{dx_{p+i}}H_{kj} = \frac{d}{dx_{p+j}}H_{ki}, \quad \ldots\ldots\ldots\ldots \quad (7.)$$

k representing any integer from 1 to p, and i, j any integers from 1 to m, and the bracketed symbols of differentiation referring to H_{kj}, H_{ki} *as transformed*. Hence, the unbracketed symbols referring to the prior state of the functions, we have

$$\begin{aligned} \left(\frac{d}{dx_{p+i}}\right) &= \frac{d}{dx_{p+i}} + \frac{dx_1}{dx_{p+i}}\frac{d}{dx_1} \ldots + \frac{dx_p}{dx_{p+i}}\frac{d}{dx_p} \\ &= \frac{d}{dx_{p+i}} + H_{1i}\frac{d}{dx_1} \ldots + H_{pi}\frac{d}{dx_p} \\ &= \Delta_i. \end{aligned}$$

In the same way

$$\left(\frac{d}{dx_{p+j}}\right) = \Delta_j;$$

so that (7.) becomes

$$\Delta_i H_{kj} - \Delta_j H_{ki} = 0. \quad \ldots\ldots\ldots\ldots \quad (8.)$$

Now if we construct, in analogy with (4.), Prop. II., the developed form of the conditional equation (5.) of the present section, we shall have

$$(\Delta_i H_{ij} - \Delta_j H_{ii})\frac{dP}{dx_1} \ldots + (\Delta_i H_{pj} - \Delta_p H_{pi})\frac{dP}{dx_p} = 0,$$

or, Σ denoting summation from $k=1$ to $k=p$,

$$\Sigma(\Delta_i H_{kj} - \Delta_j H_{ki})\frac{dp}{dx_k} = 0. \quad \ldots\ldots$$

Now as by (8.) the coefficients vanish identically, the equation, and generally the system of conditional equations of which it is the type, will be identically satisfied.

2ndly. We proceed to show that the system of m linear partial differential equations (7.), represented under the form

$$\Delta_1 P=0,\ \Delta_2 P=0,\ \ldots\ \Delta_m P=0,$$

and satisfying identically the system of conditions represented by

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0,$$

will admit of p integrals expressing distinct values of P; and the system of ordinary differential equations (6.) corresponding to the above system of partial differential equations will be expressible as a system of exact differential equations, and will by integration give the above systems of integrals.

Beginning with the first partial differential equation of the system (4.), and forming the corresponding Lagrangian system of ordinary differential equations

$$dx_{p+1}=\frac{dx_1}{H_{11}}\ldots=\frac{dx_p}{H_{p1}},$$

$$dx_{p+2}=0,\quad dx_{p+m}=0,$$

we see that the integrals of this system will be of the form

$$u_1\ =c_1\quad \ldots u_p\ =c_p,$$

$$x_{p+2}=c_{p+2}\ldots x_{p+m}=c_{p+m},$$

$u_1\ldots u_p$ being functions of all the variables $x_1,\ldots x_{p+m}$, among which, by virtue of the integrals of the second line, $x_{p+2},\ldots x_{p+m}$ may be regarded as constant. The general integral will be

$$F(u_1,\ldots u_p,\ x_{p+2},\ldots x_{p+m})=0,$$

the form of F being perfectly arbitrary.

Now the general form of *any* equation of the system (4.) is

$$\frac{dP}{dx_{p+i}}+H_{1i}\frac{dP}{dx_1}\ldots+H_{pi}\frac{dP}{dx_p}=0\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (9.)$$

Let us transform this by assuming

$$u_1\ldots u_p,\ x_{p+1}\ldots x_{p+m}$$

as independent variables. Then referring the right-hand members of the following equations to the new, the left-hand to the old system of variables, we have

$$\frac{dP}{dx_{p+i}}=\frac{dP}{dx_{p+i}}+\frac{dP}{du_1}\frac{du_1}{dx_{p+i}}\ldots+\frac{dP}{du_p}\frac{du_p}{dx_{p+i}},$$

$$\frac{dP}{dx_1}\ =\ \frac{dP}{du_1}\frac{du_1}{dx_1}\ \ldots+\frac{dP}{du_p}\frac{du_p}{dx_1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{dP}{dx_p}\ =\ \frac{dP}{du_1}\frac{du_1}{dx_p}\ \ldots+\frac{dP}{du_p}\frac{du_p}{dx_p};$$

whence, substituting in (9.),

$$\frac{dP}{dx_{p+i}}+\frac{dP}{du_1}\left(\frac{du_1}{dx_{p+i}}+H_{1i}\frac{du_1}{dx_1}\ldots+H_{pi}\frac{du_1}{dx_p}\right)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{dP}{du_p}\left(\frac{du_p}{dx_{p+i}}+H_{1i}\frac{du_p}{dx_1}\ldots+H_{pi}\frac{du_p}{dx_p}\right)=0,$$

or

$$\frac{dP}{dx_{p+i}}+(\Delta_i u_1)\frac{dP}{du_1}\ldots+(\Delta_i u_p)\frac{dP}{du_p}=0; \quad\ldots\ldots\ldots \quad (10.)$$

and in this equation we may give to i the successive values $1, 2, \ldots p$. If, then, we write

$$\frac{d}{dx_{p+i}}+(\Delta_i u_1)\frac{d}{du_1}\ldots+(\Delta_i u_p)\frac{d}{du_p}=\Delta'_i,$$

we see that the proposed transformation will have the effect of converting

$$\Delta_1,\ \Delta_2, \ldots \Delta_p$$

into

$$\Delta'_1,\ \Delta'_2, \ldots \Delta'_p$$

respectively, and the system of partial differential equations into

$$\Delta'_1 P=0,\ \Delta'_2 P=0, \ldots \Delta'_p P=0.$$

The developed form of the first of these equations is

$$\frac{dP}{dx_{p+1}}+(\Delta_1 u_1)\frac{dP}{du_1}\ldots+(\Delta_1 u_p)\frac{dP}{du_p}=0.$$

But since $u_1, \ldots u_p$ are integrals of $\Delta_1 P=0$, we have

$$\Delta_1 u_1=0, \ldots \Delta_1 u_p=0,$$

so that the equation $\Delta'_1 P=0$ reduces to

$$\frac{dP}{dx_{p+1}}=0.$$

We learn from this that x_{p+1} will not explicitly appear in P after the transformation which introduces $u_1, \ldots u_p$.

The developed form of the remaining $p-1$ equations represented by (10.) will be

$$\left.\begin{aligned}&\frac{dP}{dx_{p+2}}+(\Delta_2 u_1)\frac{dP}{du_1}\ldots+(\Delta_2 u_p)\frac{dP}{du_p}=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &\frac{dP}{dx_{p+m}}+(\Delta_m u_1)\frac{dP}{du_1}\ldots+(\Delta_m u_p)\frac{dP}{du_p}=0;\end{aligned}\right\}\quad\ldots\ldots\ldots \quad (11.)$$

and we shall next show that the variable x_{p+1} will not present itself in the coefficients $(\Delta_s u_1)$, &c.

The general form of such coefficients is $\Delta_i u_s$, where i has any value from 2 to m, and s any value from 1 to p.

Now if in (5.), which is true independently of the nature of the function P, we make

$j=1$ and $P=u_s$, we have

$$(\Delta_i\Delta_1-\Delta_1\Delta_i)u_s=0.$$

But $\Delta_1 u_s=0$; therefore, by the above,

$$\Delta_i\Delta_1 u_s=0.$$

Now $\Delta_i u_s$ is expressible at most as a function of $u_1, \ldots u_p, x_{p+1} \ldots x_{p+m}$. But this transformation converts, as has been seen, Δ_1 into $\frac{d}{dx_{p+1}}$. Thus we have

$$\frac{d}{dx_{p+1}}\Delta_i u_s=0,$$

so that $\Delta_i u_s$ is free from x_{p+1}. Thus the system (11.) is free from x_{p+1}.

Lastly, since by the above transformation $\Delta_2 \ldots \Delta_p$ are converted into $\Delta_2' \ldots \Delta_p'$, the system of conditions $(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$ is converted into $(\Delta_i'\Delta_j'-\Delta_j'\Delta_i')P=0$.

It is thus seen that the system of p partial differential equations

$$\Delta_1 P=0,\ \Delta_2 P=0, \ldots \Delta_p P=0,$$

containing $p+m$ independent variables $x_1, x_2 \ldots x_{p+m}$, and satisfying the conditions

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0,$$

is convertible into a system of $p-1$ partial differential equations,

$$\Delta_2' P=0,\quad \Delta_3' P=0,\quad \Delta_p' P=0,$$

containing $p+m-1$ independent variables $u_1 \ldots u_p, x_{p+2} \ldots x_{p+m}$, and satisfying the condition

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0.$$

And as this system possesses the same character as that upon which the previous transformation depended, it will admit of transformation into a system of $p-2$ partial differential equations containing $p+m-2$ independent variables; and so on until we arrive at a single final partial differential equation containing $p+1$ independent variables, and having therefore p distinct integrals, which will be the common integrals of the primary system of partial differential equations as well as of the system of ordinary differential equations to which they correspond.

Cor. The property of the coefficients $\Delta_i u_1$, &c. of the system (11.), of being free from the variable x_{p+1}, enables us, by properly determining the integrals of the partial differential equation $\Delta_1 P=0$, to reduce the system to a form of great simplicity.

Let $\Delta_i u_j$ be any one of those coefficients. Its developed form is

$$\left(\frac{d}{dx_{p+i}}+H_{i1}\frac{d}{dx_1} \ldots +H_{pi}\frac{d}{dx_p}\right)u_j \quad \ldots\ldots\ldots \quad (12.)$$

Now as this expression will, after the performance of the differentiations, be free from x_{p+1}, and as the differentiations are none of them with respect to x_{p+1}, we can give to x_{p+1} in it any particular value before differentiation without affecting the final result. Let us then suppose that in $H_{i1} \ldots H_{pi}$, and in u_j, x_{p+1} is made equal to 0. Now it is possible so to determine the integrals u_j as functions of the variables $x_1, \ldots x_{p+1}$, that

when $x_{p+1}=0$ each u_j shall reduce to x_j. For this purpose it is only necessary to choose as arbitrary constants a set of arbitrary values of $x_1, x_2, \dots x_p$ corresponding to $x_{p+1}=0$. Let $x'_1, x'_2, \dots x'_p$ be such arbitrary constants, and let the given system of integrals be reduced to the form

$$u_1=x'_1, \dots u_p=x'_p,$$

and the functions $u_1, \dots u_p$ will possess the required property. Changing, then, each u_j into x_j, the expression (12.) reduces to H_{p1}, and it only remains to express this in terms of $u_1, \dots u_p$; which, as $x_{p+1}=0$, is done by merely changing $x_1 \dots x_p$ into $u_1, \dots u_p$.

Thus the system (11.) is reduced to

$$\frac{dP}{dx_{p+1}}+(H_{12})\frac{dP}{dx_1}\dots+(H_{p2})\frac{dP}{dx_p}=0,$$

. . . . . .

$$\frac{dP}{dx_{p+m}}+(H_{1m})\frac{dP}{dx_1}\dots+(H_{pm})\frac{dP}{dx_p}=0,$$

where the brackets denote that in the enclosed portion x_{p+1} is to be made 0, and $x_1, x_2, \dots x_p$ converted into $u_1, u_2, \dots u_p$.

Now this form is identical, the above conversion of letters excepted, with that of the system (4.), omitting that equation of the latter system by the integration of which the forms of $u_1, \dots u_p$ are determined.

It follows from the above that, obtaining the integrals of $\Delta_1 P=0$ in such a form that the arbitrary constants shall represent the arbitrary values of $x_1 \dots x_p$ when $x_{p+1}=0$, and representing the functions which are equal to those arbitrary constants by $x'_1, x'_2, \dots x'_p$, then if in the remaining equations $\Delta_2 P=0 \dots \Delta_m P=0$ we change $x_1, \dots x_p, \frac{d}{dx'}, \dots \frac{d}{dx_p}$ to $x'_1, \dots x'_p, \frac{d}{dx'_1}, \dots \frac{d}{dx'_p}$, and x_{p+1} and $\frac{dP}{dx_{p+1}}$ to 0, the common integrals of the transformed system of $p-1$ will be the same as those of the previous system of p partial differential equations. In the same way a third system of $p-2$ partial differential equations may be formed, and so on, till we obtain a single final partial differential equation which will have the common integrals of the parent system. By this method, which is due to JACOBI and NUTANI, all the labour of the successive transformations is avoided. The successive integrals thus introduced are termed '*Haupt-integrale*.'

Instead of applying the foregoing methods, general or particular, to the final system of partial differential equations, we may apply it to the solution of the corresponding final system of ordinary differential equations. In this case they would really represent the method of solution known as the variation of parameters, and the conditions $(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$ would secure the sufficiency of that method. If in the system of ordinary differential equations (6.) we regard $x_{p+2}, \dots x_{p+m}$ as constant, we get

$$\left.\begin{aligned} dx_1-H_{11}dx_{p+1}&=0,\\ &\dots\dots\dots\\ dx_p-H_{p1}dx_{p+1}&=0. \end{aligned}\right\} \quad \dots\dots\dots\dots \quad (13.)$$

Integrate this in the form

$$u_1=c_1, \dots u_p=c_p;$$

then, treating $c_1, \dots c_p$ as functions of the variables before regarded as constant, and endeavouring to satisfy the unreduced equations (6.), we obtain, in virtue of the above conditions, a system of differential equations equal in number to the system given, but containing one variable fewer. The system (13.) by which the forms of $u_1, \dots u_p$ are here determined, is the Lagrangian auxiliary of the first partial differential equation $\Delta_1 P=0$ integrated in the other method; and so in each subsequent stage. And with respect to the other parts of the process, it obviously makes no difference whether we take as the new variables $u_1, \dots u_p$, or $c_1, \dots c_p$ under the condition (necessarily involved in the method of the variation of parameters) that they shall after integration be replaced by $u_1, \dots u_p$. But it would not have sufficed simply to refer the solution of the final system of ordinary differential equations to the method of the variation of parameters, first, because the necessity and sufficiency of the conditions which form the ground of that method and are the warrant of its success were to be shown; secondly, because the connexion of the systems of ordinary differential equations which arise in the method of parameters with the successive partial differential equations forms an essential part of the demonstration.

General Rule.

The results of the foregoing inquiry may be collected into the following General Rule:—

To find the number p of possible integrals of a system of n differential equations of the first order connecting $n+r$ variables $x_1, x_2, \dots x_{n+r}$, and to determine those integrals.

Rule.—Suppose P an integral of the given system. Determine from the given system $dx_1, dx_2, \dots dx_n$ as functions of the other differentials. Substitute these values in the equation

$$\frac{dP}{dx_1}dx_1+\frac{dP}{dx_2}dx_2 \dots +\frac{dP}{dx_{n+r}}dx_{n+r}=0,$$

and equate to 0 the coefficients of the remaining differentials. This will give a system of r partial differential equations of the form

$$\frac{dP}{dx_{n+1}}+A_{11}\frac{dP}{dx_1} \cdots +A_{1n}\frac{dP}{dx_n}=0,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{dP}{dx_{n+r}}+A_{r1}\frac{dP}{dx_1} \cdots +A_{rn}\frac{dP}{dx_n}=0,$$

r of the differential coefficients appearing each only in one equation and with coefficient unity. Representing these equations in the symbolical form

$$\Delta_1 P=0,\ \Delta_2 P=0, \dots \Delta_r P=0,$$

deduce any equation or equations of the form

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0.$$

If *all* such prove to be identities, the given system of differential equations admits of n integrals, and is reducible to a system of exact differential equations. But if *any* such equation is not an identity, it will constitute a new partial differential equation of the form

$$B_1\frac{dP}{dx_1}+B_2\frac{dP}{dx_2}\ldots+B_n\frac{dP}{dx_n}=0.$$

And this, combined with the previous ones, will enable us to form a system of $r+1$ partial differential equations, in which $\frac{dP}{dx_n}, \frac{dP}{dx_{n+1}}, \ldots \frac{dP}{dx_{n+r}}$ appear each in only one equation and with coefficient unity. Upon this system let the same process be repeated as upon the previous system of r partial differential equations, and so continually repeated until we arrive at a final system of partial differential equations such that, if that system be represented in the form

$$\Delta_1 P=0 \ldots \Delta_m P=0,$$

the condition

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$$

shall be identically satisfied for every pair.

Then, the number of such partial differential equations being m, the number of integrals of the original system of partial differential equations will be $n+r-m$, i. e. it will be equal to the number of the original variables diminished by the number of final partial differential equations.

And if by that final system we eliminate m of the differential coefficients from

$$\frac{dP}{dx_1}dx_1+\frac{dP}{dx_2}dx_2\ldots+\frac{dP}{dx_{n+r}}dx_{n+r}=0,$$

and equate to 0 the coefficients of the remaining differential coefficients, we shall have a system of $n+r-m$ differential equations expressible as exact differential equations for the determination of the integrals.

Actually to determine these, we should endeavour in the first instance to reduce the final system of differential equations, as such reduction is theoretically possible, to a system of exact differential equations. If the means of doing this are not obvious, the method of the variation of parameters or the equivalent methods of Prop. III. must be applied.

Lastly, if the process which consists in the application of the theorem $(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$ do not stop with the formation of the final system of partial differential equations, but lead to algebraic relations among the variables, the given system of differential equations will have no integrals properly so called, but it may admit of *solutions* analogous to those the theory of which has been developed by PFAFF, JACOBI, and others for the differential equation

$$X_1dx_1+X_2dx_2\ldots+X_ndx_n=0.$$

Applications.

1st. Suppose it required to find the number of integrals of the form $P=c$, which the system of differential equations

$$dz=(t+xy+xz)dx+(xzt+y-xy)dy,$$
$$dt=(y+z-3x)dx+(zt-y)dy$$

admits, and to determine such integrals.

Eliminating dz and dt from the equation

$$\frac{dP}{dx}dx+\frac{dP}{dy}dy+\frac{dP}{dz}dz+\frac{dP}{dt}dt=0,$$

and equating to 0 the coefficients of dz and dt in the result, we have

$$\frac{dP}{dx}+(t+xy+xz)\frac{dP}{dz}+(y+z-3x)\frac{dP}{dt}=0, \quad \ldots\ldots (1.)$$

$$\frac{dP}{dy}+(xzt+y-xy)\frac{dP}{dz}+(zt-y)\frac{dP}{dt}=0. \quad \ldots\ldots (2.)$$

Hence writing

$$\Delta=\frac{d}{dx}+(t+xy+xz)\frac{d}{dz}+(y+z-3x)\frac{d}{dt},$$

$$\Delta'+\frac{d}{dy}+(xzt+y-xy)\frac{d}{dz}+(zt-y)\frac{d}{dt},$$

and forming the equation

$$(\Delta\Delta'-\Delta'\Delta)P=0,$$

we have, on rejecting a common algebraic factor,

$$x\frac{dP}{dz}+\frac{dP}{dt}=0. \quad \ldots\ldots (3.)$$

By substituting in (1.) and (2.) the value of $\frac{dP}{dt}$ hence obtained, we have the system of three equations,

$$\frac{dP}{dx}+(3x^2+t)\frac{dP}{dz}=0,$$

$$\frac{dP}{dy}+y\frac{dP}{dz}=0,$$

$$\frac{dP}{dt}+x\frac{dP}{dz}=0.$$

Now if upon any two of these we repeat the same process as upon (1.) and (2.), we obtain as the result $0=0$. Thus the system of partial differential equations is complete.

As then there are three equations in this final system, while the number of original variables was four, the primary system will admit of one integral of the form $P=c$.

To obtain this integral, eliminate $\frac{dP}{dx}$, $\frac{dP}{dy}$, $\frac{dP}{dt}$ from the above equations and

$$\frac{dP}{dx}dx+\frac{dP}{dy}dy+\frac{dP}{dz}dz+\frac{dP}{dt}dt=0,$$

and equate to 0 the coefficient of $\frac{dP}{dz}$ in the result. We find

$$dz-(t+3x^2)dx-ydy-xdt=0,$$

the integral of which is

$$z-xt-x^3-y^2=c.$$

2nd. The solution of the partial differential equation

$$Rr+Ss+Tt+U(s^2-rt)=V,$$

as well as of the special equations

$$Rr+Ss+Tt=V,$$

$$Rr+Ss+Tt+U(s^2-rt)=0,$$

the theory of which constitutes an exception to that of the more general form, depends in general upon the integration of three simultaneous differential equations between five variables. To this integration the method of the foregoing sections is applicable.

The only cases for which the theory of the ultimate solution can be said to be complete, are those in which the auxiliary system of common differential equations admits either three integrals of the form $P=c$, or two integrals of that form.

We may apply the method of the foregoing sections, not only to the determination of the integrals, but also to the discovery of the *à priori* conditions connecting the coefficients R, S, T, U, V in order that each of these species of integration may be possible.

For example, the solution of the equation

$$Rr+Ss+Tt+(s^2-rt)=V$$

depending upon the integration of the system

$$dq=-m_1dx+Rdy,$$

$$dp=-m_2dy+Tdx,$$

$$dz=\ pdx\ +qdy,$$

in which m_1 and m_2 are roots of the equation

$$m^2-Sm+RT-V=0,$$

let it be required to determine the conditions under which the system admits three integrals.

Eliminating dq, dp, dz between the above equations and

$$\frac{dP}{dx}dx+\frac{dP}{dy}dy+\frac{dP}{dz}dz+\frac{dP}{dp}dp+\frac{dP}{dq}dq=0,$$

and equating to 0 the coefficients of dx and dy in the result, we obtain two partial differential equations which may be thus represented, viz.

$$\Delta P=0, \qquad \Delta' P=0,$$

in which

$$\Delta=\frac{d}{dx}-m_1\frac{d}{dq}+T\frac{d}{dp}+p\frac{d}{dz},$$

$$\Delta'=\frac{d}{dy}+R\frac{d}{dq}-m_2\frac{d}{dp}+q\frac{d}{dz}.$$

That there may be three integrals, it is here necessary that there should be but two partial differential equations in the completed system. Hence the equation

$$(\Delta\Delta'-\Delta'\Delta)\mathrm{P}=0$$

must vanish identically.

Developing this, we have the conditions

$$\Delta\mathrm{R}+\Delta' m_1=0, \quad \Delta m_2+\Delta'\mathrm{T}=0, \quad \Delta q-\Delta' p=0.$$

Now on performing the operations denoted by Δ and Δ', the last equation gives

$$m_2-m_1=0.$$

Hence referring to the quadratic, we see that

$$\mathrm{S}^2-4\mathrm{RT}+4\mathrm{V}=0. \quad \ldots\ldots\ldots \quad \text{(I.)}$$

To this must be added the two other reduced conditions,

$$\Delta\mathrm{R}+\Delta' m=0, \quad \ldots\ldots\ldots \quad \text{(II.)}$$

$$\Delta m+\Delta'\mathrm{T}=0, \quad \ldots\ldots\ldots \quad \text{(III.)}$$

m representing one of the equal roots of the reduced quadratic.

The first of the above conditions was given by AMPÈRE*. The others are probably new. Satisfied, they enable us to predict that the partial differential equation under consideration admits a complete primitive involving three constants, and a general primitive arising from the variation of those constants in subjection to any two arbitrary conditions.

3rd. We have supposed each linear partial differential equation employed in the processes of this paper to be of the form

$$\mathrm{A}_1\frac{d\mathrm{P}}{dx_1}+\mathrm{A}_2\frac{d\mathrm{P}}{dx_2}\ldots+\mathrm{A}_n\frac{d\mathrm{P}}{dx_n}=0,$$

and we have supposed each system of partial differential equations which arises, to be so reduced that each equation shall have some one of the partial differential coefficients of P entering into it alone and with a coefficient equal to unity.

The first of these conditions is virtually sufficiently general, because any linear partial differential equation can be deprived of its second member. The advantage of the second condition is that each newly-formed equation will be really new, and not an algebraic combination of the old ones.

But neither of these conditions is necessary. From two linear partial differential equations of the form

$$\Delta_1\mathrm{P}=\mathrm{H}, \quad \Delta_2\mathrm{P}=\mathrm{K},$$

in which H and K are functions of the independent variables, arises a new equation,

$$(\Delta_1\Delta_2-\Delta_2\Delta_1)\mathrm{P}=\Delta_1\mathrm{K}-\Delta_2\mathrm{H}, \quad \ldots\ldots\ldots \quad \text{(1.)}$$

which will be satisfied by all the simultaneous integrals of the equations from which it is derived.

It may be rigorously proved that, in applying this process, the generated system

* Journal de l'Ecole Polytechnique, Cahier 18e.

(including the original equations) will be complete when no new equation arises from the combination of any one of the equations with any one of the equations of the original system.

I will illustrate this by investigating the conditions of integrability of the expression

$$\varphi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}\right)dx.$$

If this expression admit of an integral V, it is easy to see that V will satisfy the two partial differential equations

$$\frac{dV}{dx}+y_1\frac{dV}{dy}+y_2\frac{dV}{dy_1}\ldots+y_n\frac{dV}{dy_{n-1}}=\varphi, \quad \ldots\ldots\ldots \quad (2.)$$

$$\frac{dV}{dy_n}=0, \quad \ldots\ldots\ldots \quad (3.)$$

in which

$$y_1=\frac{dy}{dx}, \; y_2=\frac{d^2y}{dx^2} \ldots\ldots,$$

and φ stands for

$$\varphi(x, y, y_1, \ldots y_n).$$

The above are, in fact, the partial differential equations which we should obtain by Prop. I. as the equivalents of the system of ordinary differential equations,

$$dV=\varphi dx,$$

$$dy=y_1dx, \; dy_1=y_2dx, \ldots dy_{n-1}=y_ndx.$$

If we write

$$\left(\frac{d}{dx}\right)=\frac{d}{dx}+y_1\frac{d}{dy}+y_2\frac{d}{dy_1}\ldots+y_n\frac{d}{dy_{n-1}},$$

the above partial differential equations become

$$\left(\frac{d}{dx}\right)V=\varphi\ldots(\text{I.}), \qquad \frac{dV}{dy_n}=0\ldots(\text{II.})$$

The combination of (I.) with (II.) (by the theorem (1.)), then of (I.) with the result, and so on, gives a series of equations which may be thus expressed:—

$$\frac{dV}{dy_{n-1}}=\Delta_n\varphi, \quad \ldots\ldots\ldots\ldots\ldots \quad (\text{III.})$$

$$\frac{dV}{dy_{n-2}}=\Delta_{n-1}\varphi, \quad \ldots\ldots\ldots\ldots\ldots \quad (\text{IV.})$$

. . . . .

$$\frac{dV}{dy_0}=\Delta_1\varphi, \quad \ldots\ldots\ldots\ldots\ldots \quad (\text{V.})$$

$$0=\Delta_0\varphi, \quad \ldots\ldots\ldots\ldots\ldots \quad (\text{VI.})$$

in which

$$\Delta_r=\frac{d}{dy_r}-\left(\frac{d}{dx}\right)\frac{d}{dy_{r+1}}+\left(\frac{d}{dx}\right)^2\frac{d}{dy_{r+2}}-\&c.,$$

$$\Delta_0=\frac{d}{dy}-\left(\frac{d}{dx}\right)\frac{d}{dy_1}+\left(\frac{d}{dx}\right)^2\frac{d}{dy_2}-\&c.$$

The combination of (II.) with (III.) ... (V.) gives the series of conditions

$$\frac{d}{dy_n}\Delta_n\varphi=0,\ \frac{d}{dy_n}\Delta_{n-1}\varphi=0,\ \ldots\ \frac{d}{dy_n}\Delta_1\varphi=0. \quad . \quad . \quad . \quad . \quad . \quad \text{(VII.)}$$

The conditions of integrability are expressed by (VI.) and (VII.). These satisfied, the equations (III.), (IV.), ... (V.) show that φdx can be expressed as an exact differential with respect to x, y, y_1, y_{n-1}, in the form

$$\varphi dx=(\varphi-y_1\Delta_1\varphi-y_2\Delta_2\varphi\ \ldots\ -y_n\Delta_n\varphi)dx$$
$$+\Delta_1\varphi dy+\Delta_2\varphi dy_1\ \ldots\ +\Delta_n\varphi dy_{n-1},$$

a result first established by M. Sarrus.

XXXI. (I.) *On the Dicynodont Reptilia, with a Description of some Fossil Remains brought by* H.R.H. PRINCE ALFRED *from South Africa, November* 1860. *By Professor* OWEN, *F.R.S. &c.*

Received January 23,—Read February 20, 1862.

ON the return of His Royal Highness PRINCE ALFRED from the Cape of Good Hope, in 1860, he honoured me by transmitting some fossil remains, with the following Note:—

"DEAR PROFESSOR OWEN,

"In the course of my journey in South Africa I met with two very interesting Fossil Remains; one, the larger, being the head of a *Dicynodon*: and I hope you will accept them from me as being the best specimens I obtained, upon the PRINCE CONSORT'S suggestion, on the occasion of your last Lecture, of which I retain the most agreeable recollection.

"Yours truly,

"Windsor Castle, November 15th." "ALFRED."

I feel it a duty, at the present season of grief at a National bereavement, which becomes daily more appreciated, to communicate to the Society this evidence of the lamented PRINCE CONSORT'S unfailing interest in our sciences, as manifested by his desire that Lectures on Natural History should be delivered before the Royal offspring; and also as showing that, amidst the manifold occupations of the PRINCE CONSORT, he allowed no opportunity to escape in which his influence could be, in any way, exerted to aid in the direct advancement of science.

In the present instance, this influence, through the zealous fulfilment by PRINCE ALFRED of his father's suggestion, has procured for us valuable additional evidence of the peculiar structure and characters of some rare extinct animals, which, under any circumstances, I should have felt bound to submit to the Royal Society.

Some delay has been occasioned by the necessity of removing the very hard matrix from the varied and intricate surfaces of the imbedded petrified bones,—a labour requiring to be performed under my immediate supervision, or by my own hands.

The smaller of the two fossils referred to by H.R.H. PRINCE ALFRED, has thus proved to be one of the most perfect specimens of the petrified skull of a dicynodont Reptile which, hitherto, has come under my observation. It is figured, of the natural size, in Plates XIX. & XX. figs. 1, 2, 3, 4 & 5.

Ptychognathus Alfredi, Owen.

By the angular contour of the profile (figs. 1 & 3), in which the almost straight line of the vertex (*v o*) meets the equally straight occipital line (*o b*) at a right angle, and the straight facial or naso-premaxillary line (*v n*) meets that of the vertex at an obtuse angle, the specimen belongs to the subgeneric form of the dicynodont family called *Ptychognathus*.

The occiput (fig. 4) shows the same crocodilian extent of ossification as in other members of this singular family.

The articular tubercle is formed, as in *Ptychognathus declivis**, by the basioccipital (fig. 4, 1) at the lower and middle part, and, in less proportions, by the exoccipitals (*ib.* 2) which just meet at the upper part of the tubercle. The foramen magnum is a rather narrow oval, with the small end upwards, exclusively bounded, externally, by the exoccipitals; the basioccipital appearing at the interspace of the exoccipitals immediately within the cranial cavity. The basioccipital in advance of the tubercle, has its lower surface at first concave, and then convex, lengthwise; and is more deeply concave transversely (fig. 5, 1) through the production of its sides into the short and thick hypapophyses, figs. 4 & 5, *hy*. Its junction with the basisphenoid (fig. 5, 2) is by an almost straight transverse harmonia. The hypapophyses (*hy*) are divided by a short notch from the paroccipitals (fig. 4, 4), which abut, as in *Crocodilia* and *Chelonia*, against the inner side of the tympanics (*m*), but here against their lower half, fixing these bones more firmly in their position. The paroccipitals, which, as in the Crocodile, are exogenous growths of the exoccipitals (*ib.* 2), are divided by a wider and more shallow notch from the exoccipital and mastoid, *s*. The superoccipital (fig. 4, 3) forms the upper part of the occipital surface, is of a vertically oblong or subtriangular form, with the apex downward, and terminating some distance above the foramen magnum. The sides of the superoccipital concavity are formed by the backwardly projecting and slightly diverging hinder end of the parietals (fig. 4, 7); between which and the exoccipitals and tympanics the broad and angular mastoid (*ib.* 5) are wedged, forming the upper, outer, and posterior outstanding backwardly bent ridge, for the attachment of the cervical muscles, and presenting the sutural surface to the upper end of the tympanic, *m*. The upper surface of the parietal (fig. 2, 7) presents a roughish non-articular tract of about 8 lines in breadth, at the fore part of which is the venous foramen. This median tract is made slightly concave across by the elevation of the ridges (*t*, *t*) bounding the smooth, longitudinally concave, tracts affording origin to the temporal muscle.

The frontals (11, 11) form the chief part of the flattened vertex or upper platform of the cranium (figs. 2 & 4, *f*, *t*). They extend longitudinally from the foramen parietale forward to about one-third of an inch from the prefronto-nasal ridge (fig. 2, *f*); and transversely to the superorbital ridges (*s*), of which they form about the middle third part. The median or frontal suture may be traced along a great part of their extent: the surface is slightly raised at this part, and is depressed laterally, between the median

* Quarterly Journal of the Geological Society, vol. xvi. (April 1859) p. 49, pl. 1. figs. 3–5.

rising and the slightly elevated orbital borders. There is no trace of the pair of tubercles which, in *Ptychognathus declivis**, stand transversely on the frontal platform, in the line bisecting the middle of the orbits: they have not been chipped away in clearing off the matrix, either in the present skull or in that of *Ptychognathus latirostris*†. The median rising at the frontal suture is continued backward over the part where it divides to go to each tubercle, leaving an intermediate shallow depression, in *Ptychognathus declivis*. The frontal tubercles may be, therefore, reckoned among the cranial characters distinguishing *Ptychognathus declivis* from *Ptychognathus latirostris*, and from the skull of the species under description. In this, as in the previously defined species of *Ptychognathus*, the vertex or upper surface of the skull is bounded anteriorly by a low ridge (fig. 2, *f*, *f*), extending transversely with a slight curve, convex forward, from the fore part of one superorbital border to the other. The ridge is formed by the nasals (fig. 2, 13) at the middle, and by the prefrontals (*ib.* 14) at the sides. The latter are strong, large, angular bones (figs. 1, 2 & 3, 14), dividing by a prominent tuberosity the fore from the upper borders of the orbit: the frontal surface of the prefrontal is divided by the before-described ridge from the facial surface; both are slightly concave, the latter of greater extent, reaching almost to the hinder angle of the external nostril (figs. 1 & 3, *n*), and articulating with the nasal (*ib.* 13) above, and with the lacrymal (*ib.* 73) below. The nasals (13), divided by a median suture, bend abruptly, at the prefronto-nasal ridge, from the upper to the fore or facial part of the skull, at an open angle. They are united for about an inch as they slope upon the face (fig. 2), and then diverge to form the upper border of the nostrils, receiving at the angles, so formed, the upper and hinder ends of the coalesced premaxillaries, *ib.* 22. The premaxillaries continue the facial line, begun by the nasals, straight to the upper margin of the mouth, which is directed forward. There is a slightly elevated line along the place of the obliterated medial suture (fig. 2, *m*), parallel with which is a pair of similar linear ridges (fig. 2, *p*, *p*), dividing the median from the lateral surfaces of the premaxillary: these surfaces have almost the same breadth, except that the lateral surface (fig. 1, 22) increases near the oral border; it is narrowest where it forms the fore part of the nostril, *n*.

The maxillary (figs. 1 & 3, 21) is chiefly remarkable, as in the rest of the genus, for the strong ridge-like prominence (*r r*) of the socket of the long and large canine tusk, *c*. Between this and the outer premaxillary ridge (fig. 1, *p*) the sides of the face are slightly sinuous, the convexity above gradually changing to a deeper concavity below, in the transverse or vertical direction; lengthwise the sides of the face are slightly concave as they converge towards the mouth; except at the canine alveoli, which are a little convex lengthwise at their fore part. Above, the maxillary bounds the lower part of the nostril, and there unites with the premaxillary (22) and lacrymal (73); below, it is continued from the canine-socket inward and downward, forming the edentulous sectorial border which meets the corresponding part of that border of the mandible (23), the fore part of which, in advance of the maxillary, is strongly bent upward (23′) to meet the eden-

* Quarterly Journal of the Geological Society, vol. xvi. p. 50, pl. 1. fig. 3. † Ibid. p. 51.

tulous trenchant border of the premaxillaries, *n*. The back part of the maxillary is continued beneath the orbit in a pointed form, articulating by a strong, oblique and extensive suture with the malar (fig. 1, *m*), by which and the lacrymal the maxillary is removed from the orbital border itself.

The lacrymal (*l*), forming the middle of the fore part of the orbit, extends upon the face apparently to the nostril. The malar curving back along the lower part of the orbit, the border of which it there forms, bifurcates behind to connect itself with the postfrontal (figs. 2 & 4, *i*) above and with the squamosal (fig. 1, *s*) behind; but their limitary sutures I cannot satisfactorily make out. There is no other bony bridge overspanning the temporal fossa except the normal malo-squamosal zygomatic arch. The squamosal combines with the mastoid in affording the articular sutural surface to the upper end of the tympanic.

The squamosal (fig. 1, *s*) is compressed, of 8 lines in vertical extent, by 2 in transverse; and increases vertically as it passes backward. The tympanic pedicle is of great length, broad and compressed from before backward at its upper part, becoming narrower as it descends, but gaining in thickness to where it receives the broad abutment of the paroccipital; below which it slightly expands in every direction to form the convex articular surface for the mandible. The tympanic pedicle is fixed immoveably, and by its size and connexions forms an unusually strong 'point d'appui' for the vigorous actions of the lower jaw. It consists of two bones, united by a broad overlapping squamous suture. The upper portion includes the mastoid and squamosal elements; the latter extends to near the articular condyle, along the fore part of the pedicle; the lower portion, or tympanic proper (*to*), forms the lower half of the back part of the pedicle, and expands below to form the joint.

The mandible of *Ptychognathus* (fig. 1, *m*–*sa*) resembles that of the *Chelonia* in its edentulous condition, general proportions, and comparative simplicity of structure, and that of the *Crocodilia* in the vacuity left between the dentary (*m*), angular (*a*), and surangular (*sa*) elements; but it is peculiar and ptychognathic in the sudden vertical expanse and upward curve of its symphysial end (*sy*), the vertical diameter here being three times that of the articular end. I cannot distinguish an articular from a surangular element: the suture between this and the angular extends near and runs parallel with the upper border of the surangular to the vacuity, towards which it bends. No coronoid process is developed, nor is there any coronoid or complementary ossicle as in modern Lizards, Chelonians, and Crocodiles. The splenial element (fig. 5, *sa*) extends far back, as in *Chelonia*. The dentary elements, confluent at the symphysis, are deeply notched at their narrow hinder part, the upper projection being the longest. A horizontal ridge extends from the upper part of the notch forward as far as the depression, lodging the end of the canine tusk when the mouth is shut. The vacuity is situated halfway between the two ends of the mandibular ramus, not in the posterior third as in the Crocodile. The symphysis is broad as well as high, and the rami meet there so as to form, below, an arch or curve, concave backward, fig. 5, *s*. The fore part of the symphysis is convex both ver-

tically and transversely. The longitudinal ridge on the outside of the dentary, parallels that formed by the canine alveolus of the maxillary; and the alveolar borders in both incline inwards, to meet at the oral margin behind the great tusks. The mandibular ridge resembles the maxillary one in *Oudenodon*; and one is led to speculate whether in that genus the maxillary ridge may have related to a rudimentary upper canine, and the mandibular ridge to that of a lower one, in the embryo. In both *Ptychognathus* and *Oudenodon*, however, these parallel ridges in the upper and lower jaws, with the inward convergence of the alveolar plates, recall the structure of those parts in *Scelidosaurus**, and which is indicated by the mandibles to have existed, in a certain degree, in *Hylæosaurus* and *Iguanodon*.

The fossil skull above described was obtained from a greenish sandstone of the Rhenosterberg. As it manifests specific distinctive characters, I propose to refer it to a species of *Ptychognathus*, under the name of the young Prince by whom it has been made available in advancing our knowledge of South African fossils.

The *Ptychognathus Alfredi*, like *Ptychognathus latirostris*, differs from *Pt. declivis* in the more circular orbits, the absence of the frontal tubercles, the right angle at which the occiput and vertex meet, and the greater depth and thickness of the facial part in proportion to its length.

But on comparing *Ptychognathus Alfredi* with *Ptychognathus latirostris*, the facial part of the skull presents less breadth in proportion to its length and depth, and the lower jaw is narrower in proportion to its length; the paroccipital is also relatively narrower.

The occipital region in *Ptychognathus declivis* (Plate XXI. fig. 2) shows nearly the whole extent of the masto-tympanic pedicle, and well exemplifies the singular breadth of the hinder surface of the cranium in these dicynodont reptiles. It also shows the precondyloid foramina, and the small vacuity between the exoccipital and masto-tympanic bones. The side view of the skull of *Ptychognathus latirostris* (Plate XXI. fig. 1) well exemplifies the ridged structure of the jaws and the composition of the mandible.

In the corresponding view of the skull of *Ptychognathus Alfredi* (Plate XX. fig. 3) is shown the extent of the pulp-cavity of the canine (*c*), closely conforming to that in the instances of ever-growing tusks in the mammalian class. As in former specimens of true *Dicynodon*, the section made to show the base of the tusk exhibits no trace of any germ of a successional tooth. The tusk curves forward, downward, and slightly inward. The points of both tusks have been broken away.

Dicynodon tigriceps, Owen.

The larger fossil, obtained by H.R.H. Prince Alfred from the Karoo-beds, in the district of Graaf Reinet, is a skull of a true *Dicynodon*, equalling in size that of the *Dicynodon tigriceps*†. It has been obliquely crushed, but under circumstances of sur-

* "Monograph on British Fossil Reptiles," Palæontological Society's volume for 1860, p. 12, pls. 4 & 5.

† Transactions of the Geological Society, 2nd Series, vol. vii. p. 233.

rounding support, which have kept the lower jaw in its natural connexions with the tympanic pedicle.

In the breaking up of the hard rock in which this fossil has been imbedded, the right maxillary, tympanic, and zygomatic arch have been removed; the fore part of the upper jaw, from the beginning of the sockets of the canine tusks, and the corresponding end of the lower jaw, are also broken away. The length of the remaining portion of the skull now exposed is 1 foot 6 inches; but this is somewhat more than it would naturally be, owing to the left half of the broad occipital region (Plate XXII. 4, m) having been bent backward from the transverse to almost the longitudinal position, in the line of the skull's axis, and this with so little disturbance of the connexions of its elementary bones as exemplifies, with other similar condition of the skull, the gradual operation of the disturbing force, and the condition of surrounding support that has made the pressure act upon the brittle fossil as if it had been a plastic material. For, after close observation and reflection upon all the appearances presented by this fossil, I infer that the cosmic movements affecting the matrix have operated after the sediment in which the dead body of the old reptile was buried had become, with its contents, hardened into stone.

After careful removal of the matrix from the remaining petrified bones of the right side of the skull, the occipital tubercle (1) was worked out in its whole length: it projects from the lower plane of the basioccipital (*h*) to the extent of 1 inch 8 lines, and from the foramen magnum (*f*) about 1 inch 4 lines; the vertical diameter of the base of the tubercle is 1 inch 6 lines; it slightly expands to its articular convexity. The right side of the tubercle having been broken away, the compact or close granuloid texture of the bone is here displayed.

The occipital hypapophyses (*h*) are 2 inches in length; the left paroccipital (4) expands to a breadth of 5 inches, where it abuts against the broad masto-tympanic pedicle (m). An extent of 6 inches by 4 inches of the smooth posterior surface of this singularly expanded lamelliform bone is here preserved. The small vacuity (*a*) between the par- and exoccipitals has been converted into a foramen by the meeting of extensions of ossification from 2 to 4. The foramen magnum (*f*), naturally of a vertically oval form, is here made narrower by the slow lateral squeezing of the occiput; its long diameter is 1 inch 4 lines. The lower angle of the superoccipital (3) is preserved, making an extent of the well-ossified occipital plane reaching to 4 inches above the occipital tubercle.

The removal of the outer part of the maxillary and the zygomatic arch has brought into view part of the interorbital septum, and the upper and outer part of the bony palate; structures that have not been shown by previous specimens of *Dicynodon*.

The descending cranial plate of the frontal, where it forms the inner wall of the orbit and the rhinencephalic continuation of the cranial cavity, is shown at 11, Plate XXII.

The basisphenoid (*ib.* 8) is short and deep, and sends out a process uniting with one from the pterygoid, to abut against the tympanic pedicle.

The presphenoid (*ib.* 9) projects forward as a compressed plate 10 lines in vertical dia-

meter; in its length or extent of ossification it exceeds that in the *Chelonia*, and more resembles that in the *Crocodilia*; at its base there arise some thin rays of ossification, which ascend to unite with a similarly attenuated lamelliform base of the orbitosphenoid. I infer that there was persistent cartilage in this part of the skull. The orbitosphenoid (*ib.* 18) becomes thicker as it ascends to unite with the under part of the hinder third of the frontal, and with part of the parietal. It is perforated by a foramen opticum of elliptical form, 6 lines in long diameter.

Below and anterior to the presphenoid is seen a small part of the vomer (*ib.* 13), where it expands laterally to join the palatine (*ib.* 22) and pterygoid, *ib.* 24. The pterygoid, about 5 inches in length, contracts as it extends backward, bounds above or mesially the outer part of a long elliptical foramen (posterior nostril, *ib. n*), and then bends downward and outward to join the basisphenoid and abut against the lower end of the tympanic.

There appears to have been a part of the interorbital space unossified, about 3 inches in length, $1\frac{1}{2}$ inch broad below, but suddenly contracting to a width of $\frac{1}{4}$ an inch at the upper part.

The exposed bottom of the right socket of the canine tusk (*ib.* fig. 1, *c*) shows the similarly exposed pulp-cavity of the beginning of that tusk, which measures $1\frac{1}{2}$ inch across. The walls of the tusk increase in thickness to about 3 lines in an extent of $1\frac{1}{4}$ inch. A greater extent of the left socket (Plate XXII. fig. 2) is preserved, showing the concentrically lamelliform structure of the base of that tusk.

Sufficient of the mandible is preserved to demonstrate its characteristic dicynodontal composition. The alveolar border of the dentary element is toothless. The ramus rapidly augments in depth towards and at the symphysis, where a portion is broken away. The posterior part of the dentary (*ib.* fig. 1, 32) is notched or bifurcates to form an oblong vacuity at the middle of the mandibular ramus. Traces of a long surangular (*ib.* fig. 1, 30), which developes no coronoid process, of a broader or deeper angular (*ib.* 31), and also of a splenial (*ib.* 29), are discernible amongst the elements of the lower jaw.

As a whole, the present instructive specimen exemplifies the near equality in size of some of the strange extinct two-tusked reptiles of South Africa to the existing Walrus; it also shows that in the structure of the bony palate the *Dicynodon* combines, as in other parts of the skull, crocodilian with chelonian and lacertian characters.

Explanation of the Plates.

PLATE XIX.

Fig. 1. Side view of skull of *Ptychognathus Alfredi*: nat. size.
Fig. 2. Upper view of the same skull: nat. size.

PLATE XX.

Fig. 3. Side view, with section of tusk, of the same skull: nat. size.
Fig. 4. Back view of the same skull.
Fig. 5. Under view of ditto.

PLATE XXI.

Fig. 1. Side view of skull of *Ptychognathus latirostris*: nat. size.
Fig. 2. Back view of *Ptychognathus declivis*.

PLATE XXII.

Dicynodon tigriceps.

Fig. 1. Side view of skull: ½ nat. size.
Fig. 2. Socket and base of canine tusk: nat. size.

(II.) *On the Pelvis of the* DICYNODON. *By Professor* OWEN, *F.R.S. &c.*

The following description is taken from a part of the petrified skeleton of a *Dicynodon* equalling in size the species, *D. tigriceps*, OW., to which the cranium described in the preceding paper belongs, and exemplifying the structure of the pelvic part of the trunk of that extinct animal.

This instructive fossil is from the same locality in the Graaf Reinet district as that from which the specimen of the skull of the *Dicynodon tigriceps* was obtained which is described in the seventh volume of the 2nd Series of the Geological Transactions *. It formed the nucleus of a huge nodule of greyish-blue argillo-ferruginous limestone, transmitted from that locality by A. G. BAIN, Esq., F.G.S., and is now in the British Museum.

In a front view of this specimen (Plate XXIII. fig. 1) seven successive vertebræ are seen; but there is an appearance of slight dislocation at the expanded and co-adapted articular ends of the two anterior of these (D & S_1), which indicates that they were not anchylosed together; the rest seem to have coalesced, although traces of the intervertebral articulations remain. One-half of the last vertebra (Plate XXIV. S_4) is broken away. The length of the six entire centrums is 1 foot 2 inches.

The first vertebra (Plate XXIII. fig. 1, D) supports a pair of long, comparatively slender curved ribs (*pl*), articulated to strong outstanding transverse processes (*d*). These were the last or hindmost pair of free ribs, and indicate great breadth in that part of the trunk.

The second vertebra (*ib.* S_1), which is the first of the sacral series, sends out a pair of broad and thick parapophyses (*ib. p*), to each of which is attached a longer and broader pleurapophysis (*ib. pl*). This rapidly expands as it extends outward, and underlaps, or

* 4to. 1855, p. 233.

passes anterior to, the iliac bone (*ib.* ∞). It resembles in shape the human scapula, but is much thicker. The hinder and inner border of this sacral rib is thick, and smoothly rounded; the front border is thinner, and is slightly concave; the outer border appears to have been straight, but is somewhat mutilated. This expanded termination of the rib rests on the ventral side of the ilium, concealing much of that bone in a front view of the pelvis. It has not been anchylosed therewith; decomposition of the ligamentous uniting matter, and subsequent partial dislocation, have allowed the matrix to insinuate itself between the sacral rib and the ilium, as seen at *a*, fig. 2, Plate XXIII.

Of the short, thick, transversely extended ribs of the five succeeding sacral vertebræ, the left of the penultimate (*pl*) is the best preserved towards the pelvic cavity; it presents a smooth convex or rounded surface about an inch in breadth, is slightly bent with the concavity towards the outlet of the pelvis, and abuts against the ischium.

The bodies of the sacral vertebræ (s_1.....s) are contracted at the middle, and slender there in proportion to the pelvis, but are rapidly and much expanded at their articular ends; consequently they are very concave lengthwise, both below and at the sides; but are smooth and convex transversely, yielding a semicircular transverse section. The parapophyses go off near the base of the neural arch. The first centrum (s_1) is nearly 3 inches in length, and as much in breadth at the articular end, but is only 1 inch 3 lines across the middle; the rest (s.....s) slightly diminish in length as they approach the tail.

The ilium (Plates XXIII. and XXIV. ∞) is a strong straight triangular bone, at least 10 inches in length from the upper border of the acetabulum (Plate XXIII. fig. 2, *c*); above which it is contracted to a breadth of 3½ inches and a thickness of 2 inches, and then expands to a breadth of 8½ inches, measured along its oblique anterior border or labrum, Plate XXIV. fig. 4, *l l'*. The front part of the anterior two-thirds of the ilium expands into a rough flattened surface, 6 inches in length, and 3 inches in breadth anteriorly (*ib.* fig. 4, *r*), to which the back surface of the expanded first sacral rib is ligamentously attached. The inner surface of the ilium extends 6 inches behind this articulation, and is almost flat, but rather sinuous: coarse bony ridges or rays of ossification appear on this surface, near the labrum (*s'*), diverging thereto. The outer surface of the ilium is moderately concave transversely. This anterior expanded part of the ilium passes behind and in advance of the last pair of free ribs; the relations of which, and of the first sacral ribs, remind one of those of the answerable vertebræ in the pelvis of birds. The ribs of the second and third sacral vertebræ also abut against the ilium.

The ischium (Plates XXIII. and XXIV. ∞), behind or beyond the acetabulum, forms a short and very thick prominence (*ib.* fig. 2, *t*), which receives the abutment of the ribs of the fifth sacral (Plate XXIV. $pl\,s$): below or behind this it receives a similar abutment from the ribs of the last sacral vertebra ($pl\,s$). The space, corresponding to the 'great ischiadic foramen' in Edentate mammals, is thus divided into two vacuities. The ischium becomes thinner, but is of great breadth where it forms the lower wall of the pelvis and converges towards its fellow to form, with the pubis, the long symphysis (Plate XXIII. fig. 2, *y*); here it again increases in thickness. A fracture, with a slight

dislocation on the left side, has taken place where the suture with the pubis may originally have existed; but on the right side no trace of such suture is visible; and the perfect state of the surface of the pelvis at this part demonstrates as complete a confluence of the two bones, with each other and the ilium, to form a large 'os innominatum,' as in mammals.

The pubis (Plates XXIII. and XXIV. α_3) is remarkable for the bold and broad anterior convexity (b) of its iliac half, the inner part of which is perforated by an elliptical aperture (figs. 1 & 4, f), answering to that in the pubis of the Monitor*. The outer part of the bone is produced into a short obtuse process (r), less developed proportionately than that in the Monitor's pubis. Beyond this process the inferior border of the pubis bends downward and forward, rendering the antero-inferior surface of the half of the bone next the symphysis concave. The symphysis makes a slight angle near its beginning (Plate XXIII. fig. 1, y), projecting towards the pelvic cavity: the suture, there, is obliterated. The inferior or symphysial wall of the pelvis measures 14 inches across in a straight line from the border of one acetabulum to that of the other. The length of the symphysis (*ib.* fig. 2, α_3, y, α_4) in a straight line cannot have been less than 8 inches; but the upper or anterior margin is wanting (*ib.* fig. 1, α_3). Its external contour is first concave, then convex, lengthwise. The broad, subquadrate ischio-pubic walls of the pelvis on each side the symphysis are slightly concave, outwardly, both vertically and transversely; the hinder three-fourths of these appear to have been formed by the ischium (α_4): there is no 'obturator' space or vacuity, merely the outlet of the pubic perforation appears externally. Both ischia and pubis combine to form a continuous tract of bone at the symphysis, which presents a thick protuberance at its lower or outer part near its termination (Plate XXIII. α_4). The thick posterior border of the ischium is concave below the acetabulum, lengthwise, expanding into an angular tuberosity.

The outlet of the pelvis (Plate XXIV. fig. 3) is of a semielliptic form, 9 inches in transverse and 4 inches in the fore-and-aft diameters. If the inlet or brim of the pelvis (Plate XXIII. fig. 1) be defined by the smooth thick convex border of the first sacral ribs (pl), it presents an oval form, and measures 11 inches in transverse and 10 inches in fore-and-aft diameters, the latter being taken from the middle of the first sacral vertebra to the symphysis.

From the study of the above-described most interesting portion of the dicynodont skeleton, we learn—

1st. That there were no lumbar vertebræ, *i. e.* none bearing the technical anatomical characters of such†; but that free ribs continued to be developed to the pelvic or sacral series.

2nd. That the sacral series includes six vertebræ.

3rd. That the ilium, ischium, and pubis coalesce into an 'os innominatum.'

* Cuvier, 'Ossemens Fossiles,' pl. 17. fig. 39, *b*.

† This negative character is open to the same kind of objection as that relative to the 'hippocampus minor' in animals below man.

4th. That the junction of the ossa innominata with the vertebral column is effected in two ways—by an overlapping or squamous syndesmosis, and by the usual abutments: thus the anterior bony wall or surface of the pelvis, analogous to that formed by the expansion of the iliac bones in mammals, is here formed by the expanded ribs of the first sacral vertebra.

5th. That the ischium of the right side joins that of the left, and the right pubis joins the left pubis; and that both pairs of pelvic hæmapophyses are coextended and confluent, not only along a continuous ischio-pubic symphysis, as in mammals, but so as to obliterate the intervening vacuities called '*foramina ovalia* seu *obturatoria*,' thereby repeating the character of the connate abdominal hæmapophyses in the chelonian plastron.

In the comparison of this new and, at present, unique type of pelvic structure, it is interesting to observe, in connexion with the mammalian tusks in the skull, a mammalian condition of the ischio-pubic symphysis. I lay less stress on the degree of coalescence expressed by the term 'innominate bone,' because in some lizards I have observed a like confluence of iliac, ischial, and pubic bones, yet never with that amount of expansion of the iliac element which the *Dicynodon* shows, and in which, again, may be discerned a mammalian characteristic. If the remains of the huge reptiles of the extinct Dinosaurian order had not revealed to us an extent of sacrum so much surpassing that of all living Saurians, one would have laid more stress on the character of the six sacral vertebræ in the *Dicynodon*, as repeating that in some mammalia. But in this modification we may not be justified in inferring more than that, like the *Megalosaurus* and *Iguanodon*, a heavy trunk was in part supported on a pair of huge hind limbs, and the weight thereupon transferred by a larger proportion of the vertebral column in the *Dicynodon* than in the prone, crawling Crocodiles and Lizards of the present day.

As the lacertian characters prevail in the skull of the *Dicynodon*, so likewise do they in the pelvis: the backward production of the iliac bones, their confluence with the ischium and pubis, never met with in Crocodiles or Chelonians, and, above all, the oblique perforation of the pubis near its acetabular expansion, are all repetitions of structures known only, among existing Reptilia, in the lizard tribes. But the massive and entire anterior or ventral bony walls of the pelvis, the thick tumid acetabular halves of the pubic bones, and the great expanse of the over- or rather under-lapping pleurapophyses of the first sacral vertebra, are dicynodontal specialities, and suggest immense strength in this part of the massive framework of these strange extinct Reptilia.

Explanation of the Plates.

PLATE XXIII.

Dicynodon tigriceps.

Fig. 1. Front view of pelvis: $\frac{1}{4}$ nat. size.
Fig. 2. Side view of pelvis: $\frac{1}{4}$ nat. size.

PLATE XXIV.

Fig. 3. Back view of sacrum and pelvis: ¼ nat. size.
Fig. 4. Os innominatum, inner surface: ¼ nat. size.
The outline restored by dots, where abraded.

(III.) *Notice of a Skull and parts of the Skeleton of* Rhynchosaurus articeps.

By Professor Owen, *F.R.S. &c.*

The rocks in South Africa containing the Dicynodont Reptilia, appear, by other fossils, especially of some plants, to belong to the Triassic period. It is in the New Red Sandstones of Europe and of our own island, that reptilian remains have been discovered which offer the nearest approach, though the gap is wide, to the dicynodont type.

A lizard which, in biting, with trenchant edentulous jaws, may also have pierced its prey by a pair of produced weapons analogous to the tusks of *Dicynodon*, has left its remains in the New Red Sandstone of the Grinsill quarries near Shrewsbury. In this singular species (*Rhynchosaurus articeps**) the premaxillary bones, by their shape and structure—the bony tissue of the produced tips acquiring the hardness and almost the texture of dentine—and by the production of their sharp end (Plate XXV. fig. 2, 22) beyond the mandible (*ib.* 23), may have inflicted wounds and served a purpose, like those of the upper curved tusks of the *Dicynodon*.

I take the present opportunity of briefly noting a second example of this exceedingly rare fossil reptile, found, like the first, in the Grinsill sandstone, and liberally transmitted to me for examination and description by the Directors of the Museum of Natural History at Shrewsbury. This fossil consists of the skull almost entire (Plate XXV. figs. 1 and 2), with about fifteen of the anterior vertebræ (*ib.* figs. 3 and 4) more or less mutilated; and in the same slab of stone are parts or impressions of several ribs, a humerus (*ib.* fig. 3, 43), antibrachial (*ib.* 44, 45) and metapodial (*ib.* 47) bones. All these parts agree sufficiently in size and other characters with the first-discovered specimen to be referable to the same species.

The apparent greater breadth of the cranium is due to its having been crushed almost flat (fig. 1), which has correspondingly expanded the mandibular rami (fig. 3, 21, 23); but the force has been applied either so gradually, or, whether gradually or suddenly, with such equable support from the surrounding matrix, that there has occurred little fracture or dislocation.

The temporal muscles have encroached upon the sides of the parietal (fig. 1, 7), so as to develope a straight median crest along that bone. The outer and back part of the temporal fossa is bounded by a strong triangular mastoid (*ib.* 8), which unites by an oblique suture with the long postfrontal, 12. The prefrontals are seen at 14 in fig. 1. A portion of the lower or true zygomatic arch, not preserved in the first specimen, is

* Transactions of the Cambridge Philosophical Society, vol. vii.

seen at 24, in fig. 2. The long, slender, curved and dense premaxillaries (22) are in the same good state of preservation. The lower jaw is likewise preserved *in situ*, as when the mouth is shut. There is the same compound structure and entire or imperforate outer surface of the ramus as in the original specimen. The symphysis is very short, fig. 3, *y*. A long and slender bone, on the inner side of the articular end of each ramus, appears to be a part or 'cornu' of the hyoid apparatus, fig. 3, 40, 40. The vertebræ are biconcave, with broad horizontally flattened zygapophyses and a moderately high, subquadrate compressed, fore-and-aft extended spine. The ribs acquire thoracic length at about the eighth vertebra from the head, and are longitudinally grooved as in the Ichthyosaur. There is an impression of a broad scapula (*ib.* 21) and broader coracoid (*ib.* 23) near the remains of the proximal half of a humerus, *ib.* 25. The humerus has an expanded proximal end, and a concave outline behind; it has also a large medullary cavity with a compact wall. The radius (*ib.* 24) and the ulna (*ib.* 25) are distinct. The remains of apparently a metatarsus show two of the larger and two of the rather more slender of these bones. All bespeak a reptile capable of progression on dry land, as well of swimming in the sea; of one that might leave impressions of its foot-prints on a tidal shore.

The skull of a rare New Zealand Lizard is figured in Plate XXV. fig. 5, as coming nearest to the Rhynchosaur in the proportions of the divided premaxillaries (*ib.* 22), each with a large and long tooth, which, were it completely confluent with the bone, would add still more to its resemblance to the New Red Sandstone fossil.

Explanation of the Plate.

PLATE XXV. *Rhynchosaurus articeps.*

Fig. 1. Upper view of skull.
Fig. 2. Side view of skull.
Fig. 3. Under view of the mandible and fore part of skeleton.
Fig. 4. Portion of the vertebral column.
Fig. 5. Side view of the skull of *Rhynchocephalus* (recent).

All the figures are of the natural size.

XXXII. *On the Differential Coefficients and Determinants of Lines, and their Application to Analytical Mechanics.* By A. COHEN, *Esq.* *Communicated by Professor* STOKES, *Sec. R.S.*

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CHAPTER I.

1. I PROPOSE in these pages to prove the principal theorems of dynamics in a manner which appears to me both simpler and more methodical than that in which they are generally proved; and I believe that I shall be able, by applying a few conceptions which spring naturally from the principles of higher algebra and statics, to give a clear interpretation to most of the more complicated formulæ in dynamics, as well as to the several analytical steps which lead to those formulæ.

2. There are many reasons why the diagonal AD, which is constructed on the straight lines AB, AC, should be considered as the *sum* of those two lines. Those reasons may be found developed in DE MORGAN'S 'Double Algebra,' in WARREN 'On Imaginary Quantities,' and in the Tract of BENJAMIN GOMPERTZ 'On Imaginary Quantities.'

I shall therefore call AD (AD being the diagonal of the parallelogram constructed on AB and AC) the *complete sum* of AB and AC, and the two lines AB and AC will be called the *components* of AD. Moreover, denoting AB, AC, AD by P, Q, R respectively, I shall express their relation to one another by the equation

$$R=(P)+(Q).$$

I shall likewise denote by $(-Q)$ a line equal and opposite to Q, and define $(P)-(Q)$ to be the same as $(P)+(-Q)$, calling it *the complete difference* of P and Q.

All lines which have the same length and direction will be considered as equal to one another, so that any line is equivalent to a line through the origin having the same length and direction.

3. It evidently follows from the above definitions, that the complete sum of AB and BC is AC, and the complete difference of AB and AC is BC.

4. Suppose now Q to represent a line which varies with the time t both in length and direction. The *complete difference* of the two consecutive values of Q after an increment of time Δt may be called the *complete increment* of Q, and may be denoted by $\Delta(Q)$. Moreover, if we divide the length of $\Delta(Q)$ by Δt, and take the limit of that ratio, then the line which has that limit for its length, and which has for its direction the direction of $\Delta(Q)$, when Δt diminishes without limit, will be called the complete differential coefficient of Q, and will be denoted by $D_t(Q)$.

5. It will be sometimes found convenient to denote a line of length r, which is parallel or perpendicular to a line P, by $(r \parallel \text{to P})$ or $(r \perp^r \text{to P})$. Moreover, if n represent a

numerical quantity, then nP may be used to denote a line which is in the direction of P, and whose length bears to that of P the ratio of n to 1.

6. The following Lemmas, which will be of constant use, are all but self-evident:—

I. If R=(P)−(Q), then (R)−(P)+(Q)=0. In short, the ordinary rule of signs holds good.

II. $(n\mathrm{P})\pm(n\mathrm{Q})=n\{(\mathrm{P})\pm(\mathrm{Q})\}$.

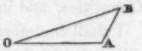
III. The projection on any line or plane of the complete sum or difference of two lines is equal to the sum or difference of their respective projections on the line or plane.

7. Whenever there is no risk of any mistake, the brackets may be omitted in the above and similar formulæ, and the *complete* sum or difference of lines may be spoken of simply as their sum or difference.

What has been hitherto said may be of course extended to all magnitudes whatsoever which can be adequately represented by straight lines, such as forces, velocities, axes of couples, axes of rotation, and accelerations, &c.

8. The application which can be made in dynamics of this conception of the complete differential coefficient of a line, will become at once apparent from the following considerations.

Suppose a particle to be moving from A to B. Let O be any fixed point. Then the particle's velocity is, according to its very definition, represented in magnitude and direction by the limit of $\frac{\mathrm{AB}}{\Delta t}$. But AB is the complete difference of O B and O A, or the complete increment of the radius vector O A, and therefore the velocity, being the limit of $\frac{\mathrm{AB}}{\Delta t}$, is the complete differential coefficient of the radius vector.



In the next place let O A and O B in the last figure represent in magnitude and direction the successive velocities of a particle at times t and $t+\Delta t$ respectively. Then, since OB=(OA)+(AB), it follows that, if with the velocity OA we compound the velocity AB, we shall obtain the velocity OB, and therefore the particle's acceleration is represented by the limit of $\frac{\mathrm{AB}}{\Delta t}$, or by the complete differential coefficient of the velocity OA. Hence we have the following proposition:—

If a particle's velocity and acceleration be represented by straight lines, the velocity will be represented by the complete differential coefficient of the radius vector drawn from a fixed point to the particle, and the acceleration will be represented by the complete differential coefficient of the velocity. Or more briefly, *the velocity is the complete differential coefficient of the radius vector, and the acceleration is the complete differential coefficient of the velocity, and is therefore the second complete differential coefficient of the radius vector.* So that if R denote the radius vector, and V and F denote respectively the velocity and acceleration, we have

$$\mathrm{V}=\mathrm{D}_t(\mathrm{R}),\quad \mathrm{F}=\mathrm{D}_t(\mathrm{V})=\mathrm{D}_t^2(\mathrm{R}).$$

9. Such then being the connexion which exists between the differential coefficient of the radius vector and the velocity and acceleration of a particle, I will proceed to prove some of the principal propositions concerning the differential coefficients of lines, and to apply them to the dynamics, or rather the kinematics of a particle.

The first proposition is the following:—

If P, Q, R represent straight lines, and if we have

$$(P)\pm(Q)=R,$$

then

$$D_t(P)\pm D_t(Q)=D_t(R).$$

For let P, Q, R after an interval of time Δt become P′, Q′, R′ respectively, then we have

$$(P')\pm(Q')=R',$$

and therefore, by Lemma I. of section 6, it follows that

$$\{(P')-(P)\}\pm\{(Q')-(Q)\}=(R')-(R),$$

or

$$\Delta(P)\pm\Delta(Q)=\Delta(R).$$

Therefore by Lemma II. of section 6, we have

$$\left(\frac{\Delta(P)}{\Delta t}\right)\pm\left(\frac{\Delta(Q)}{\Delta t}\right)=\frac{\Delta(R)}{\Delta t};$$

and taking the limit of both sides of this equation, we obtain

$$D_t(P)\pm D_t(Q)=D_t(R).$$

Similarly it may be shown that

$$D_t\{(P)\pm(Q)\pm(R)\}=D_t(P)\pm D_t(Q)\pm D_t(R).$$

Moreover, denoting the second complete differential coefficient by D_t^2, it follows that

$$\begin{aligned}D_t^2\{(P)\pm(Q)\pm(R)\}&=D_t\{D_t(P)\pm D_t(Q)\pm D_t(R)\}\\&=D_t^2(P)\pm D_t^2(Q)\pm D_t^2(R).\end{aligned}$$

10. Suppose now a line Q to have Q_x, Q_y, Q_z for its components parallel to the axes of coordinates Ox, Oy, Oz; it is evident from Lemma III. of section 6, that

$$Q=(Q_x)+(Q_y)+(Q_z).$$

It follows, therefore, from the preceding section, that

$$D_t(Q)=D_t(Q_x)+D_t(Q_y)+D_t(Q_z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

and

$$D_t^2(Q)=D_t^2(Q_x)+D_t^2(Q_y)+D_t^2(Q_z). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (II.)$$

These equations are true whether the axes of coordinates are fixed or move. But supposing the axes to be *fixed* axes, let q_x, q_y, q_z be the respective lengths of Q_x, Q_y, Q_z. Then it is evident that, as the direction of Q_x does not vary, $D_t(Q_x)$ is a line whose direction is that of Q_x or Ox, and whose length is $\frac{dq_x}{dt}$; and similarly, $D_t^2(Q_x)$ is a line whose direction is that of Ox, and whose magnitude is $\frac{d^2q_x}{dt^2}$. Similar results hold good for

$D_i(Q_x)$, $D_i^2(Q_x)$, &c. Therefore the two equations (I.) and (II.) evidently show that the components of $D_i(Q)$ and $D_i^2(Q)$ parallel to Ox are respectively equal to $\frac{dq_x}{dt}$ and $\frac{d^2q_x}{dt^2}$.

11. It is easy to apply the above results to the velocity and acceleration of a particle. For let Q in the last section stand for the radius vector of a moving particle, then the components of the radius vector are respectively equal to x, y, and z; and since the velocity is the complete differential coefficient, and the acceleration is the second complete differential coefficient of the radius vector, it follows from the last section that the components parallel to Ox of the velocity and of the acceleration are respectively equal to $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$. So that if v_x, v_y, v_z be the components of the velocity, and f_x, f_y, f_z be the components of the acceleration, we have the elementary formulæ

$$v_x=\frac{dx}{dt}, \text{ and similarly } v_y=\frac{dy}{dt}, \quad v_z=\frac{dz}{dt};$$

$$f_x=\frac{d^2x}{dt^2}, \text{ and similarly } f_y=\frac{d^2y}{dt^2}, f_z=\frac{d^2z}{dt^2}.$$

12. Our next proposition will arise from investigating the complete differential coefficient of a line Q, which varies both in magnitude and direction with the time t.

Let Q at time t be the line O A, and let it become O B at time $t+\Delta t$, so that we have

$$\Delta(Q)=(OB)-(OA)=AB.$$

Produce OA to C, making OC=OB, and draw BC. Let OA and OB have for their respective lengths q and $q+\Delta q$, and let angle $BOA=\alpha$.

Then

$$\Delta(Q)=AB=(AC)+(CB).$$

Therefore, by Lemma II. of section 6,

$$\frac{\Delta(Q)}{\Delta t}=\left(\frac{AC}{\Delta t}\right)+\left(\frac{CB}{\Delta t}\right). \quad \ldots\ldots\ldots\ldots \quad (I.)$$

Now diminish Δt indefinitely and take the limit of the last equation. The limit of $\frac{\Delta(Q)}{\Delta t}$ is $D_i(Q)$, the complete differential coefficient of Q. The limit of $\frac{AC}{\Delta t}$ is evidently a line whose length is the limit of $\frac{AC}{\Delta t}$, or $\frac{dq}{dt}$, and whose direction is that of OA or of Q. Finally, since COB is an isosceles triangle, $\frac{CB}{\Delta t}$ has for its limit a line whose length is OA limit of $\frac{\alpha}{\Delta t}$, and whose direction is perpendicular to OA or Q, and in the plane in which Q is moving at time t; and if ϖ be the rate at which the direction of Q is varying at time t, ϖ=limit of $\frac{\alpha}{\Delta t}$. Therefore, taking the limit of equation (I.), we obtain

$$D_i(Q)=\left(\frac{dq}{dt} \parallel \text{ to } Q\right)+(q\varpi \perp^r \text{ to } Q),$$

the latter line ($q\varpi \perp^r$ to Q) being in the plane in which Q is moving at time t. This is the fundamental proposition concerning the differential coefficient of a line, and may be stated in the following form:—

The complete differential coefficient of a line Q, *whose length is* Q *and whose direction is at time* t *varying with an angular velocity* ϖ, *is the complete sum or is compounded of two lines, one being* $\frac{dq}{dt}$ *in the direction of* Q, *and the other being* $q\varpi$ *in a direction perpendicular to* Q *and in the plane in which* Q *is moving at time* t. The former of these two lines would evidently be the complete differential coefficient of Q, if the *length* of Q *only* varied, and the latter would be its complete differential coefficient if the *direction* of Q *only* varied; and in this sense it may therefore be said that the complete differential coefficient of a line is the *complete sum of the two partial differential coefficients* obtained by varying separately the length and the direction of Q. One of these partial differential coefficients may be called the *length-differential coefficient*, and the other the *direction-differential coefficient* of Q, and the complete sum of these two constitutes the complete differential coefficient of Q.

13. Let Q in the preceding section stand for the velocity of a moving particle. Then $D_t(Q)$ will be the particle's acceleration, q will be the velocity v, and the direction of Q will be that of the tangent to the particle's path. Finally, ϖdt will be the angle between two consecutive tangents, so that $\varpi dt = \frac{ds}{\rho}$, ds being an element of the particle's path, and ρ the absolute radius of curvature. Therefore $\varpi = \frac{1}{\rho}\frac{ds}{dt} = \frac{v}{\rho}$. It follows then at once from the last section, that $D_t(Q)$, the particle's acceleration, is compounded of $\frac{dv}{dt}$ along the tangent, and $v\varpi$ or $\frac{v^2}{\rho}$ perpendicular to the tangent and in the plane in which the radius vector is moving at time t. In other words, the resolved part of the acceleration along the tangent is $\frac{dv}{dt} = \frac{d^2s}{dt^2}$, and the resolved part along the absolute radius of curvature is $\frac{v^2}{\rho}$.

14. The same fundamental proposition of section 12 enables us to investigate $D_t^2(Q)$, the second complete differential coefficient of a line Q, if we suppose that line to move always in the same plane. We have, namely,

$$D_t(Q) = \left(\frac{dq}{dt} \parallel \text{to Q}\right) + (\varpi q \perp^r \text{to Q}).$$

Now, in order to find $D_t^2(Q)$, we must ascertain the complete differential coefficients of $\left(\frac{dq}{dt} \parallel \text{to Q}\right)$ and of $(\varpi q \perp^r \text{to Q})$. The complete differential coefficient of the former line $\left(\frac{dq}{dt} \parallel \text{to Q}\right)$ is, by section 12,

$$\left(\frac{d^2q}{dt^2} \parallel \text{to Q}\right) + \varpi\left(\frac{dq}{dt} \perp^r \text{to Q}\right).$$

Again, the complete differential coefficient of the line ($q\varpi \perp^r$ to Q) is similarly the complete sum of the line $\left(\frac{d}{dt}(q\varpi) \perp^r \text{to Q}\right)$ and of a line whose length is $q\varpi^2$ and whose direction is perpendicular to the line ($q\varpi \perp^r$ to Q), and whose direction is therefore evidently opposite to that of Q. Hence $D_i^2(Q)$ is the complete sum of

$$\left(\frac{d^2q}{dt^2} \parallel \text{to Q}\right) \text{ and } \left(\varpi\frac{dq}{dt} \perp^r \text{to Q}\right) \text{ and } \left(\frac{d}{dt}(q\varpi) \perp^r \text{to Q}\right) \text{ and } (-q\varpi^2 \parallel \text{to Q}).$$

Therefore

$$D_i^2(Q) = \left\{\left(\frac{d^2q}{dt^2} - q\varpi^2\right) \parallel \text{to Q}\right\} + \left\{\left(q\varpi + \frac{d}{dt}(q\varpi)\right) \perp^r \text{to Q}\right\}.$$

In other words, the components of $D_i^2(Q)$ parallel and perpendicular to Q are respectively

$$\frac{d^2q}{dt^2} - q^2\varpi \text{ and } q\varpi + \frac{d}{dt}(q\varpi).$$

15. This last result may be easily applied to the dynamics of a particle. For, let Q stand for the radius vector of a particle moving in a given plane. Let that radius vector have r for its length and ϖ for its angular velocity; then, since the acceleration equals the second differential coefficient of the radius vector, it follows at once from the last section, that the components of the acceleration parallel and perpendicular to the radius vector are respectively

$$\frac{d^2r}{dt^2} - r\varpi^2 \text{ and } r\varpi + \frac{d}{dt}(r\varpi), \text{ or } \frac{1}{r}\frac{d}{dt}(r^2\varpi).$$

16. This last result is, however, but a particular instance of the connexion which exists between the actual motion of a particle, and its motion relatively to axes which move in the same plane as the particle moves. It will be found that that connexion may be easily deduced from the solution of the following problem:—

"Supposing the axes of coordinates Ox and Oy to move about O in the plane of xy with an angular velocity ϖ at time t, it is required to find the complete differential coefficient of a line Q which moves in that plane, the lengths of Q's components along the moving axes being given."

Let Q have for its components Q_x and Q_y, and let the respective lengths of these be q_x and q_y. Then, by Lemma III. of section 6, we have

$$Q = (Q_x) + (Q_y).$$

Whence it follows that

$$D_i(Q) = D_i(Q_x) + D_i(Q_y).$$

Now since Q_x and Q_y vary in direction as well as magnitude, and since the angular velocity of their change of direction is ϖ, we have, by the fundamental proposition in section 12,

$$\left.\begin{aligned} D_i(Q_x) &= \left(\frac{dq_x}{dt} \parallel \text{to } Ox\right) + (\varpi q_x \perp^r \text{to } Ox), \\ D_i(Q_y) &= \left(\frac{dq_y}{dt} \parallel \text{to } Oy\right) + (\varpi q_y \perp^r \text{to } Oy). \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(I.)}$$

But since the lines ($\varpi q_x \perp^r$ to Ox) and ($\varpi q_y \perp^r$ to Oy) are respectively proportional and perpendicular to Q_x and Q_y, and since the latter lines have Q for their complete sum, it evidently follows that the former two lines have for their complete sum a line which is perpendicular to Q, and whose length is ϖq.

Hence

$$D_t(Q)=\left(\frac{dq_x}{dt} \parallel \text{to } Ox\right)+\left(\frac{dq_y}{dt} \parallel \text{to } Oy\right)+(\varpi q \perp^r \text{to Q}). \quad . \quad . \quad . \quad . \quad \text{(II.)}$$

17. This last formula is true whether the axes be rectangular or oblique, and may be made the basis of all the formulæ of relative motion in one plane.

It may be observed that the line

$$\left(\frac{dq_x}{dt} \parallel \text{to } Ox\right)+\left(\frac{dq_y}{dt} \parallel \text{to } Oy\right)$$

is what would be the complete differential coefficient of Q if the coordinate axes were fixed; and it may therefore be called the complete differential coefficient relative to the moving axes, or, more briefly, the *relative differential coefficient* of Q. So that the above formula shows that *the complete differential coefficient of* Q is *its relative differential coefficient together with a line* ($\varpi q \perp^r$ to Q), the latter line being drawn towards the direction in which the axes are revolving.

18. If the axes of coordinates be rectangular, then the line ($\varpi q_x \perp^r$ to Ox) is evidently the same as ($\varpi q_x \parallel$ to Oy), and the line ($\varpi q_y \perp^r$ to Oy) is the same as ($-\varpi q_y \parallel$ to Ox); and therefore, looking at the equations (I.) in section 16, we see that

$$\begin{aligned} D_t(Q)&=D_t(Q_x)+D_t(Q_y) \\ &=\left(\left(\frac{dq_x}{dt}-\varpi q_y\right) \parallel \text{to } Ox\right)+\left(\left(\frac{dq_y}{dt}+\varpi q_x\right) \parallel \text{to } Oy\right). \end{aligned}$$

In other words, the components of $D_t(Q)$ parallel to Ox and Oy are respectively

$$\frac{dq_x}{dt}-\varpi q_y \text{ and } \frac{dq_y}{dt}+\varpi q_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III.)}$$

The same result may be also deduced from formula (II.) in the same section, if we resolve the line ($\varpi q \perp^r$ to Q) along the rectangular axes of x and y.

19. Let us apply the above formulæ first to the velocity of a particle.

Suppose, then, a particle to move in a given plane, and that the rectangular axes of coordinates in that plane revolve about the origin with an angular velocity ϖ at time t. Let v_x and v_y be the components of the particle's velocity along the moving axes. Then, since the velocity is the complete differential coefficient of the radius vector, and since x and y are the components of that radius vector, it follows from the formulæ (III.) of the last section, that

$$v_x=\frac{dx}{dt}-y\varpi,$$

$$v_y=\frac{dy}{dt}+x\varpi.$$

If the radius vector be chosen as axis of x, then $x=r$, $y=0$; therefore $v_x=\frac{dx}{dt}$ along the radius vector, $v_y=r\varpi$ perpendicular to the radius vector, where ϖ is the angular velocity of the radius vector.

20. Let us now apply the same formulæ (III.) of section 18 to the acceleration of a particle. Let v_x and v_y, as before, denote the components of the velocity, and let f_x and f_y denote the components of the acceleration of the particle. Then, since the acceleration is the complete differential coefficient of the velocity which has v_x, v_y for its components, it follows at once from the formulæ (III.), that

$$f_x=\frac{dv_x}{dt}-v_y\varpi,$$

$$f_y=\frac{dv_y}{dt}+v_x\varpi.$$

Suppose now the axis of x to be the radius vector, then we have already shown that $v_x=\frac{dr}{dt}$, $v_y=r\varpi$. Therefore by substituting these values in the last formulæ, we see that f_x, the acceleration along the radius vector, is $\frac{d^2r}{dt^2}-r\varpi^2$, and f_y, the acceleration perpendicular to the radius vector, is $\frac{dt}{d}(r\varpi)+\varpi\frac{dr}{dt}=\frac{1}{r}\frac{d}{dt}(\varpi r^2)$, which is the same result as was obtained in section 15.

21. Returning to the more general case, we have, as before,

$$f_x=\frac{dv_x}{dt}-v_y\varpi,$$

$$f_y=\frac{dv_y}{dt}+v_x\varpi;$$

and substituting in these the values already obtained for v_x and v_y, namely, $\frac{dx}{dt}-y\varpi$, $\frac{dy}{dt}+x\varpi$, we find

$$\left.\begin{aligned} f_x&=\frac{d^2x}{dt^2}-2\frac{dy}{dt}\varpi-\varpi^2x-y\frac{d\varpi}{dt},\\ f_y&=\frac{d^2y}{dt^2}+2\frac{dx}{dt}\varpi-\varpi^2y+x\frac{d\varpi}{dt},\end{aligned}\right\} \quad \ldots\ldots\ldots \text{(IV.)}$$

which are the formulæ for the components of the acceleration in terms of the coordinates of the particle.

22. The last formulæ (IV.) may also be obtained in the following manner.

Let Q_x and Q_y represent respectively the components of the radius vector Q along the axes of x and y. Then

$$Q=(Q_x)+(Q_y).$$

Therefore

$$D_t^2(Q)=D_t^2(Q_x)+D_t^2(Q_y).$$

But since the axes of x and y revolve at time t with an angular velocity ϖ, and since Q_x and Q_y have x and y for their respective lengths, it follows from section 14, that

$$D_t^2(Q_x)=\left\{\left(\frac{d^2x}{dt^2}-x\varpi^2\right)\parallel \text{to } Ox\right\}+\left\{\frac{1}{x}\frac{d}{dt}(x^2\varpi)\perp' \text{to } Ox\right\},$$

and that

$$D_t^2(Q_y)=\left\{\left(\frac{d^2y}{dt^2}-y\varpi^2\right)\parallel \text{to } Oy\right\}+\left\{\frac{1}{y}\frac{d}{dt}(y^2\varpi)\perp' \text{to } Oy\right\}.$$

Whence it is easy to see that the components of $D_t^2(Q)$, or of the particle's acceleration, are

$$\left.\begin{aligned} f_x&=\frac{d^2x}{dt^2}-x\varpi^2-\frac{1}{y}\frac{d}{dt}(\varpi y^2),\\ f_y&=\frac{d^2y}{dt^2}-y\varpi^2+\frac{1}{x}\frac{d}{dt}(\varpi x^2),\end{aligned}\right\} \quad \ldots\ldots\ldots \text{(V.)}$$

which equations are clearly the same as those obtained in the preceding section.

We shall soon prove similar formulæ for the more general case of a particle and axes of coordinates moving in any manner whatsoever in space of three dimensions, and therefore, in order to prevent needless repetition, we shall postpone the further discussion and complete interpretation of the equations (IV.) or (V.).

23. It is, however, interesting here to observe that all the results already obtained may be readily deduced from the principles of what Professor De Morgan has called "Double Algebra." According to those principles, namely, the radius vector R whose length is r and whose inclination to a fixed line is θ, is symbolically represented by $re^{\theta\sqrt{-1}}$, so that we have

$$R=re^{\theta\sqrt{-1}}.$$

Therefore

$$D_t(R)=e^{\theta\sqrt{-1}}\left(\frac{dr}{dt}+r\frac{d\theta}{dt}\sqrt{-1}\right);$$

and the last expression represents the complete sum of

$$\left(\frac{dr}{dt}\parallel \text{to R}\right) \text{ and } \left(r\frac{d\theta}{dt}\perp' \text{to R}\right).$$

This result is the same as that arrived at in section 12.

Again, $D_t^2(R)=D_t(D_t(R))=$

$$e^{\theta\sqrt{-1}}\left(\frac{d^2r}{dt^2}+\sqrt{-1}\frac{d}{dt}\left(r\frac{d\theta}{dt}\right)+\frac{d\theta}{dt}\sqrt{-1}\left(\frac{dr}{dt}+r\frac{d\theta}{dt}\sqrt{-1}\right)\right)$$

$$=e^{\theta\sqrt{-1}}\left(\frac{d^2r}{dt^2}-r\frac{d\theta^2}{dt}+\sqrt{-1}\left(\frac{d}{dt}\left(r\frac{d\theta}{dt}\right)+\frac{dr}{dt}\frac{d\theta}{dt}\right)\right);$$

and this expression represents the complete sum of a line

$$\frac{d^2r}{dt^2}-r\frac{d\theta^2}{dt}\parallel \text{to R and a line } \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)+\frac{dr}{dt}\frac{d\theta}{dt}\perp' \text{to R}.$$

This result is the same as that arrived at in section 14.

Finally, in order to obtain the formulæ for relative motion, we have merely to put

$$R=re^{(\theta+\alpha)\sqrt{-1}},$$

where θ is the angle made by R with the moving axis of x, and α is the angle made by that moving axis with a fixed line. It follows then that

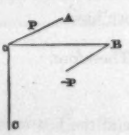
$$D_t(R)=e^{\alpha\sqrt{-1}}\frac{d}{dt}\left(re^{\theta\sqrt{-1}}\right)+r\frac{d\alpha}{dt}\sqrt{-1}\,e^{(\theta+\alpha)\sqrt{-1}}.$$

Now it is evident that $e^{\alpha\sqrt{-1}}\frac{d}{dt}\left(re^{\theta\sqrt{-1}}\right)$ represents *the relative differential coefficient* of R, and $r\frac{d\alpha}{dt}\sqrt{-1}\,e^{(\theta+\alpha)\sqrt{-1}}$ represents a line $r\frac{d\alpha}{dt}\perp^r$ to R. We thus obtain the same result as in section 16.

By differentiating again it would also be easy to deduce the result of section 20, if we observe that $e^{\alpha\sqrt{-1}}\frac{d^2}{dt^2}\left(re^{\theta\sqrt{-1}}\right)$ represents the particle's relative acceleration whose components are $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$.

Chapter II.

24. In order to extend the formulæ which we have proved for the motion of a particle in one plane to the motion of a particle in space, it will be found very convenient to make use of a conception which presents itself in statics, as soon as the equilibrium of a solid body is treated of in that science.



Let O A and O B be any two straight lines drawn from the origin O. If then O A represent a force P, and if we apply at B a force $-$P, we shall obtain *a couple*. Let O C be the *axis* of that couple. We know then from statics that, if O A and O B have for their projections on the axes of coordinates X, Y, Z and x, y, z, then O C has for its projections

$$zY-yZ,\quad xZ-zX,\quad yX-xY. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

Now the relation which the line O C bears to the lines O A and O B is one which not only presents itself in statics, but which also plays a very important part in the differentiation of lines, and in the dynamics both of a particle and of a body. For this reason it will be proper to treat of the relation in question quite independently of statical considerations; and since the expressions (I.), which are the projections of O C, are evidently what are called *determinants*, I shall call the line O C *the determinant of* O B *to* O A.

Hence we have the following definition:—

"The determinant of a line Q to a line P is a line which is equal to twice the area of the triangle of which the lines P and Q drawn from the origin are sides, and which is perpendicular to that area, and the line is moreover drawn in such a direction that, to

an eye looking along it towards the origin, the revolution of Q towards P appears to be a revolution in the positive direction."

The determinant of Q to P may be briefly denoted by

$$\det (Q, P).$$

It is evident from the above definition, that the determinant of Q to P is a line equal and opposite to the determinant of P to Q.

Moreover, if the projections of P on the axes of coordinates be p_x, p_y, p_z, and those of Q be q_x, q_y, q_z, then it follows from the formulæ (I.), that the determinant of Q to P or det (Q, P) has for its projections

$$q_y p_z - q_z p_y, \quad q_z p_x - q_x p_z, \quad q_x p_y - q_y p_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (II.)$$

25. The connexion which exists between the notion of a determinant of lines and the elementary conceptions of dynamics may be easily made apparent. For suppose a particle at the extremity B of O B to be revolving about the line O A with an angular velocity represented in magnitude by O A, then if O C be drawn perpendicular to the plane A O B, and equal to twice the area A O B, it is evident that O C will represent the linear velocity of the particle. But O C is then by definition the same thing as the *determinant of* O A *to* O B. Whence it follows *that the determinant of* O A *to* O B *represents the velocity of the point* B, *due to a rotation whose axis and angular velocity are represented by* O A.

This result, together with the result of the preceding section, may then be recapitulated in the following manner. If V represent the velocity of a particle at the extremity of the radius vector R, and the particle rotates about an axis which is represented by the line Ω, then, if the angular velocity is represented by the length of Ω, we have

$$V = \det (\Omega, R).$$

Secondly, if P represent a force at the origin and R represent the radius vector at the extremity of which a force $-P$ acts, then the axis of the couple $(P, -P)$ is $\det (P, R)$ or $\det (R, -P)$; so that $\det (R, P)$ is what French writers call "the moment-axis of a force P with respect to the origin."

26. Such, then, being the connexion between determinants and statical and dynamical conceptions, I will proceed to prove some of the more important propositions concerning the determinants of lines.

The most important theorem concerning the determinants of lines is the following:—"If P, P′, and Q be three straight lines drawn from the origin, then

$$\det (P, Q) + \det (P', Q) = \det \{(P) + (P'), Q\}."$$

This proposition might be easily proved by geometry, but it is at once deducible from statics. For, consider two couples having a common arm Q, and having forces P and P′ respectively acting at the extremity of Q at the origin O. The resultant of those two couples will be a couple having the same arm Q, and having for its force acting at O the resultant of P and P′, or $(P)+(P')$. Now it is proved in statics that the *axis* of this

resultant couple is the complete sum of the axes of two component couples. Therefore, substituting for those axes the equivalent determinants, we see that the determinant of (P)+(P′) to Q is the complete sum of the determinant of P to Q, and of the determinant of P′ to Q. Thus we have

$$\det(P, Q)+\det(P', Q)=\det\{(P)+(P'), Q\}. \quad \text{(I.)}$$

And it may similarly be shown that

$$\det(P, Q)-\det(P' Q)=\det\{(P)-(P'), Q\}. \quad \text{(II.)}$$

The same proposition follows also easily from a consideration of the linear expressions (II.) in section 24 for the projections of a determinant, and is in fact equivalent to a fundamental theorem concerning algebraic determinants, which theorem may be found in SALMON'S 'Lessons on Higher Algebra,' section 19, page 9.

27. The proposition proved in the last section will be found to be of constant use in explaining and shortening analytical processes in mechanics. One useful application can be made of it in proving "the parallelogram of angular velocities." For taking Q to be the radius vector of a particle, and P and P′ to represent two axes and angular velocities of rotation then the formula (I.) of the last section translates itself at once by means of section 25 into the following proposition:—"The linear velocity of a particle due to a rotation whose axis and angular velocity are represented by the line P, compounded with the linear velocity due to a rotation similarly represented by the line P′, is equivalent to the linear velocity due to a rotation represented by the complete sum of P and P′." And this is evidently the same as the proposition called "the parallelogram of angular velocities of rotation."

28. Let us next investigate the complete differential coefficient of det (P, Q).

We will first premise that, if m be any numerical quantity, it follows evidently from the definition of a determinant, that

$$\det(mP, Q)=m\det(P, Q)=\det(P, mQ). \quad \text{(I.)}$$

Suppose now that P and Q after an interval of time Δt become respectively (P)+(ΔP), (Q)+(ΔQ), the sign + here denoting the *complete* sum. Then the complete increment of det (P, Q) is

$$\left.\begin{aligned}&\det(P+\Delta P, Q+\Delta Q)-\det(P, Q)=\\ &\quad\det(P+\Delta P, Q+\Delta Q)-\det(P+\Delta P, Q)+\det(P+\Delta P, Q)-\det(P, Q).\end{aligned}\right\} \quad \text{(II.)}$$

But it follows from formula (II.) of section 26, that

$$\det(P+\Delta P, Q+\Delta Q)-\det(P+\Delta P, Q)=\det(P+\Delta P, \Delta Q);$$

and similarly,

$$\det(P+\Delta P, Q)-\det(P, Q)=\det(\Delta P, Q).$$

Substitute, then, these values in the above equation (II.), and divide both sides by Δt by means of the above formula (I.), and finally let Δt diminish without limit. We thus obtain for the complete differential coefficient of det (P, Q)

$$\det(P, D_t(Q))+\det(D_t(P), Q).$$

We have therefore the equation

$$D_t\{\det(P, Q)\} = \det\{P, D_t(Q)\} + \det\{D_t(P), Q\}.$$

The same equation may also be proved by considering the algebraical determinants which represent the projections of det (P, Q); and it may in fact be easily deduced from the following identical equation,

$$\frac{d}{dt}(q_x p_y - q_y p_x) = \left(\frac{dq_x}{dt} p_y - \frac{dq_y}{dt} p_x\right) + \left(q_x \frac{dp_y}{dt} - q_y \frac{dp_x}{dt}\right).$$

29. It may be here observed that the formulæ (I.) and (II.) in section 26, and the formulæ (I.) and (II.) in section 28, show that there exists an intimate *symbolical* connexion between det (P, Q) and the product P, Q. In fact the only difference between their symbolical properties consists in P and Q *not* being commutative in the expression det (P, Q), and being so in the expression for the product.

30. There is one more proposition which is often very useful in analytical dynamics. Let it be required to find det (R, Q′), where Q′ itself equals det (P, Q). Let the required line det (R, Q′) be denoted by U. Then, by the definition of a determinant, U is perpendicular on R and on Q′, which last line is itself perpendicular on the plane containing P and Q. Hence it follows that U is perpendicular on R and in the plane containing P and Q.

We have still to find the magnitude of U. For this purpose let the angle which R makes with the plane containing P and Q be ψ, so that ψ is the complement of the angle between R and Q′.

Moreover, let θ be the angle between P and Q, and let the magnitudes of P, Q, Q′, R be denoted by p, q, q', r respectively. Then, since U is det (R, Q′), it follows from the definition of a determinant, that the length of U equals $rq' \sin\left(\frac{\pi}{2} - \psi\right)$ or $rq' \cos\psi$. Similarly, $q' = pq \sin\theta$. Hence the length of U equals $pqr \sin\theta \cos\psi$.

There are two cases especially which frequently occur in dynamics, first, when R is identical with Q, and secondly, when R is perpendicular on Q.

Let us first take the case of R being identical with Q; then $\psi = 0$ and $r = q$. Therefore the required determinant is a line in the plane containing P and Q, and perpendicular on R or Q, and its length equals $pq^2 \sin\theta$.

If, moreover, Q is perpendicular on P, then the required determinant is in the direction of P, and its length equals pq^2, since $\theta = \frac{\pi}{2}$. So that, if P is perpendicular on Q, we see that det {Q, det (P, Q)} is a line pq^2 in the direction of P; and therefore evidently det {Q, det (Q, P)} is a line pq^2 opposite to P.

Let us now take the case of R being perpendicular on Q. Then it might be easily proved by spherical trigonometry, that $\sin\theta \cos\psi$ equals the cosine of the angle between R and P. But we will prove this by analysis, because in doing so we shall meet with formulæ which will be of use in the sequel.

Let, then, the components of P, Q, Q', R parallel to any three axes of coordinates be denoted by p_x, p_y, p_z, q_x, &c., q'_x, &c., r_x, &c. Then, if we denote the components of $U=\det(R, Q')$ by u_x, u_y, u_z we have, by section 24,

$$u_x=q_y r_z-q_z r_y; \qquad \text{(I.)}$$

and since $Q'=\det(P, Q)$, we have

$$q'_x=q_y p_z-q_z p_y$$
$$q'_y=q_z p_x-q_x p_z$$
$$q'_z=q_x p_y-q_y p_x$$

Hence, substituting the values of q'_x and q'_y in (I.), we get

$$u_x=p_x(q_y r_y+q_z r_z)-q_x(p_y r_y+p_z r_z).$$

Now by hypothesis R is perpendicular on Q; hence

$$q_x r_x+q_y r_y+q_z r_z=0;$$

therefore

$$q_y r_y+q_z r_z=-q_x r_x.$$

Therefore

$$u_x=-q_x(p_x r_x+p_y r_y+p_z r_z).$$

But as q and r denote the magnitudes of P and R, it is evident that

$$p_x r_x+p_y r_y+p_z r_z=pr\cos\varphi,$$

where φ denotes the angle between P and R. Therefore

$$u_x=-prq_x\cos\varphi.$$

Similarly

$$u_y=-prq_y\cos\varphi,$$
$$u_z=-prq_z\cos\varphi.$$

Therefore the line U of which u_x, u_y, u_z are the components is a line in direction opposite to Q, and whose length equals $prq\cos\varphi$, q being the length of Q. Hence if R be perpendicular on Q, then $\det(R, \det(P, Q))$ equals $-pqr\cos\varphi$ in the direction of Q.

It follows from the above proposition, that, if Q' or $\det(P, Q)$ represent a force or acceleration which acts at the extremity of the radius vector R, and if Q be perpendicular on R, then the moment-axis of that force or acceleration about the origin is $-pqr\cos\varphi$ in the direction of Q, and the moments of such force or acceleration about the coordinate axes are respectively $-pr\cos\varphi\, q_x$, $-pr\cos\varphi\, q_y$, $-pr\cos\varphi\, q_z$; and as φ is the angle between P and the radius vector R, $pr\cos\varphi=xp_x+yp_y+zp_z$, if x, y, z be the coordinates of the extremity of the radius vector.

Chapter III.

31. We are now in a condition to treat fully of the motion of a particle in space of three dimensions; and it will be found that the propositions which have just been proved concerning the determinants of lines, will enable us to show how all the results

arrived at as to a particle's motion in one plane may be extended to motion in space generally.

32. Suppose Q to represent a line drawn from the origin, varying both in direction and magnitude in any manner whatsoever, and let it be required to investigate $D_t(Q)$ the complete differential coefficient of Q.

Let the length of Q be q at time t, and let the direction of Q be revolving at time t about a line whose direction is that of the line represented by Ω, and let the length of Ω be the angular velocity ϖ, with which Q's direction is revolving at time t.

It has been already shown in section 12 that $D_t(Q)$ is in all cases the complete sum of the two partial differential coefficients which are obtained by varying separately the length and direction of Q. Now the former partial differential coefficient is evidently $\left(\frac{dq}{dt} \parallel \text{to } Q\right)$, and the other partial differential coefficient is, by section 25, equal to $\det(\Omega, Q)$. Hence we have the following fundamental equation,

$$D_t(Q) = \left(\frac{dq}{dt} \parallel \text{to } Q\right) + \det(\Omega, Q). \quad \ldots\ldots\ldots\ldots \quad (I.)$$

33. It is not difficult to deduce from the last equation the expression for $D_t^2(Q)$, the second complete differential coefficient of Q. In order to find that expression we must take the complete differential coefficient of each of the expressions of which the right-hand member of equation (I.) is composed. For this purpose represent for a moment the line $\left(\frac{dq}{dt} \parallel \text{to } Q\right)$ by Q_1. Then it follows from the fundamental formula of the preceding section, that

$$D_t^2(Q) = D_t(Q_1) + D_t(\det(\Omega, Q)).$$

Now the formula (I.) of the last section gives evidently

$$D_t(Q_1) = \left(\frac{d^2q}{dt^2} \parallel \text{to } Q_1\right) + \det(\Omega, Q_1),$$

or

$$\left(\frac{d^2q}{dt^2} \parallel \text{to } Q\right) + \det(\Omega, Q_1).$$

Moreover we have, according to section 28 of the preceding Chapter,

$$D_t\{\det(\Omega, Q)\} = \det\{D_t(\Omega), Q\} + \det\{\Omega, D_t(Q)\}.$$

But since

$$D_t(Q) = (Q_1) + \det(\Omega, Q),$$

it follows from section 26 of the preceding Chapter, that

$$\det(\Omega, D_t(Q)) = \det(\Omega, Q_1) + \det\{\Omega, \det(\Omega, Q)\}.$$

Therefore, collecting the above results, we obtain

$$D_t^2(Q) = D_t(Q_1) + D_t\{\det(\Omega, Q)\} =$$

$$\left(\frac{d^2q}{dt^2} \parallel \text{to } Q\right) + 2\det(\Omega, Q_1) + \det(D_t(\Omega), Q) + \det\{\Omega, \det(\Omega, Q)\}.$$

The two last terms of this expression are evidently what would be $D_t^2(Q)$ if $\frac{dq}{dt}$ were zero, that is to say, if Q did not vary in *magnitude*; and $\left(\frac{d^2q}{dt^2} \parallel \text{to } Q\right)$ is evidently what $D_t^2(Q)$ would be if Q did not vary in *direction*; so that we have the following proposition:—

"$D_t^2(Q)$ is the complete sum of the two partial second differential coefficients obtained by varying separately the length and the direction of Q, together with 2 det (Ω, Q_1), where Q_1 is the line $\left(\frac{dq}{dt} \parallel \text{to } Q\right)$."

34. Suppose Q to be R the radius vector of a moving particle, the length of which radius vector is r, then $D_t^2(Q)$ is the particle's acceleration; Q_1 is $\left(\frac{dr}{dt} \parallel \text{to R}\right)$, and is therefore the velocity along the radius vector. If, then, we denote this by R_1, the equation arrived at in the last section shows that the acceleration is compounded of

$$\left(\frac{d^2r}{dt^2} \parallel \text{to R}\right) + 2 \det(\Omega, R_1),$$

and of what would be the particle's acceleration if R did not vary in magnitude, that is to say, if the particle simply revolved about the origin. And this latter acceleration is again compounded of det $(D_t(\Omega), R)$ and det $\{\Omega, \det(\Omega, R)\}$. The last line is, by section 30, a line drawn from the extremity of R, or from the particle, perpendicular to and towards Ω, and whose magnitude is $\varpi^2 p$, p being the length of that perpendicular.

35. The above result is, however, but a particular instance of the theory of the motion of a particle relatively to axes which revolve about the origin, a subject which we are now in a condition to treat of very simply in its utmost generality. That theory will be found to depend upon the solution of the following problem:—

"Supposing the axes of coordinates Ox, Oy, Oz to revolve round the origin O about an axis Ω at time t with angular velocity ϖ (which is the length of Ω), it is required to find the complete differential coefficient of a line Q, the components of Q along the coordinate axes being given."

Let Q have for its components Q_x, Q_y, Q_z, and let the respective lengths of these be q_x, q_y, q_z. Then evidently

$$Q = (Q_x) + (Q_y) + (Q_z).$$

Therefore

$$D_t(Q) = D_t(Q_x) + D_t(Q_y) + D_t(Q_z).$$

But, by the fundamental formula of section 32, we have

$$D_t(Q_x) = \left(\frac{dq_x}{dt} \parallel \text{to } Ox\right) + \det(\Omega, Q_x),$$

$$D_t(Q_y) = \left(\frac{dq_y}{dt} \parallel \text{to } Oy\right) + \det(\Omega, Q_y),$$

$$D_t(Q_z) = \left(\frac{dq_z}{dt} \parallel \text{to } Oz\right) + \det(\Omega, Q_z).$$

Now we know, from section 26 of Chapter II., that

$$\begin{aligned}&\det(\Omega, Q_x)+\det(\Omega, Q_y)+\det(\Omega, Q_z)\\ &=\det\{\Omega, (Q_x)+(Q_y)+(Q_z)\}\\ &=\det(\Omega, Q).\end{aligned}$$

Therefore

$$D_t(Q)=\left(\frac{dq_x}{dt} \parallel \text{to } Ox\right)+\left(\frac{dq_y}{dt} \parallel \text{to } Oy\right)+\left(\frac{dq_z}{dt} \parallel \text{to } Oz\right)+\det(\Omega, Q).$$

This formula is true whether the axes be rectangular or oblique, and may be made the basis of all the formulæ of relative motion.

It may be observed that, if the coordinate axes did not move, $D_t(Q)$ would be equivalent to

$$\left(\frac{dq_x}{dt} \parallel \text{to } Ox\right)+\left(\frac{dq_y}{dt} \parallel \text{to } Oy\right)+\left(\frac{dq_z}{dt} \parallel \text{to } Oz\right).$$

So that the line represented by the last expression may be called the differential coefficient of Q *relatively to the moving axis*, or, more briefly, *the relative differential coefficient of* Q. The above formula of the last section therefore shows that *the complete differential coefficient of* Q *is the relative differential coefficient of* Q *together with det* (Ω, Q).

This proposition exactly corresponds with the proposition in section 17 of Chapter I.

36. Assuming now the axes of coordinates to be rectangular, we know, from formulæ (II.) in section 24 of Chapter II., that $\det(\Omega, Q)$ has for its components

$$\begin{aligned}&q_z\varpi_y-q_y\varpi_z \text{ parallel to } Ox,\\ &q_x\varpi_z-q_z\varpi_x \text{ parallel to } Oy,\\ &q_y\varpi_x-q_x\varpi_y \text{ parallel to } Oz.\end{aligned}$$

Therefore it follows from the preceding section, that the components of $D_t(Q)$ are

$$\begin{aligned}&\frac{dq_x}{dt}+q_z\varpi_y-q_y\varpi_z,\\ &\frac{dq_y}{dt}+q_x\varpi_z-q_z\varpi_x,\\ &\frac{dq_z}{dt}+q_y\varpi_x-q_x\varpi_y.\end{aligned}$$

These formulæ are in fact simply the analytical expression of the fundamental proposition in the preceding section, and correspond exactly to the formulæ in section 18 of Chapter I.

37. Let us now apply the above formulæ to dynamics, and first to the velocity of a particle.

Suppose, then, a particle to move in space in any manner whatsoever, and suppose that the rectangular axes of coordinates revolve about a line Ω at time t with angular velocity ϖ, ϖ being the length of Ω. Let v_x, v_y, v_z be the components of the particle's velocity, and ϖ_x, ϖ_y, ϖ_z the components of Ω. Then, since the velocity is the complete differential coefficient of the radius vector R of the particle, and since x, y, z are the components of

R, it follows at once from the formulæ of the preceding section, that

$$v_x=\frac{dx}{dt}+z\varpi_y-y\varpi_z,$$

$$v_y=\frac{dy}{dt}+x\varpi_z-z\varpi_x,$$

$$v_z=\frac{dz}{dt}+y\varpi_x-x\varpi_y.$$

These formulæ simply express the fact, that the absolute velocity is equivalent to the relative velocity together with $\det(\Omega, R)$.

38. Let us next apply the formulæ to the acceleration of a particle. Let, as before, v_x, v_y, v_z be the components of the particle's velocity V, and let f_x, f_y, f_z be the components of the particle's acceleration. Then, since the acceleration is the complete differential coefficient of the velocity, of which v_x, v_y, v_z are the components, it follows at once from section 36, that

$$f_x=\frac{dv_x}{dt}+v_z\varpi_y-v_y\varpi_z,$$

$$f_y=\frac{dv_y}{dt}+v_x\varpi_z-v_z\varpi_x,$$

$$f_z=\frac{dv_z}{dt}+v_y\varpi_x-v_x\varpi_y.$$

If we substitute in the last equations the values obtained for v_x, v_y, v_z in the preceding section, we obtain

$$f_x=\frac{d^2x}{dt^2}+z\frac{d\varpi_y}{dt}-y\frac{d\varpi_z}{dt}+2\left(\frac{dz}{dt}\varpi_y-\frac{dy}{dt}\varpi_z\right)+(y\varpi_z-z\varpi_y)\varpi_y-(x\varpi_z-z\varpi_x)\varpi_x,$$

and similar formulæ for f_y and f_z.

These are the ordinary formulæ. It would not be difficult to deduce their real meaning from their analytical form; but it will be better first to prove the result of such interpretation in a different and more direct manner.

39. We have already seen that, if V denote the particle's absolute velocity, and R the radius vector, and if V_1 denote the relative velocity which has $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ for its components, then

$$V=V_1+\det(\Omega, R).$$

Let then F denote the particle's absolute acceleration, and let F_1 denote the particle's relative acceleration which has $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ for its components; then

$$F=D_t(V)=D_t(V_1)+D_t\det(\Omega, R). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

Now it follows from the fundamental proposition in section 35, that

$$D_t(V_1)=F_1+\det(\Omega, V_1).$$

Moreover we know from section 28, that

$$D_t \det(\Omega, R) = \det(D_t(\Omega), R) + \det(\Omega, D_t(R)).$$

But since $D_t(R) = V = V_1 + \det(\Omega, R)$, therefore

$$\det(\Omega, D_t(R)) = \det(\Omega, V_1) + \det\{\Omega, \det(\Omega, R)\}.$$

Hence, collecting the above results, and substituting them in the equation (I.), we find

$$F = F_1 + 2\det(\Omega, V_1) + \det(D_t(\Omega), R) + \det\{\Omega, \det(\Omega, R)\}.$$

It may be observed as to this formula, that if $V_1 = 0$, that is to say, if the particle had no *relative* motion, and moved as if rigidly connected with the axes of coordinates, then the two first terms of the last equation would vanish; and therefore its other two terms are what the acceleration would be if the particle had no relative motion, and they represent what may therefore be conveniently termed the particle's *system-acceleration*. French writers have given to this acceleration the name of "accélération d'entraînement;" it is the acceleration of a point which is in the position of the moving particle, and which is supposed to be rigidly connected with the system of moving axes, and I therefore propose to call it "system-acceleration." Using then this expression, we have the following proposition:—"The acceleration of a particle is equivalent to its acceleration relatively to a system of axes revolving about a fixed point, together with the system-acceleration corresponding to the particle and together with an acceleration equal to $2\det(\Omega, V_1)$, V_1 being the particle's relative velocity, and Ω the axis about which the system is revolving at time t." Or, more briefly, a particle's absolute acceleration equals the complete sum of its relative acceleration and of its system-acceleration together with $2\det(\Omega, V_1)$. Such is the brief expression of CORIOLI's beautiful and very useful proposition concerning relative motion.

40. We have just seen that the particle's system-acceleration is compounded of $\det(D_t(\Omega), R)$ and $\det\{\Omega, \det(\Omega, R)\}$. As regards the latter line, it is clear from section 30 that it is in the direction of the line drawn from the particle perpendicular to and towards the axis Ω, and that its magnitude is $\varpi^2 p$, p being the length of that perpendicular. It is therefore equal and opposite to what is usually called "the centrifugal force."

As regards the other line $\det(D_t(\Omega), R)$, it is of course at once determined as soon as $D_t(\Omega)$ is known. Now if $\varpi_x, \varpi_y, \varpi_z$ be the components of Ω, it follows from the fundamental proposition in section 35 that $D_t(\Omega)$ is equivalent to

$$\left(\frac{d\varpi_x}{dt} \parallel \text{to } Ox\right) + \left(\frac{d\varpi_y}{dt} \parallel \text{to } Oy\right) + \left(\frac{d\varpi_z}{dt} \parallel \text{to } Oz\right) + \det(\Omega, \Omega).$$

But it is clear, from the very definition of a determinant, that $\det(\Omega, \Omega)$ is zero. Hence we see that the components of $D_t(\Omega)$ are $\frac{d\varpi_x}{dt}, \frac{d\varpi_y}{dt}, \frac{d\varpi_z}{dt}$. This is an important proposition often used in the dynamics of a rigid body, and generally proved by means of a good deal of analytical work. It is usually expressed in the following manner. If

$\varpi_1, \varpi_2, \varpi_3$ be the components of ϖ along fixed axes, and $\varpi_x, \varpi_y, \varpi_z$ be its components along moving axes which coincide with the former at time t, then $\frac{d\varpi_1}{dt}=\frac{d\varpi_x}{dt}, \frac{d\varpi_2}{dt}=\frac{d\varpi_y}{dt}, \frac{d\varpi_3}{dt}=\frac{d\varpi_z}{dt}$. It is evident that this amounts to saying that $D_t(\Omega)$ has $\frac{d\varpi_x}{dt}, \frac{d\varpi_y}{dt}, \frac{d\varpi_z}{dt}$ for its components; and we have just seen how that proposition follows at once from the fundamental theorem in section 35, and from the self-evident fact that $D_t(\Omega, \Omega)=0$.

41. Recapitulating then the results of the two last sections, we see that a particle's system-acceleration is equivalent to $\det(D_t(\Omega), R)$ minus the centrifugal force, and that the absolute acceleration of the particle is compounded of the relative acceleration of the particle, its system-acceleration, and $2\det(\Omega, V_1)$, V_1 being the particle's relative velocity.

If we now look back on the analytical expressions obtained in section 38 for the components of the absolute acceleration, it will be easy to see their full meaning. The expression $\frac{d^2x}{dt^2}$ is the component of the relative acceleration. The expression $z\frac{d\varpi_y}{dt}-y\frac{d\varpi_z}{dt}$ is the component of $\det(D_t(\Omega), R)$, since, as we have seen, $D_t(\Omega)$ has for its components $\frac{d\varpi_x}{dt}, \frac{d\varpi_y}{dt}, \frac{d\varpi_z}{dt}$. The expression $2\left(\frac{dz}{dt}\varpi_y-\frac{dy}{dt}\varpi_z\right)$ is the component of $2\det(\Omega, V_1)$, since V_1 has for its components $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. Finally, it may be easily shown by analytical geometry, that the expression

$$(y\varpi_z-z\varpi_y)\varpi_y-(x\varpi_z-z\varpi_x)\varpi_z$$

is the component of the line $\varpi^2 p$ drawn from the point (x, y, z) on the line whose direction-cosines are proportional to $\varpi_x, \varpi_y, \varpi_z$, p being the length of that perpendicular. Hence it is manifest that the analytical formulæ in question merely express the proposition enunciated at the commencement of this section.

It may, finally, be observed that the above results might also have been easily deduced from the formula in section 34 for the acceleration along the radius vector in exactly the same manner as the corresponding analytical formulæ for the relative motion of a particle in one plane were deduced in section 22 from the formula for the acceleration of the particle along the radius vector.

42. If the origin of coordinates also moves, it is evident that the particle's actual acceleration is the resultant of the acceleration of the origin and the acceleration relatively to the origin. Hence substituting for the latter acceleration the expression already found for it, it is easy to see that the particle's actual acceleration is, as before, the resultant of the relative acceleration, an acceleration represented by $2\det(\Omega, V_1)$, and the particle's system-acceleration, but that the system-acceleration is now the resultant of the acceleration of the origin, and of the system-acceleration relatively to the origin, for which latter system-acceleration we have already obtained the expression. Now in whatever way a system moves, the motion may be decomposed into a motion of translation and a motion of rotation. Hence we see that a particle's absolute accelera-

tion is in all cases the resultant of the relative acceleration, the system-acceleration, and an acceleration equal to $2\det(\Omega, V)$, where Ω is the axis about which the system is turning at the time, and V is the relative velocity of the particle.

This is the most general form of CORIOLI's theorem.

43. One of the most important illustrations of the theory of relative motion is the motion of a heavy particle relatively to a system which revolves uniformly about a fixed axis; for this includes the case of a falling body and the pendulum, where the earth's motion is taken into account.

Suppose then a particle of mass m to have for its actual weight W', and for its apparent weight W, so that a force $-W$ would keep the particle in relative equilibrium or apparently at rest. Then evidently $\left(\frac{W'}{m}\right)-\left(\frac{W}{m}\right)$ is equivalent to the particle's system-acceleration.

Let then the particle be acted on by a force P over and above the weight W', and let the particle's actual acceleration be F, its relative acceleration F_1, its system-acceleration $\overline{F}_s$. Then clearly

$$F=\left(\frac{P}{m}\right)+\left(\frac{W'}{m}\right).$$

But by CORIOLI's theorem

$$F=(F_1)+(F_s)+2\det(\Omega, V);$$

and we have just seen that the system-acceleration $F_s=\left(\frac{W'}{m}\right)-\left(\frac{W}{m}\right)$. Hence it follows that

$$(F_1)+2\det(\Omega, V)=\left(\frac{P}{m}\right)+\left(\frac{W}{m}\right): \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

$\frac{W}{m}$ is the apparent acceleration of gravity, and is generally denoted by g.

44. The above formula is quite general; but in most cases g may be considered as constant both in magnitude and direction, its direction being the vertical direction at the point of reference or origin.

We have then the formula

$$F_1=\left(\frac{P}{m}\right)+(g)-2\det(\Omega, V_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (II.)$$

This simple formula enables us to solve easily all problems concerning the motion of a heavy particle relatively to a spectator on the earth. The formula shows that the relative acceleration is found, just as if the earth did not move, by substituting the apparent for the actual force of gravity, and by adding on a force $-2m\det(\Omega, V_1)$, where V_1 is the particle's apparent velocity.

45. Let us take the vertical downwards as axis of z; let the axis of x be the horizontal line drawn from north to south, and let the axis of y be the horizontal line drawn from west to east. Then the equation to the earth's axis is evidently $\frac{x}{\cos\lambda}=\frac{z}{\sin\lambda}$, if λ denote the latitude of the spectator's position.

Therefore $\varpi_x=\varpi\cos\lambda$, $\varpi_y=0$, $\varpi_z=\varpi\sin\lambda$.

Moreover we know that the components of det (Ω, V_1) are

$$\frac{dz}{dt}\varpi_y-\frac{dy}{dt}\varpi_z=-\varpi\sin\lambda\frac{dy}{dt},$$

$$\frac{dx}{dt}\varpi_z-\frac{dz}{dt}\varpi_x=\varpi\left(\frac{dx}{dt}\sin\lambda-\frac{dz}{dt}\cos\lambda\right),$$

$$\frac{dy}{dt}\varpi_x-\frac{dx}{dt}\varpi_y=\varpi\cos\lambda\frac{dy}{dt}.$$

Let then the force P, which, besides gravity, acts on the particle, have for its components X, Y, Z, then, as F_1, the relative acceleration, has for its components $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, it evidently follows from equation (II.) of the preceding section, by revolving along the axes, that

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=\frac{X}{m}+2\varpi\sin\lambda\frac{dy}{dt},\\ \frac{d^2y}{dt^2}&=\frac{Y}{m}+2\varpi\left(\frac{dz}{dt}\cos\lambda-\frac{dx}{dt}\sin\lambda\right),\\ \frac{d^2z}{dt^2}&=\frac{Z}{m}+g-2\varpi\cos\lambda\frac{dy}{dt}.\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(\text{III.})$$

On multiplying these equations by $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ respectively, and adding, we find

$$\frac{dx}{dt}\frac{d^2x}{dt^2}+\frac{dy}{dt}\frac{d^2y}{dt^2}+\frac{dz}{dt}\frac{d^2z}{dt^2}=\frac{X}{m}\frac{dx}{dt}+\frac{Y}{m}\frac{dy}{dt}+\frac{Z}{m}\frac{dz}{dt}+g\frac{dz}{dt}.$$

This equation may also at once be deduced from the formula (II.) if we resolve along the direction of the particle's relative motion, and observe that det (Ω, V_1) is perpendicular on that direction which coincides with the direction of V_1.

By integrating the last equation, we see that the equation of *vis viva* applies to the particle's relative motion just as if the particle's relative motion were its actual motion, with this difference only, that for the actual force of gravity the apparent force of gravity must be substituted.

If the particle be a free particle acted on by no forces but gravity, then X=0, Y=0, Z=0, and the equations (III.) are linear, and are therefore easily integrated.

Moreover if v_1 be the relative velocity, the equation of *vis viva* gives

$$v_1^2=2g(2-h),\ h \text{ being a constant.}$$

46. If the particle be suspended by a string from a point fixed to the earth, then if that point be taken as the origin, and X, Y, Z be the components of the string's tension, we evidently have $Zy-Yz=0$, $Xz-Zx=0$, $Yx-Xy=0$; and substituting those values in the preceding equations (III.), we shall obtain two independent equations, which, together with the equation $x^2+y^2+z^2=$ a constant, will determine the particle's relative motion.

But those resulting equations can be found far more simply and directly in the following manner.

For this purpose let us revert to the fundamental formula

$$F_1=\left(\frac{P}{m}\right)+(g)-2\det(\Omega, V). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

Now section 30 of Chapter II. shows us how to find the moment-axis with respect to the origin of $\det(\Omega, V)$. If, namely, ϖ and v be the magnitudes of Ω and V, and φ be the angle between Ω and the radius vector, then the moment-axis of $\det(\Omega, V)$ is $-\varpi v r\cos\varphi$. Therefore the moment about the axis of x of $-2\det(\Omega, V)$ is $2\varpi r\cos\varphi\frac{dx}{dt}$; and similarly, its moments about the axis of y and z are respectively $2\varpi r\cos\varphi\frac{dy}{dt}$ and $2\varpi r\cos\varphi\frac{dz}{dt}$. Moreover, since φ is the angle between the radius vector and Ω, it is evident that

$$\varpi r\cos\varphi=\varpi_x x+\varpi_y y+\varpi_z z=\varpi(x\cos\lambda+z\sin\lambda),$$

λ being the latitude of the origin.

Let us now take the moments of F_1 about the axes of coordinates, and equate them to the moments of those components of F_1 which are given in the formula (I.).

The moments of $\frac{P}{m}$ about the axes are zero in this case of the pendulum. The moments of g about the axes of x, y, and z are respectively gy, $-gx$, and 0; and the moments of $-2\det(\Omega, V)$ we have just found. Hence we obtain at once the following three equations:—

$$\left.\begin{aligned} y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}&= gy+2\varpi\frac{dx}{dt}(x\cos\lambda+z\sin\lambda),\\ z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}&=-gx+2\varpi\frac{dy}{dt}(x\cos\lambda+z\sin\lambda),\\ x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}&= 2\varpi\frac{dz}{dt}(x\cos\lambda+z\sin\lambda). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (II.)$$

These are the three equations given in HANSEN'S elaborate 'Theorie der Pendel-Bewegung,' and which are generally obtained by means of very complicated analysis.

One simple equation can be deduced by means of the proposition contained in section 45; for the principle of *vis viva* gives

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=2g(z)+\text{a constant}.$$

Moreover $x^2+y^2+z^2$ is a constant, and these two equations, combined with any one of the equations (II.), determine the particle's motion.

Chapter IV.

47. As soon as we pass from the statics or dynamics of a particle to the statics or dynamics of a system of particles or of a rigid body, we find that two forces which are equal and parallel to one another are not equivalent to one another, and that we have to take into account the *position* as well as the magnitude and direction of a force. Notwithstanding this, we are enabled by means of an elementary principle of statics to confine our operation and our notation to lines passing through one and the same point. For suppose a force P to act at a point m of a rigid body, and apply at the origin O two equal and opposite forces P and $-$P, then P at m is equivalent to P at O and the couple whose forces are P at m and $-$P at O. Let the *axis* of this couple be denoted by G; the couple, being completely determined by G, may be called *the couple* G. It is extremely convenient to have a name for the line G, indicating briefly its connexion with the force P at m, and I shall adopt that given to it by French writers*, and shall call G *the moment-axis about* O *of the force* P *at* m.

It has been proved in section 25 of Chapter II., that the line G, being the axis of the couple whose forces are $-$P at O and P at m, is equal to $\det(-\mathrm{P}, \mathrm{R})$, where R is the radius vector of the particle. Hence we have

$$\mathrm{G}=\det(-\mathrm{P}, \mathrm{R})=\det(\mathrm{R}, \mathrm{P}).$$

We thus see that the force P at m is completely represented and determined by the two lines P and G drawn from the origin, G being the moment-axis with respect to the origin of P at m, and being equal to $\det(\mathrm{R}, \mathrm{P})$.

48. Suppose now that we have a system of forces P_1, P_2, &c. acting respectively at points m_1, m_2, &c. of a rigid body. Then it is clear from statics that the given system of forces is equivalent to a force P at the origin O and a couple whose axis is G, where P is the *complete sum* of the forces P_1, P_2, &c. supposed to be collected at the origin, and G is the complete sum of the *moment-axes* (G_1, G_2, &c.) (about the origin) of the forces P_1 at m_1, P_2 at m_2, &c.

49. We will now apply the above considerations to dynamics. Since the acceleration is the complete differential coefficient of the velocity, it is evident that the line which represents the *moving force* of the particle is the complete differential coefficient of the line which represents the particle's momentum; or, more briefly, *the moving force is the complete differential coefficient of the momentum.*

Let now P represent the moving force, and U the momentum of a particle m, P and U denoting straight lines; then, if we treat the moving force and momentum as if they were statical forces, it is clear that P at m is equivalent to P at the origin O and a couple G, where G is the moment-axis about O of P at m; and similarly, the momentum U at m is equivalent to U at O and a couple of momenta whose axis is H, where H is the moment-axis about O of U at m. We have just seen that P is the complete differential coefficient of U, and we will now prove that in like manner G is the complete

* See Delaunay's 'Mechanics,' page 254.

differential coefficient of H. If, namely, R denotes the radius vector of the particle, G=det (R, P), and similarly H=det (R, U). Now, if we differentiate the last equation, H=det (R, U), we obtain, according to section 28,

$$D_t(H)=\det(R, D_t(U))+\det(D_t(R), U). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

But $D_t(R)$ is identical with the particle's velocity, and is therefore in the direction of the momentum U. Whence it follows, from the very definition of a determinant, that

$$\det(D_t(R), U)=0.$$

Therefore the above equation (I.) becomes, since $D_t(U)=P$,

$$D_t(H)=\det(R, D_t(U))=\det(R, P)=G.$$

This is an important result. It shows that the moment-axis about any point of the moving force of a particle is the complete differential coefficient of the momentum, and that therefore the *moment* of the moving force *about any line* is the differential coefficient of the moment of the momentum.

The above result may also be easily deduced from the identical equation

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=\frac{d}{dt}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right).$$

50. The proposition which we have just proved may be easily extended to a system of moving forces and momenta of the particles of a rigid body. For, according to section 43, the system of moving forces is reducible to a moving force at the origin O, and a couple G. And the system of momenta may be similarly reduced to a momentum U, and a momentum-couple whose axis is H. Now we have seen that P is the complete sum of the moving forces, and that each moving force is the complete differential coefficient of the corresponding momentum. It therefore evidently follows that P is the complete differential coefficient of the complete sum of the momenta, or of U. Hence $P=D_t(U)$. Moreover we have seen that G is the complete sum of the moment-axes about O of the moving forces, and that each of these moment-axes is the complete differential coefficient of the moment-axis of the corresponding momentum. Hence it follows that G is the complete differential coefficient of the complete sum of the moment-axes of the momenta. Hence $G=D_t(H)$.

This result may be also easily proved by means of the identical equations

$$\Sigma\left(m\frac{d^2x}{dt^2}\right)=\frac{d}{dt}\Sigma\left(m\frac{dx}{dt}\right),$$

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=\frac{d}{dt}\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right).$$

51. The science of the dynamics of a rigid body is founded upon D'ALEMBERT'S principle, which asserts that the moving forces of a body's particles are together statically equivalent to the impressed forces acting on the body. If therefore these external forces be reduced to a force P at the origin O and a couple G, then P and G are equal

respectively to what was denoted in the preceding section by P and G; and we have therefore

$$P=D_t(U), \qquad G=D_t(H).$$

In other words, if we treat the momenta as statical forces, and reduce the system of momenta of a body's particles to a momentum U at O and a momentum-couple whose axis is G, then the external forces acting on the body are equivalent to the force $D_t(U)$ at O and the couple of forces whose axis is $D_t(H)$.

Since $G=D_t(H)$, the resolved part of G along any fixed line will be the differential coefficient of the resolved part of H along that line; or, in other words, the sum of the moments of the external forces about any line equals the differential coefficient of the sum of the moments of the momenta of the body's particles.

The above results may be easily deduced from the ordinary equations

$$\Sigma(X)=\Sigma\left(m\frac{d^2x}{dt^2}\right)=\frac{d}{dt}\Sigma\left(m\frac{dx}{dt}\right), \text{ \&c.},$$

$$\Sigma(Zy-Yz)=\Sigma m\left(\frac{d^2z}{dt^2}y-\frac{d^2y}{dt^2}z\right)=\frac{d}{dt}\Sigma m\left(\frac{dz}{dt}y-\frac{dy}{dt}z\right), \text{ \&c.}$$

But it will be generally found far better not to use those six equations at all, and simply to bear in mind the fact which they alone express, namely, that $P=D_t(U)$, $G=D_t(H)$.

52. It will be convenient to recapitulate once for all the notation and phraseology I shall constantly use in the sequel. The system of momenta of a body's particles, or what may be called *the body's momenta-system*, is reducible, if we treat the momenta as forces, to a momentum at a point O, and a couple of momenta. The former I call *the body's momentum*, and denote it by U; the latter I call *the body's momentum-couple* about O, and denote its axis by H. U and H may both be represented by straight lines through the origin O. It is to be observed that U remains the same wherever O be taken, but that H changes with the position of O.

The components of U and H along the axes of coordinates will be denoted by U_x, U_y, U_z, H_x, H_y, H_z respectively, the *magnitudes* of all these quantities being represented by the corresponding small letters. Thus h_x will be equal to the resolved part of H along the axis O*x*, and will therefore equal the sum of the moments of the momenta about O*x*.

The system of moving forces is reducible to a moving force $D_t(U)$ at O, and a couple whose axis is $D_t(H)$; and it follows from D'Alembert's principle that, if the external forces be reduced to a force P at O and a couple G, then

$$P=D_t(U), \qquad G=D_t(H).$$

53. The next step will be to investigate the expressions for U and H.

In the first place, U can be easily found. For, let R denote the radius vector of a particle of mass *m*; then, if Σ denote the operation of taking the *complete* sum of lines, we have

$$\begin{aligned} U&=\Sigma m D_t(R) \\ &=D_t\Sigma(mR). \end{aligned}$$

Now, by a well-known proposition, it is evident that $\Sigma(m\mathrm{R})=\mathrm{M}\overline{\mathrm{R}}$, where M is the mass of the body and $\overline{\mathrm{R}}$ is the radius of the body's centre of gravity. Hence

$$\mathrm{U}=\mathrm{MD}_t(\overline{\mathrm{R}});$$

and therefore, if $\overline{\mathrm{V}}$ denote the velocity of the centre of gravity in magnitude and direction, $\mathrm{U}=\mathrm{M}\overline{\mathrm{V}}$, or, in other words, U, *the body's momentum, is the momentum of the body's mass collected at the body's centre of gravity.*

54. In the next place we have to find H. The investigation will be very much facilitated by the following consideration. A body's motion is said to be compounded of motions α, β, γ, if the velocity of each of the body's particles may be considered as the resultant of the respective velocities due to the motions of α, β, γ *separately*. In such case the momentum of a particle will evidently be the resultant of the momenta due to each of the motions α, β, γ separately; and since the resultant of the momenta of all the particles will be the same in whatever way we group them together, it is evident that we have the following proposition:—

"The resultant of the momenta of a body's particles, or the body's momenta-system, is the resultant of the momenta-systems due to each of the motions α, β, γ."

Thus a body's motion may be decomposed into a motion of rotation and translation. Hence the body's momenta-system may be found by compounding the momenta-system due to the motion of rotation with that due to the motion of translation.

Again, a motion of rotation may be decomposed into rotations about three axes. Hence the momenta-system of a body which rotates about a fixed point is the resultant of the momenta-systems respectively due to the separate motions of rotation about the three axes.

55. Let us then first investigate H for a body having simply a motion of translation.

Let v be the velocity of translation in the direction of a line AB at time t, then the momentum of a particle of mass m is mv in the direction of AB. Hence the momenta-system consists of a number of momenta parallel to one another, and proportional to the masses of the respective particles. Their resultant is therefore $\mathrm{M}v$ at the centre of gravity, M being the body's mass. Hence the momenta-system is reducible to $\mathrm{M}v$ at the centre of gravity. Therefore H, the axis of the body's momentum-couple about O, is the moment-axis about O of $\mathrm{M}v$ at the centre of gravity, and is zero, if the point O coincides with the centre of gravity. In the latter case, since $\mathrm{H}=0$, therefore $\mathrm{D}_t(\mathrm{H})=0$, therefore $\mathrm{G}=0$, or the moment of the external forces about any line through the centre of gravity is zero for a body which has simply a motion of translation.

56. The next simplest case is that of a body rotating about a line. Take that line as axis of z, and suppose the body of mass M to be revolving at time t about that line with angular velocity ω. It may be easily shown in the ordinary way, that the sum of the moments of the momenta about the axes of x, y, z are respectively

$$-\omega\Sigma(mxz), \quad -\omega\Sigma(myz), \quad \omega\Sigma(mr^2).$$

So that, using the notation of section 52, we have

$$h_x=-\varpi\Sigma(mxz),\quad h_y=-\varpi\Sigma(myz),\quad h_z=\varpi\Sigma(mr^2).$$

If the axis of z be a principal axis, we have

$$\Sigma(mxz)=0,\quad \Sigma(myz)=0,$$

therefore

$$h_x=0,\quad h_y=0,\quad h_z=\varpi\Sigma(mr^2);$$

and consequently H is a line in the direction of Oz, and equal to the product of ϖ and the moment of inertia about Oz.

57. If the axis about which the body rotates is neither a fixed line nor a principal axis, it is more convenient to express H, the body's momentum-couple, in the following manner.

The body's rotation about the point O may be considered as compounded of motions of rotation about the principal axes at O. Let ϖ_1, ϖ_2, ϖ_3 be the angular velocities of those component rotations, and let A, B, C be the respective moments of inertia about the principal axes. We have shown in the preceding section that the body's momentum-couples due to the three separate rotations about the principal axes would have for their respective axes lines along the principal axes and equal to $A\varpi_1$, $B\varpi_2$, $C\varpi_3$. It follows, therefore, from section 54, that H, the body's momentum-couple, is the resultant of the three couples, whose axes are repectively $A\varpi_1$, $B\varpi_2$, $C\varpi_3$. In other words, H, the axis of the body's momentum-couple, has $A\varpi_1$, $B\varpi_2$, $C\varpi_3$ for its components along the principal axes.

58. The results of the last two sections may be also proved in the following more direct manner.

Take any rectangular axes as axes of coordinates. Let ϖ_x, ϖ_y, ϖ_z be the components, along those axes, of the body's angular velocity of rotation. Let v_x, v_y, v_z be the components of the velocity of a particle of mass m, whose coordinates are x, y, z.

If then h_x, h_y, h_z be the components of H, we have

$$h_x=\Sigma m(yv_z-zv_y).$$

But $\quad v_z=y\varpi_x-x\varpi_y,\quad v_y=x\varpi_z-z\varpi_x.$

Therefore $\quad h_x=\varpi_x\Sigma m(y^2+z^2)-\varpi_y\Sigma(myx)-\varpi_z\Sigma(mxz).$

If, then, the moments of inertia about the axes of x, y, z be denoted by A, B, C respectively, and if we denote $\Sigma(myz)$ by A′, $\Sigma(mxz)$ by B′, $\Sigma(myx)$ by C′, the last equation becomes

$$h_x=A\varpi_x-C'\varpi_y-B'\varpi_z;$$

and similarly,

$$h_y=B\varpi_y-A'\varpi_z-C'\varpi_x,$$
$$h_z=C\varpi_z-B'\varpi_x-A'\varpi_y.$$

These results are true, whatever rectangular axes of coordinates be taken; but if they be principal axes, then A′=0, B′=0, C′=0, and therefore we have, as before,

$$h_x=A\varpi_x,\quad h_y=B\varpi_y,\quad h_z=C\varpi_z.$$

59. We are now in a condition to solve easily the problem of the motion of a body rotating about a fixed line, or about a fixed point under the action of any forces.

First, let us take the case of a body of mass M revolving about a fixed line. Take that line as axis of z, and choose for the axes of x and y any lines *fixed in the body* which are perpendicular to one another and to Oz.

It was proved in section 52 that H, the axis of the body's momentum-couple, has for its components

$$h_x = -\varpi\Sigma(mxz), \quad h_y = -\varpi\Sigma(myz), \quad h_z = Mk^2\varpi, \quad \ldots\ldots \text{(I.)}$$

Mk^2 standing for the moment of inertia about Oz.

Moreover we know from section 48 that U, the body's momentum, is $M\overline{V}$, where $\overline{V}$ is the velocity of the centre of gravity. Now if $\bar{x}$, $\bar{y}$, $\bar{z}$ be the coordinates of the centre of gravity, $\overline{V}$ has evidently for its projections on Ox, Oy, Oz, $-\varpi\bar{y}$, $\varpi\bar{x}$, 0 respectively. Hence the components of U are equal to

$$u_x = -M\varpi\bar{y}, \quad u_y = M\varpi\bar{x}, \quad u_z = 0; \quad \ldots\ldots\ldots \text{(II.)}$$

knowing, then, the components of U and H, we can easily find the components of $D_t(U)$ and $D_t(H)$. Using the notation of section 52, $P = D_t(U)$ and $G = D_t(H)$, and the components of P and G may be denoted by P_x, P_y, P_z, and G_x, G_y, G_z respectively.

In the problem now before us, the axis of z does not move; hence evidently

$$P_z = \frac{d}{dt}(u_z) \text{ and } G_z = \frac{d}{dt}(h_z). \quad \ldots\ldots\ldots \text{(III.)}$$

But as to the axes of x and y, they revolve about the axis of z with an angular velocity ϖ at time t. Hence by the elementary formulæ of section 18 in Chapter I., we have

$$\left.\begin{aligned} P_x &= \frac{d}{dt}(u_x) - u_y\varpi, \quad P_y = \frac{d}{dt}(u_y) + u_x\varpi. \\ \text{Similarly,} \\ G_x &= \frac{d}{dt}(h_x) - h_y\varpi, \quad G_y = \frac{d}{dt}(h_y) + h_x\varpi. \end{aligned}\right\} \quad \ldots\ldots \text{(IV.)}$$

Let, then, external forces acting on the body be reduced to a force at O whose components are X, Y, Z, and to a couple the components of whose axis are L, M, N. Let the reactions of the fixed axis be similarly reduced to a force whose components are X′, Y′, Z′, and to a couple the components of whose axis are L′, M′. Then, by D'Alembert's principle,

$$X + X' = P_x, \text{ \&c.},$$
$$L + L' = G_x, \text{ \&c.}$$

Therefore, substituting the values (I.) and (II.) in equations (III.) and (IV.), we obtain the following six equations:—

$$X + X' = -M\bar{y}\frac{d\varpi}{dt} - M\varpi^2\bar{x},$$

$$Y + Y' = M\bar{x}\frac{d\varpi}{dt} - M\varpi^2\bar{y},$$

$$Z + Z' = 0,$$

$$L+L' = -\Sigma(mxz)\frac{d\varpi}{dt}+\varpi^2\Sigma(myz),$$

$$M+M' = -\Sigma(myz)\frac{d\varpi}{dt}-\varpi^2\Sigma(mxz),$$

$$N = \quad Mk^2\frac{d\varpi}{dt}.$$

These six equations are those ordinarily given in text-books, and their full import and meaning is now apparent. The first three have for their right-hand members the components of $D_t(U)$, where U has for its components $-M\varpi\bar{y}$, $M\varpi\bar{x}$, 0.

The last three have for their right-hand members the components of $D_t(H)$, where H has for its components $-\varpi\Sigma(mxz)$, $-\varpi\Sigma(myz)$, $Mk^2\varpi$. And the six equations are at once obtained by applying the elementary formulæ of section 18.

60. It is, however, in the solution of problems, far better to avoid using those six equations, and simply to remember that the body's momentum U is $M\varpi\bar{r}$ in the direction of the velocity of the centre of gravity ($\bar{r}$ being its distance from the axis), and that the body's momentum-couple H has for its components $-\varpi\Sigma(mxz)$, $-\varpi\Sigma(myz)$, and $Mk^2\varpi$. Then the complete differential coefficients of U and H can be found at once according to the ordinary rules; and those complete differential coefficients are by D'ALEMBERT's principle respectively identical with the force and the axis of the couple to which the forces acting on the body may be reduced.

Take for example the following well-known problem:—

"Under what circumstances will there be no pressure on the fixed axis, supposing no external forces to act on the body?"

Since there are no external forces nor pressures which act on the body, it follows that $D_t(U)$ and $D_t(H)$ must each equal zero. Therefore U and H are lines of constant magnitude and direction. Now the direction of U is that of the velocity of the centre of gravity, and would therefore vary, unless the centre of gravity were at rest. Hence the first condition is that the fixed axis passes through the centre of gravity.

Again, since H is a line of constant length and direction, its components along and perpendicular to the fixed axis Oz must be lines of constant length and direction. Hence $h_z = Mk^2\varpi$ must be constant. Therefore ϖ is constant, or the body revolves with uniform angular velocity.

Moreover the components of H perpendicular to Oz are $h_x = -\varpi\Sigma(mxz)$, $h_y = -\varpi\Sigma(myz)$; and we have just seen that the resultant of these two components must be a line of constant length and direction. But as ϖ is constant, it is clear that that resultant has always the same components along the variable axes of x and y, and would therefore move with the latter, unless those components were always zero. Hence the second condition is that $\Sigma(mxz)=0$, $\Sigma(myz)=0$; in other words, the fixed axis must be a principal axis. It is evident also that the two conditions are sufficient, for they make H and U constant lines, and therefore they make $D_t(H)$ and $D_t(U)$ vanish, and consequently, by D'ALEMBERT's principle, there are no forces acting on the body.

61. We now come to the case of a body moving about a fixed point O. Let ϖ be, at time t, the angular velocity about the instantaneous axis, let A, B, C be the moments of inertia about the principal axes at O, and let ϖ_x, ϖ_y, ϖ_z be the components of ϖ along these axes.

H, the axis of the body's momentum-couple about O, has, we have already seen, for its components

$$h_x = A\varpi_x, \quad h_y = B\varpi_y, \quad h_z = C\varpi_z.$$

If, then, we denote by Ω the instantaneous axis, we know, from the fundamental propositions in sections 35 and 36, that $D_t(H)$ is equivalent to $\frac{dh_x}{dt} \parallel$ to Ox, $\frac{dh_y}{dt} \parallel$ to Oy, $\frac{dh_z}{dt} \parallel$ to Oz, together with the determinant of Ω to H, Ω denoting the instantaneous axis; and moreover, that this determinant has for its components

$h_x\varpi_y - h_y\varpi_x$, or $(C-B)\varpi_y\varpi_z$ parallel to Ox;
$h_z\varpi_x - h_x\varpi_z$, or $(A-C)\varpi_z\varpi_x$ parallel to Oy;
$h_y\varpi_z - h_z\varpi_y$, or $(B-A)\varpi_y\varpi_z$ parallel to Oz.

Therefore the components of $D_t(H)$ are

$$A\frac{d\varpi_x}{dt} + (C-B)\varpi_y\varpi_z,$$

$$B\frac{d\varpi_y}{dt} + (A-C)\varpi_z\varpi_x,$$

$$C\frac{d\varpi_z}{dt} + (B-A)\varpi_x\varpi_y.$$

But by D'ALEMBERT'S principle $D_t(H)$ is the same as G, the axis of the couple resulting from the external forces. If, therefore, L, M, N be the components of G, L, M, N must be respectively equal to the components of $D_t(H)$. Hence we have

$$L = A\frac{d\varpi_x}{dt} + (C-B)\varpi_y\varpi_z,$$

$$M = B\frac{d\varpi_y}{dt} + (A-C)\varpi_z\varpi_x,$$

$$N = C\frac{d\varpi_z}{dt} + (B-A)\varpi_x\varpi_y.$$

We thus see that these well-known equations of EULER are found at once by resolving $D_t(H)$ along the principal axes, where H is the axis of the body's momentum-couple and has $A\varpi_x$, $B\varpi_y$, $C\varpi_z$ for its components, and that they merely express the fact that G, the resultant of L, M, N, is identical with $D_t(H)$.

62. The theory of the motion of a body about a fixed point can be more simply investigated, and the problems connected with that theory can generally be more easily solved, by merely bearing in mind that $G = D_t(H)$ than by using EULER'S equations, which merely express that fact in *one particular form*; for that form is not always the most convenient form, and is in all cases apt to conceal the fact which it embodies.

Take, for instance, the problem of a body rotating about a fixed point, no external forces acting on it. Here $D_t(H)=0$, therefore H is a line of constant length and direction. The motion of the body must therefore entirely depend upon the fact that, *whilst the body moves about the principal axes with angular velocities* ω_x, ω_y, ω_z, *the line* H, *whose projections on those axes are respectively* $A\omega_x$, $B\omega_y$, $C\omega_z$, *remains throughout the body's motion the same in magnitude and direction.*

The length of H is evidently $\sqrt{A^2\omega_x^2+B^2\omega_y^2+C^2\omega_z^2}$. But the length of H is constant, say equal to h. Therefore

$$A^2\omega_x^2+B^2\omega_y^2+C^2\omega_z^2=h^2. \quad \ldots\ldots\ldots \quad (I.)$$

Moreover, from section 35, we see that $D_t(H)$ is equivalent to $A\frac{d\omega_x}{dt} \parallel$ to Ox, $B\frac{d\omega_y}{dt} \parallel$ to Oy, $C\frac{d\omega_z}{dt} \parallel$ to Oz, together with $\det(\Omega, H)$; and since the last line $\det(\Omega, H)$ is perpendicular on the instantaneous axis Ω, it follows that the resolved part of $D_t(H)$ along the instantaneous axis equals

$$\frac{\omega_x}{\omega}A\frac{d\omega_x}{dt}+\frac{\omega_y}{\omega}B\frac{d\omega_y}{dt}+\frac{\omega_z}{\omega}C\frac{d\omega_z}{dt}.$$

But this must equal zero, since $D_t(H)$ equals zero, and since consequently its resolved part along any line is zero. Hence we have

$$A\omega_x\frac{d\omega_x}{dt}+B\omega_y\frac{d\omega_y}{dt}+C\omega_z\frac{d\omega_z}{dt}=0.$$

Therefore

$$A\omega_x^2+B\omega_y^2+C\omega_z^2 \text{ is constant, equal, say, to } k^2. \quad \ldots\ldots \quad (II.)$$

We have already seen that H is a line of constant direction; and since its direction-cosines are proportional to $A\omega_x$, $B\omega_y$, $C\omega_z$, it follows that the plane whose moment has direction-cosines which are proportional to the last three quantities is a fixed plane. This plane is the invariable plane. From this fact and the two equations (I.) and (II.), POINSOT's celebrated illustration of the motion of a body which rotates about a fixed point may be easily deduced in the ordinary manner; but it is unnecessary to discuss the problem further, as it must be already sufficiently apparent that the body's motion entirely depends upon the fact that H is a line of fixed length and direction.

63. On looking at EULER's equations, we find that, when A=B=C, they take the simple form

$$L=A\frac{d\omega_x}{dt},$$

$$M=B\frac{d\omega_y}{dt},$$

$$N=C\frac{d\omega_z}{dt}.$$

Moreover, when only A=B, then the third of EULER's equations becomes $N=C\frac{d\omega_z}{dt}$. It may be interesting to trace the real meaning of these results.

Let, as before, G be the resultant of L, M, N, and let H be the axis of the body's momentum-couple about the fixed point, and let Ω represent the instantaneous axis. Then $G=D_t(H)$, and $D_t(H)$ is equivalent to $A\frac{d\varpi_x}{dt} \parallel$ to Ox, $B\frac{d\varpi_y}{dt} \parallel$ to Oy, $C\frac{d\varpi_z}{dt} \parallel$ to Oz, together with $\det(\Omega, H)$.

Now, if $A=B=C$, then H, which has for its components $A\varpi_x$, $B\varpi_y$, $C\varpi_z$, evidently coincides in direction with Ω, which has for its components ϖ_x, ϖ_y, ϖ_z. Therefore it follows, from the very definition of a determinant, that $\det(\Omega, H)=0$. It is therefore because $\det(\Omega, H)=0$ when $A=B=C$, that the components of $D_t(H)$ are simply $A\frac{d\varpi_x}{dt}$, $B\frac{d\varpi_y}{dt}$, $C\frac{d\varpi_z}{dt}$, and that EULER's equations take so simple a form.

Secondly, suppose only $A=B$. It is evident, from what has been just said, that $N=C\frac{d\varpi_z}{dt}$ + the resolved part along Oz of $\det(\Omega, H)$. Now the equation to the line H is

$$\frac{x}{A\varpi_x}=\frac{y}{B\varpi_y}=\frac{z}{C\varpi_z}.$$

If then $A=B$, the projection of H on the plane of xy evidently coincides with the projection of Ω on that plane. Therefore the lines H, Ω, and the axis of z lie in the same plane. Hence it follows that the line $\det(\Omega, H)$, which by definition is perpendicular on Ω and on H, is also perpendicular on the axis of z, and has therefore no component along that axis. We thus see that the reason why $N=C\frac{d\varpi_z}{dt}$, when $A=B$, is that in that case $\det(\Omega, H)$ is perpendicular to the axis of z.

64. In those cases in which there are more sets than one of principal axes at the fixed point, it is sometimes convenient to take moments about a set of principal axes, which *are not fixed in* the body.

There is no difficulty in applying the same method to such cases. Let ϖ_x, ϖ_y, ϖ_z be the angular velocities of the body about the principal axes Ox, Oy, Oz, and suppose those axes not to move with the body as if rigidly connected with it, but to move at time t about an instantaneous axis Ω' with an angular velocity equal to the length of Ω', and let the components of Ω' along the principal axes be ϖ'_x, ϖ'_y, ϖ'_z.

Let H, as before, represent the body's momentum-couple. We have seen that H has for its components

$$h_x=A\varpi_x, \quad h_y=B\varpi_y, \quad h_z=C\varpi_z.$$

Therefore, according to the fundamental proposition in section 35, $D_t(H)$ is equivalent to $\frac{d}{dt}(A\varpi_x) \parallel$ to Ox, $\frac{d}{dt}(B\varpi_y) \parallel$ to Oy, $\frac{d}{dt}(C\varpi_z) \parallel$ to Oz, together with $\det(\Omega', H)$; and looking at the formulæ of section 36, we see that $\det(\Omega', H)$ has for its components

$C\varpi_x\varpi'_y - B\varpi_y\varpi'_z$ parallel to Ox,

$A\varpi_z\varpi'_x - C\varpi_x\varpi'_z$ parallel to Oy,

$B\varpi_y\varpi'_x - A\varpi_x\varpi'_y$ parallel to Oz.

Therefore, if L be the moments of inertia of the forces about the principal axes, we have by D'ALEMBERT's principle,

$$L=\frac{d}{dt}(A\varpi_x)+C\varpi_z\varpi'_y-B\varpi_y\varpi'_z,$$

$$M=\frac{d}{dt}(B\varpi_y)+A\varpi_x\varpi'_z-C\varpi_z\varpi'_x,$$

$$N=\frac{d}{dt}(C\varpi_z)+B\varpi_y\varpi'_x-A\varpi_x\varpi'_y.$$

It is clear that, when there is more than one set of principal axes at the fixed point, either all three or at least two of the quantities A, B, C must be equal to one another. Suppose then A=B, then the axis of z, Oz is *fixed in* the body, and therefore ϖ'_x and ϖ'_y are clearly the same as ϖ_x and ϖ_y. And the last equations, therefore, become

$$L=A\frac{d\varpi_x}{dt}+\varpi_y(C\varpi_z-A\varpi'_z),$$

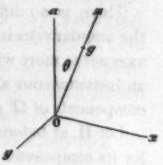
$$M=A\frac{d\varpi_y}{dt}+\varpi_x(A\varpi'_z-C\varpi_z),$$

$$N=C\frac{d\varpi_z}{dt}.$$

If we put $\varpi'_z=\varpi_z+\frac{d\chi}{dt}$, the above equations become the same as those which are given in ROUTH's 'Dynamics,' page 134, where they are deduced from EULER's equations.

65. To the above equations, however, the same remark applies as has already been made with regard to EULER's equations. They merely express the fact that $D_t(H)$ has L, M, N for its components; and it is far better in most problems to start with that simple fact, and, without using those equations, to choose any axes which the nature of the problem may suggest.

Take, for instance, the problem of the top spinning upon a perfectly rough plane.



Let O be the fixed point, g the top's centre of gravity. Take Og as axis of z. Draw Oa vertically, and take as axis of x a line perpendicular to Oz and in the plane zOa, and take as axis of y a line perpendicular on the plane zOx, and therefore perpendicular on Og. The axes of coordinates are evidently principal axes.

The components of H, which determines the body's momentum-couple, are $A\varpi_x$, $B\varpi_y$, $C\varpi_z$, ϖ_x, ϖ_y, ϖ_z being the angular velocities about the principal axes, and A being the moment of inertia about Ox and about Oy, and C being the moment of inertia about Og. We have chosen Og so as to be perpendicular on plane aOx; consequently the resolved part of H along Oa is the sum of the resolved parts of $A\varpi_x$, $C\varpi_z$, and equals therefore, if we denote angle aOz by θ,

$$-A\varpi_x\sin\theta+C\varpi_z\cos\theta.$$

Moreover, since Oa has a fixed direction, the differential coefficient of the last expression is evidently the resolved part along Oa of $D_t(H)$. But this must equal zero by

D'ALEMBERT's principle, since there are no forces acting on the top which have any moments about the vertical. Hence we have

$$\frac{d}{dt}(C\varpi_z \cos\theta - A\varpi_x \sin\theta) = 0. \quad \ldots\ldots\ldots \quad (L.)$$

Therefore

$$C\varpi_z \cos\theta - A\varpi_x \sin\theta \text{ is constant.}$$

Moreover, since two of the principal moments of inertia are equal to one another, it follows from section 58 that the sum of the moments of the external forces about Oz, the axis of unequal moment of inertia, is equal to $C\frac{d\varpi_z}{dt}$. Hence, as the forces have no moment about Oz, we have $C\frac{d\varpi_z}{dt} = 0$. Therefore ϖ_z is constant. This relation, together with the equation (L.) and the equation of *vis viva*, solve the problem. We, namely, obtain the three equations

$$C\varpi_z \cos\theta - A\varpi_x \sin\theta = h,$$

$$\varpi_z = \alpha,$$

$$A\varpi_x^2 + A\varpi_y^2 + C\varpi_z^2 = -2gb\cos\theta + c,$$

where $b = Og$, and c is some constant.

66. In some few cases it may be convenient to take moments about lines which are fixed in the body but which are not principal axes. Let then H have for its components along the rectangular axes of coordinates h_x, h_y, h_z respectively, then we know, from section 53, that

$$h_x = A\varpi_x - B'\varpi_z - C'\varpi_y, \quad \text{where } A' = \Sigma(myz),$$
$$h_y = B\varpi_y - C'\varpi_x - A'\varpi_z, \qquad B' = \Sigma(mxz),$$
$$h_z = C\varpi_z - A'\varpi_y - B'\varpi_x, \qquad C' = \Sigma(myx).$$

Therefore, if L, M, N be the moments of the forces about the axes, we have, as before,

$$L = \frac{dh_x}{dt} + h_z\varpi_y - h_y\varpi_z,$$

$$M = \frac{dh_y}{dt} + h_x\varpi_z - h_z\varpi_x,$$

$$N = \frac{dh_z}{dt} + h_y\varpi_x - h_x\varpi_y.$$

If, on the other hand, the axes of Ox, Oy, Oz are not fixed in the body, but rotate with an angular velocity whose components are ϖ'_x, ϖ'_y, ϖ'_z, then we have, in a similar manner,

$$L = \frac{dh_x}{dt} + h_z\varpi'_y - h_y\varpi'_z, \text{ \&c.}$$

The above equations are somewhat more general than those given by LIOUVILLE in his Journal of 1858, and are, as we have just seen, at once obtained by applying the fundamental formulæ of section 36.

67. I will now show how the principle of *vis viva* may be easily proved for a body

moving about a fixed point without assuming the principle of virtual velocities, and is in fact a very simple deduction from D'ALEMBERT'S principle.

Let $O\,i$ be the instantaneous axis, about which the body is rotating at time t with an angular velocity w. Let m be a particle of the body, let its mass be m, its distance from $O\,i$, r, and its velocity v in the direction $m\,n$.

Suppose now a force P to act at m. Then, since $m\,n$ is perpendicular on the plane $i\,O\,m$, it is clear from statics that the moment of P about $O\,i$ is the moment of the resolved part of P along $m\,n$. It is therefore, if φ be the angle which the direction of P makes with $m\,n$, equal to $r\mathrm{P}\cos\varphi$, or $\frac{v}{w}\mathrm{P}\cos\varphi$.

Suppose then P to be the moving force of the particle m. Then its resolved part along the velocity $m\,n$ is of course $m\frac{dv}{dt}$, so that $\mathrm{P}\cos\varphi = m\frac{dv}{dt}$, and therefore the moment of the moving force about $O\,i$ equals $m\frac{v}{w}\frac{dv}{dt}$. Consequently the sum of the moments of the moving forces about $O\,i$ equals $\frac{1}{w}\Sigma\left(mv\frac{dv}{dt}\right)$. But this sum, by D'ALEMBERT'S principle, equals the sum of the moments of the external forces about $O\,i$. Now we have already seen that the moment of any force P about $O\,i$ is $\frac{v}{w}\mathrm{P}\cos\varphi$, so that, if P represent an external force acting on the body, the sum of the moments about $O\,i$ of the forces acting on the body equals $\frac{1}{w}\Sigma(\mathrm{P}v\cos\varphi)$. Hence we have

$$\frac{1}{w}\Sigma(\mathrm{P}v\cos\varphi)=\frac{1}{w}\Sigma\left(mv\frac{dv}{dt}\right).$$

Therefore

$$\Sigma(mv^2) \quad =2\int dt\,\Sigma(\mathrm{P}v\cos\varphi).$$

This equation embodies the principle of *vis viva*; for it is evident that, if the components of P be X, Y, Z, then

$$\mathrm{P}v\cos\varphi=\mathrm{X}\frac{dx}{dt}+\mathrm{Y}\frac{dy}{dt}+\mathrm{Z}\frac{dz}{dt}.$$

68. The same result may also be obtained by analysis; and it may be worth while to notice that each step in the analytical proof is exactly equivalent to the corresponding step in the above geometrical proof. This correspondence between the steps in analytical and geometrical demonstrations is one of the most striking features of modern analytical geometry, and would, as we have already attempted to show, present itself generally in analytical mechanics, if more attention were paid to the interpretation of the equations and formulæ which are employed.

The analytical proof is as follows:—

Let X, Y, Z be the components of any one of the forces acting on the body, and suppose that force to act at a point (x, y, z) of mass m. Let w_x, w_y, w_z be the angular velocities

of rotation about the axes of coordinates, which are here supposed to be fixed in space. Then it is clear that

$$\frac{dx}{dt}=z\varpi_y-y\varpi_z, \quad \frac{dy}{dt}=x\varpi_z-z\varpi_x, \quad \frac{dz}{dt}=y\varpi_x-x\varpi_y.$$

Therefore

$$\Sigma\left(\mathrm{X}\frac{dx}{dt}+\mathrm{Y}\frac{dy}{dt}+\mathrm{Z}\frac{dz}{dt}\right)$$

may be put into the form

$$\Sigma(\mathrm{X}z-\mathrm{Z}x)\varpi_y+\Sigma(\mathrm{Y}x-\mathrm{X}y)\varpi_z+\Sigma(\mathrm{Z}y-\mathrm{Y}z)\varpi_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I.})$$

But, by D'ALEMBERT's principle,

$$\Sigma(\mathrm{X}z-\mathrm{Z}x)=\Sigma m\left(\frac{d^2x}{dt^2}z-\frac{d^2z}{dt^2}x\right),$$

$$\Sigma(\mathrm{Y}x-\mathrm{X}y)=\Sigma m\left(\frac{d^2y}{dt^2}x-\frac{d^2x}{dt^2}y\right),$$

$$\Sigma(\mathrm{Z}y-\mathrm{Y}z)=\Sigma m\left(\frac{d^2z}{dt^2}y-\frac{d^2y}{dt^2}z\right).$$

Substituting these expressions in (I.), we obtain

$$\varpi_y\,\Sigma m\left(\frac{d^2x}{dt^2}z-\frac{d^2z}{dt^2}x\right)+\varpi_z\,\Sigma m\left(\frac{d^2y}{dt^2}x-\frac{dx^2}{dt^2}y\right)+\varpi_x\,\Sigma m\left(\frac{d^2z}{dt^2}y-\frac{d^2y}{dt^2}z\right),$$

which again can be put into the form

$$\Sigma m\left\{\frac{d^2x}{dt^2}(z\varpi_y-y\varpi_z)+\frac{d^2y}{dt^2}(x\varpi_z-z\varpi_x)+\frac{d^2z}{dt^2}(y\varpi_x-x\varpi_y)\right\},$$

$$=\Sigma m\left(\frac{d^2x}{dt^2}\frac{dx}{dt}+\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt}\frac{dz}{dt}\right).$$

Hence it follows that

$$\Sigma\left(\mathrm{X}\frac{dx}{dt}+\mathrm{Y}\frac{dy}{dt}+\mathrm{Z}\frac{dz}{dt}\right)=\Sigma m\left(\frac{d^2x}{dt^2}\frac{dx}{dt}+\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt^2}\frac{dz}{dt}\right),$$

and therefore

$$\Sigma m\left(\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right)=2\int dx(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz),$$

which is the equation of *vis viva*.

It may be observed that the same proof may be quite easily extended to a body moving freely, by decomposing the original motion into a motion of translation with the velocity of the centre of gravity and a rotation about the centre of gravity.

69. The two proofs which have just been given of the principle of *vis viva* are both founded on the fact that the sum of the moments of the moving forces of the body's particles about the instantaneous axis is equal to $\frac{1}{\varpi}\Sigma\left(mv\frac{dv}{dt}\right)$, or to $\frac{1}{2\varpi}\frac{d}{dt}\Sigma(mv^2)$. This

fact is the reason why the equations of *vis viva* can be obtained by multiplying EULER'S three equations by ϖ_x, ϖ_y, ϖ_z respectively, and by adding together the products so obtained; for in performing those operations we are in reality finding the sum of the moments of the forces about the instantaneous axis whose direction-cosines are $\frac{\varpi_x}{\varpi}, \frac{\varpi_y}{\varpi}, \frac{\varpi_z}{\varpi}$.

70. If G_1 be the sum of the moments of the impressed forces about the instantaneous axis, then by D'ALEMBERT'S principle G_1 equals the sum of the moments of the moving forces about the instantaneous axis. Hence it follows from the preceding section that

$$G_1=\frac{1}{2\varpi}\frac{d}{dt}\Sigma(mv^2).$$

This equation is often useful.

For instance, since $\Sigma(mv^2)=\Sigma(m\varpi^2r^2)$, where r is the distance of a particle from the instantaneous axis, it follows that $\Sigma(mv^2)=I\varpi^2$ if I denote the body's moment of inertia about the instantaneous axis. Therefore

$$G_1=\frac{1}{2\varpi}\frac{d}{dt}(I\varpi^2)$$

$$=\frac{d\varpi}{dt}+\frac{\varpi}{2}\frac{dI}{dt}.$$

Now, if the instantaneous axis were fixed in space, we should evidently have

$$G_1=I\frac{d\varpi}{dt}.$$

Therefore the only cases in which we can take moments about the instantaneous axis as if it were fixed in space are when $\frac{\varpi}{2}\frac{dI}{dt}=0$, or when the moment of inertia about the instantaneous axis is constant. This proposition is useful in solving problems concerning rolling cones, and is usually deduced by analysis from EULER'S equations.

71. I will give two more examples of the advantage of the method I have employed in these pages.

Let ϖ_x, ϖ_y, ϖ_z be the angular velocities of rotation of a body about three rectangular axes which are fixed in and move with the body, and let a, b, c be the direction-cosines with respect to those axes of a line which is *fixed in space*. Take on the latter line a point P at a unit of distance from the origin. The velocity of the fixed point P is zero. Now, as its components along the moving axes are a, b, c respectively, it follows from one of our elementary propositions that

$$\frac{da}{dt}+c\varpi_y-b\varpi_z$$

equals the component of P's velocity along Ox, and therefore equals zero.

Hence

$$\frac{da}{dt}=b\varpi_z-c\varpi_y,$$

a formula which is generally deduced as the result of somewhat long analytical work.

72. Secondly, in order to give a striking example of the manner in which the theory of the determinants of lines explains and shortens analytical processes, I will give the following direct proof of EULER's equations.

Take the principal axes at the fixed point as the axes of coordinates. Let v_x, v_y, v_z be the components of the velocity V, and f_x, f_y, f_z those of the acceleration F of a particle (x, y, z) of mass m.

We have seen that the fact of the acceleration being the complete differential coefficient of the velocity leads at once to the three following equations:—

$$\left.\begin{aligned} f_x&=\frac{dv_x}{dt}+v_z\varpi_y-v_y\varpi_z,\\ f_y&=\frac{dv_y}{dt}+v_x\varpi_z-v_z\varpi_x,\\ f_z&=\frac{dv_z}{dt}+v_y\varpi_x-v_x\varpi_y. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1.)$$

Putting then for brevity's sake f'_x for $v_x\varpi_y-v_y\varpi_x$, f'_y for $v_x\varpi_z-v_z\varpi_x$, f'_z for $v_y\varpi_x-v_x\varpi_y$, we have

$$\left.\begin{aligned} f_x&=\frac{dv_x}{dt}+f'_x,\\ f_y&=\frac{dv_y}{dt}+f'_y,\\ f_z&=\frac{dv_z}{dt}+f'_z. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (2.)$$

Now the sum of the moments of the moving forces about Ox equals

$$\Sigma m(f_zy-f_yz)=\Sigma m\left(\frac{dv_z}{dt}y-\frac{dv_y}{dt}z\right)+\Sigma m(f'_zy-f'_yz) \quad \ldots\ldots \quad (3.)$$

Let us first investigate the expression $\Sigma m\left(\frac{dv_z}{dt}y-\frac{dv_y}{dt}z\right)$. Evidently

$$v_z=y\varpi_x-x\varpi_y, \text{ and } v_y=x\varpi_z-z\varpi_x.$$

And as the axes move with the body, x, y, z do not vary with the time, and we therefore obtain

$$\frac{dv_z}{dt}=y\frac{d\varpi_x}{dt}-x\frac{d\varpi_y}{dt}, \quad \frac{dv_y}{dt}=x\frac{d\varpi_z}{dt}-z\frac{d\varpi_x}{dt}.$$

Therefore

$$\Sigma m\left(\frac{dv_z}{dt}y-\frac{dv_y}{dt}z\right)=\frac{d\varpi_x}{dt}\Sigma m(y^2+z^2)-\frac{d\varpi_y}{dt}\Sigma(mxy)-\frac{d\varpi_z}{dt}\Sigma(mxz).$$

But as the axes of coordinate axes are principal axes, $\Sigma(mxy)=0$, $\Sigma(mxz)=0$. Therefore

$$\Sigma m\left(\frac{dv_z}{dt}y-\frac{dv_y}{dt}z\right)=\frac{d\varpi_x}{dt}\Sigma m(y^2+z^2)=\mathrm{A}\frac{d\varpi_x}{dt}, \quad \ldots\ldots\ldots \quad (4.)$$

A being the moment of inertia about Ox.

We have still to investigate the expression $\Sigma m(f'_z y-f'_y z)$ where $f'_z=v_y\varpi_x-v_x\varpi_y$, and $f'_y=v_x\varpi_z-v_z\varpi_x$.

We obtain by substitution

$$\Sigma m(f'_z y-f'_y z)=\varpi_x\Sigma m(v_y y+v_z x)-\varpi_y\Sigma(mv_x y)-\varpi_z\Sigma(mv_z x).$$

But since

$$v_x=z\varpi_y-y\varpi_z,$$
$$v_y=x\varpi_z-z\varpi_x,$$
$$v_z=y\varpi_x-x\varpi_y,$$

it is evident that for principal axes we have

$$\Sigma(mv_x y)=0, \qquad \Sigma(mv_z x)=0,$$
$$\Sigma(mv_z y)=-\varpi_z\Sigma(my^2), \qquad \Sigma(mv_y x)=\varpi_y\Sigma(mz^2).$$

Hence

$$\begin{aligned}\Sigma m(f'_z y-f'_y z)&=\varpi_y\varpi_z(\Sigma(my^2)-\Sigma(mz^2))\\&=\varpi_y\varpi_z(\Sigma m(y^2+x^2)-\Sigma m(z^2+x^2))\\&=\varpi_y\varpi_z(\mathrm{C}-\mathrm{B}),\end{aligned}$$

where C and B are the moments of inertia about the axes of z and y.

Substituting then this last expression and the expression in (4.) in equation (3.), we see that the sum of the moments of the moving forces about Ox equals

$$\mathrm{A}\frac{d\varpi_x}{dt}+(\mathrm{C}-\mathrm{B})\varpi_y\varpi_z.$$

Hence, by D'ALEMBERT'S principle, if L, M, N be the sum of the moments of the impressed forces about the axes of x, y, z, we have

$$\mathrm{L}=\mathrm{A}\frac{d\varpi_x}{dt}+(\mathrm{C}-\mathrm{B})\varpi_y\varpi_z.$$

Similarly,

$$\mathrm{M}=\mathrm{B}\frac{d\varpi_y}{dt}+(\mathrm{A}-\mathrm{C})\varpi_z\varpi_x,$$

$$\mathrm{N}=\mathrm{C}\frac{d\varpi_z}{dt}+(\mathrm{B}-\mathrm{A})\varpi_x\varpi_y.$$

I will now show the full import of each of the steps in the above analytical proof.

We have seen, in section 39 of Chapter III., that the acceleration of the particle is the result of the acceleration $\det(\mathrm{D}_t(\Omega), \mathrm{R})$, and the acceleration $\det(\Omega, \mathrm{V})$. If then the particle's mass be m, its moving force is represented by

$$m\det(\mathrm{D}_t(\Omega), \mathrm{R})+m\det(\Omega, \mathrm{V}).$$

Now the above analytical proof merely shows that the sum of the moments about the coordinate axes, of the moving forces of which m det $(D_t\Omega, R)$ is the type, are respectively A $\frac{d\varpi_x}{dt}$, B $\frac{d\varpi_y}{dt}$, C $\frac{d\varpi_z}{dt}$, and that the sum of the moments about the coordinate axes of the moving forces of which m det (Ω, V) is the type are respectively $(C-B)\varpi_y\varpi_z$, $(A-C)\varpi_z\varpi_x$, $(B-A)\varpi_y\varpi_x$.

73. I will next proceed to show how these results can be obtained far more briefly by applying the propositions concerning the determinants of lines. In the first place, if we put, for $D_t(\Omega)$, P, and suppose P to have for its components p_x, p_y, p_z parallel to the axes, then m det $(D_t(\Omega), R)$ has for its components parallel to x and to y,

$$m(yp_z-xp_y) \text{ and } m(xp_z-zp_x),$$

and therefore the moment of m det $(D_t(\Omega), R)$ about the axis of x equals

$$m\{(yp_z-xp_y)y-(xp_z-zp_x)z\}=m(z^2+y^2)p_x-mxyp_y-mxzp_z.$$

Therefore, if we take the sum of these for all the particles and remember that the axes are principal axes, that sum will equal $p_x\Sigma m(z^2+y^2)$. Now we have already proved that p_x, the component parallel to the axis of x of $D_t(\Omega)$, $=\frac{d\varpi_x}{dt}$. Hence we see that the sum of the moments about Ox of the moving forces of which m det $(D_t(\Omega), R)$ is the type is A $\frac{d\varpi_x}{dt}$, and that this follows from the properties of principal axes, and from the fact that the component of $D_t(\Omega)$ parallel to the axis of x is $\frac{d\varpi_x}{dt}$.

In the second place, we have already seen in section 30 of Chapter II. that the moment about the axis of x of det (Ω, V) equals $-r\varpi v_x \cos\varphi$, v_x being the component of V, and φ the angle between the radius vector and V.

But $-r\varpi v_x \cos\varphi$ equals evidently $-v_x(x\varpi_x+y\varpi_y+z\varpi_z)$, and v_x equals $z\varpi_y-y\varpi_z$. If then we observe that $\Sigma(mxv_x)=0$, $\Sigma(myv_x)=-\Sigma(my^2)\varpi_z$, $\Sigma(mzv_x)=\Sigma(mz^2)\varpi_y$, it follows easily from the above that the sum of the moments about Ox of the moving forces, of which m det (Ω, V) is the type, equals $\Sigma(mz^2)\varpi_y\varpi_z-\Sigma(my^2)\varpi_z\varpi_y=(C-B)\varpi_y\varpi_z$. This last proposition may be also proved in a different manner, which will show its connexion with the proof first given of EULER'S equations.

The moment-axis, with respect to the origin, of the acceleration det (Ω, V) is det $\{R, \det(\Omega, V)\}$, which, as we see from section 30 of Chapter II., equals a line opposite to V and of length $\varpi vr\cos\varphi$; but it follows from the same section that, since Ω is perpendicular on V, det $\{\Omega, \det(R, V)\}$ equals a line opposite to V and of length $\varpi vr\cos\varphi$. Hence we have

$$\det\{R, \det(\Omega, V)\}=\det\{\Omega, \det(R, V)\}.$$

Let then Σ denote the operation of taking the *complete* sum of lines. Then it follows from the last equation that

$$\Sigma m \det\{R, \det(\Omega, V)\}=\Sigma m \det\{\Omega, \det(R, V)\}$$
$$=\det(\Omega, \Sigma m \det(R, V)).$$

Now $\Sigma m \det(R, V)$ equals the complete sum of the moment-axes of the momenta, or is, in other words, the axis of the body's momentum couple H whose components are $A\varpi_x$, $B\varpi_y$, $C\varpi_z$. Hence we see that

$$\Sigma m \det(R, \det(\Omega, V)) = \det(\Omega, H),$$

and therefore the sum of the moments about Ox of the moving force, of which $m \det(\Omega, V)$ is the type, is the component parallel to x of $\det(\Omega, H)$, and equals therefore $C\varpi_z\varpi_y - B\varpi_y\varpi_z = (C-B)\varpi_y\varpi_z$.

XXXIII. *A Supplement to Two Papers published in the Transactions of the Royal Society, "On the Science connected with Human Mortality;" the one published in* 1820, *and the other in* 1825. *By* Benjamin Gompertz, *F.R.S., F.R.A.S. &c.*

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In offering to the Royal Society the ensuing Supplement to my two former papers on the Law of Mortality, with subsequent remarks on invalidism, I am anxious to acknowledge that I have derived great advantage from the encouragement and persuasion of my esteemed brother-in-law, Sir Moses Montefiore, Bart., given me to endeavour to compile and publish some of my later observations on the subject; knowing that, though I felt flattered by the attention originally shown by scientific gentlemen to these papers, they appeared to me capable of advantageous illustrations. Therefore I may venture to hope that if this Supplement merit the attention of those interested in this branch of science, I may consider that he has added a mite further to entitle him to the good wishes of those who applaud him for his constant endeavours to promote the general interest of mankind—endeavours which he has shown to extend through Europe and Asia in the cause of humanity, and to be exercised at home in various ways, among which I notice his attention to the practice of Life, Fire, and Marine Assurance; he being the President of the Alliance British and Foreign Life and Fire Assurance Company; of which I was the founding Actuary, and in which Institution, though retired from it, I feel greatly interested; it having been established about the year 1824 by the late N. M. de Rothschild, Esq., the late John Irving, Esq., the late Samuel Gurney, Esq., and Francis Baring, Esq., and himself conjointly with other gentlemen, and he being also President of the Alliance Marine Assurance Society, founded at the same time by them with him.

Art. 1. In the year 1820 the Royal Society did me the honour to publish in their Transactions a paper of mine on the Analysis and Notation applicable to the valuation of Life Contingencies, in which I introduced a new and general notation, which appeared to me far more extensively useful, and more explanatory of its object, than any other notation I had met with; and in that paper I think I introduced a new manner of dealing with the subject, by offering an analysis, with examples of the extensive use of it, applicable to some of the most intricate questions which had up to that period met with anything like a proper solution; and showed, by selections from the treatise of Life Annuities of my late learned and much-respected friend, Francis Baily, Esq., a mode of solution of all the problems in chapter 8 of that work, depending on a particular

* Subsequently revised by the author, with the insertion of some additional matter.

order of survivorship; problems previously considered many years before, and presented by my late friend WILLIAM MORGAN, Esq., of the Equitable Society, to the Royal Society, and published in their valuable Transactions; and which had been since considered, in a learned work on Life Annuities, by my late respected friend JOSHUA MILNE, Esq., with some ingenious notation with respect to those contingencies. But still, the solutions given to many of the problems, though there were but three lives concerned, were of such an intricate practical form, as to be in my opinion perfectly useless; especially on considering that it was necessary to obtain, by Tables of single and joint lives, by necessary interpolations, the required data; as the differences to be used for the interpolations, in consequence of the great irregularity of the numbers of those Tables, are so irregular as to throw great doubt on the necessary accuracy of the results. And I think the examples I gave of my method could leave no doubt as to the comparative simplicity which resulted from it, and consequently comparative utility of my analysis; an analysis which applies where there are more than three lives concerned, and, in fact, where there are any number of lives to be considered. And I may refer the reader to my solutions in that tract, to enable him to make the comparison.

There were various other subjects in that paper, and one I mention in particular, which is the problem to determine what would be the law of mortality between two lives A, B, so that, should it be known that they are both extinct, it would be an equal chance which of them had died first; because that assumption is made, in some of the solutions above alluded to, by former writers, and for a short period would, in fact, be approximatively true; and the solution of the problem showed that it could only be accurately true where there was for each life a uniform equal decrement, though not necessarily the same for both, or else a decrement for each life proceeding in geometrical proportion; the former law being in fact only an extreme case of the latter law. I may mention that there are some omissions in the printed solution of this problem, which may lead the reader, if he does not enter properly into the analysis, to think it faulty; and that the paper on the whole stands in need of some errors in the printing of it, and in one or two places of an incorrect portion of the manuscript sent to the printer, being pointed out.

Art. 2. Since that paper was written, I ventured to communicate a paper to the Royal Society, which it did me the honour to print in their Transactions of 1825, as a letter to my late friend FRANCIS BAILY, Esq., on the nature of the functions expressive of the law of human mortality expressed by the equation $L_x = d.g^{\overline{1}}$; where L_x, according to my notation in my first paper, is the number of persons living, at the age x, out of the number L_0 who were born x years previously. And as I use, from great preference, the fluxional notation of our great NEWTON, instead of the furtive notation used on the Continent and now much used in England, my d does not denote the differential character. I say I use the notation in preference, because I consider the fluxional calculus far more luminous. But as strictures have been cast on me on that account by some readers of my paper, I will not apologize for a small digression from my subject, to state some

among the causes of my preference. I call the differential notation furtive, on I think a moral ground, and also on the ground of its introducing an interruption and an inconvenience in practice. The moral ground is, that it appears to give LEIBNITZ a greater claim to originality, to the prejudice of NEWTON, than I think he is justly entitled to. And the other ground is, it steals from the alphabet a letter—and one which it is most convenient to retain, in order to keep up the regular order of notation—to use it for a purpose of different intent to that for which it was originally used; and may introduce confusion. And with respect to the superior advantage of the fluxional calculus over the differential calculus, I observe that if x and y be rectangular coordinates of a curve in a plane, and z the length of the curve from a given point in it to the point of which x and y are coordinates, the fluxional calculus gives $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}$, which is strictly true; and may be proved to be so without the introduction of infinitely small quantities: but the differential calculus gives $dz=\sqrt{\overline{dx}^2+\overline{dy}^2}$, true only on consideration of infinitely small quantities; and even with that consideration it cannot be proved luminously to be true; because $\sqrt{\overline{dx}^2+\overline{dy}^2}$ only expresses the length of the chord of an infinitely small arc, and not of the arc itself, as they have no part common with each other, but at the points of intersection: but in the fluxional calculus $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}$, which, I think, is a much neater and a much more commodious expression, $\dot{z}$, $\dot{x}$, $\dot{y}$ only express finite values; namely, the velocities in the several directions, at the point to which x and y are coordinates, with which the point describing the curve is moving in the relative directions parallel to the axes of x and y, and that of the tangent to the curve at the point; in the same way as if all causes which might incurvate the future path of the point were to cease; and similar observations may be made with respect to the luminous character of the fluxional analysis, when compared with the differential analysis, in the application of it to physical subjects. Thus if f be a force acting on a body, $\dot{v}$ the velocity which is generated by its momentary impulse, that is to say its single impulse, by which is meant the finite space the body would describe in consequence of it in the finite time $\dot{t}$, but not the variable portions of space it would describe if that force were considered to be an infinitely small action, as it were continually active during infinitely small portions of time; the fluxional calculus gives $f\dot{t}=\dot{v}$, and is correctly true, however large $\dot{v}$ and $\dot{t}$ are. But the differential analysis gives $f.dt=dv$, which is correct only if dt and dv are infinitely small, and is then only to be considered so in virtue of the hypothesis that infinitely small quantities of the second and higher degree may be omitted.

But whilst I am endeavouring to clear away the shadowing clouds which may obstruct the brilliant light of NEWTON'S lamp from being duly perceived by the scientific eye, I am willing to acknowledge that LEIBNITZ'S differential d has, in many instances, done great service in his own hands, in the hands of EULER, LAGRANGE, LAPLACE, and of a host of scientific men whose names cannot be pronounced without gratitude and reverence.

And I observe that other letters, such as f, φ, ψ, &c., when used as characteristics

instead of representatives of value, have their valuable service as well as inconvenience to be attended to; though I prefer much the Continental use of a letter when used as a characteristic, to be used, as it is in many cases, as a letter underscored by some one or more letters, denoting the quantities of which that letter may be the functional characteristic; as, for instance, to write $L_{\underline{a}}$ to express the function of a, which may be the age of a person of whom there may be the number $L_{\underline{a}}$ living. And having thus intruded by a digression on the reader's attention, I will venture to hope that my still continuing the digression will be thought to have some interesting excuse for me, as a scientific amusement to the reader; as a person walking for the sake only of healthy exercise in a beautiful garden, may find a pleasure and an advantage to notice the elegant flowers, and even the noxious weeds which the ground produces. I will state that mathematicians who have enlightened the world with the most beautiful discoveries use notations which are incorrect, often ambiguous, often furtive, and often contradictory. Thus in the notation of partial fluxions or partial differentials, I consider $\frac{\dot{y}}{\dot{x}}$, $\frac{\ddot{y}}{\dot{x}^2}$, &c., $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ as both incorrect and furtive, and that they may mislead; instead of which I use the expressions $\frac{\dot{y}}{\dot{x}}$, $\frac{\ddot{y}}{\dot{x}^2}$, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. The incorrectness of the former has been in some degree avoided by WARING, LAPLACE, and many other mathematicians of eminence, by using the expressions $\left(\frac{\dot{y}}{\dot{x}}\right)$, $\left(\frac{\ddot{y}}{\dot{x}^2}\right)$, $\left(\frac{dy}{dx}\right)$, $\left(\frac{d^2y}{(dx)^2}\right)$, &c., which removes some part of my objection, but not the whole of it; but has the disadvantage of occupying too much space, and is furtive, as it steals the excellent use of the parentheses, which have been employed for useful purposes. And now, requesting my reader to pardon me for a digression which some readers may consider uncalled for, I will proceed in the path in which I hope to lead him with some satisfaction.

Art. 3. In the equation $L_{\underline{x}}=d.\overline{g}|^{x}$, where $L_{\underline{x}}$ signifies the number of persons living at the age x, the letters d, g, q in my paper in the 'Transactions' of 1825 express quantities apparently nearly constant during a very long period of life—say, for instance, in the Carlisle Table of Mortality between the ages 10 and 60; but if we commence our limit at a different age, and terminate it at some other far distant age, those quantities apparently constant would have different values of apparent constancy; and here the product $d.g$, which would appear in the equation by taking $x=0$, would be an arbitrary quantity depending on the arbitrary value of $L_{\underline{0}}$ as a base, expressive of the arbitrary number we intend to set out with as the persons living at the age 0; and if, which is not the case accurately, the formula were universally true from birth to any other age, $d.g$ would express the number of persons born on which we intended the formula to be constructed; for we should have $L_{\underline{0}}=d.g$; but as g is not arbitrary, d would be the arbitrary quantity, and we might take $d=1$, if we did not object to fractions in the number of the births, to set out with; and in that case the formula would stand generally $L_{\underline{x}}=\overline{g}|^{x}$; if it were universally true from birth to any age; which, as I observed

above, it is not; and it is never in fact true if g and q are to be considered absolutely constant; as is evident from the example of the method I used in my investigation, by what I call the vital rule of three. And as the Table from which the data are to be obtained is moreover likely to contain irregularities, and undoubtedly does contain such, which are not contained in the real law of mortality, it follows that if even the equation $L_x = d.\overline{g}^x$, where d, g, q are supposed to be constant, did exactly express the real law of mortality, the values given to g and q by this vital rule of three would not be exactly the same for every selection, and therefore I should not have expected that any tolerable mathematician in reading my paper could suppose that I meant to state that I had given their exact values. But wherever the three lives were selected, had the Table been accurate from whence I selected the data, and were the law for constant values for d, g, q consistent throughout with the real law of mortality, their respective values would have come out the same.

Art. 4. But neither of these requisites is to be depended on, as is proved by the two formulæ given in my paper of 1825 for the Carlisle mortality, Art. 10; the one where the vital rule of three is based on the selection of the ages 20, 40, and 60, and the other where it is based on the ages 40, 60, and 100. The first of these equations is

$$\lambda L_x = 3{\cdot}88631 - \lambda^{-1}(\overline{2}{\cdot}75536 + {\cdot}0126x),$$

or its equal,

$$\lambda L_x = 3{\cdot}88631 - \lambda^{-1}({\cdot}0126x - 1{\cdot}24464), \text{ or very nearly}$$
$$3{\cdot}88631 - \lambda^{-1}{\cdot}0126(x - 100);$$

λ standing for the common logarithm of, and λ^{-1} the reverse, or the number whose common logarithm is &c., and the other, namely, from the selection of the ages 40, 60, 100, gives the equation

$$\lambda L_x = 3{\cdot}79657 - \lambda^{-1}(\overline{3}{\cdot}7467 + {\cdot}02706x)$$
$$= 3{\cdot}79657 - \lambda^{-1}({\cdot}02706x - 2{\cdot}2533).$$

And when the middle age of the three selections above is 40, we have

$$\lambda d = 3{\cdot}88631,\ \lambda q = {\cdot}0126;\ \lambda(-\lambda g) = -1{\cdot}24464,$$

sufficiently near $-1{\cdot}26$; and when the middle age is 60, of the three selections, we have $\lambda d = 3{\cdot}79657$, $\lambda q = {\cdot}02706$; sufficiently near the double of the former value; and $\lambda(-\lambda g) = -2{\cdot}2533$, also equal to nearly the double value in the former case, but not exactly in that ratio, but nearly in the ratio of 2 to $1\frac{1}{10}$; but still, considering the nature of the data which furnish the values, in order to follow the hints these numbers give, I am inclined to think but lightly of the small variation from that proportion, and to suppose that the law of mortality, instead of being $\lambda L_x = \lambda d - \lambda^{-1}(\lambda(-\lambda g) + x.\lambda q)$, when d, g, q are constant, which we have proved are only apparently constant, though for a long time exhibiting a strong appearance of constancy, should be, according to the above hints, $\lambda L_x = \lambda d_x - \lambda^{-1}(\lambda q_x.x - h)$, with the addition of some small formulæ to be sought for, where h is a constant from birth to extreme old age, and λd_x, λq_x are

functions of x to be discovered to meet the cases of comparison of the formula with the Tables. Here I retain the d and q with the prefix λ for the sake of convenient reference to the old formula; and it appears that these functions λd_x and λq_x are subject to such slow variation by the variation of x, that in forty years in the above examples λd_x only varies from 3·88631 to 3·79657, and λq_x from ·0126 to about its double; the prefix λ signifying common logarithm of, and λ^{-1} the anti-common logarithm of. Now supposing t and v to be very small quantities, and that λd_x is a function of $1+vx$, and λq_x a function of $1+tx$, then, provided v and t are sufficiently small, whatever these functions may be, we consider that if in consequence of the smallness of t and v we may in the development of the functions according to the powers of x be satisfied with the first power, we may assume these functions of any form we please to suit our convenience. I will then for that end suppose $\lambda d_x = C\mathfrak{C}^x$, $\lambda q_x = \lambda q_0 . e^x$; C standing for λd_0, and $\mathfrak{C}$ and e will be quantities differing very little from unity, the one being $=1+v$, where, as above shown, v must be a very small negative fraction, and the other $=1+t$, where t is a very small affirmative fraction, and then the approximate law of mortality will be

$$\lambda L_x = C.\mathfrak{C}^x - \lambda^{-1}(e^x . \lambda q_0 . (x-h)),$$

where C, $\mathfrak{C}$, e, q_0 are all four constant quantities from birth to extreme old age, quantities to be discovered from the Table of Statements of the living at different ages; C and $\mathfrak{C}$ first to be determined by two convenient statements, and e, q and h by three statements, by the method I call the vital rule of three. First, for finding $\mathfrak{C}$ and C take two values of x, one $x=m$, the other $x=n$, saying $C.\mathfrak{C}^m = \lambda d_m$, $C\mathfrak{C}^n = \lambda d_n$; and therefore by division $\mathfrak{C}^{n-m} = \frac{\lambda d_n}{\lambda d_m}$, and

$$\therefore \quad \lambda\mathfrak{C} = \frac{\lambda\lambda d_n - \lambda\lambda d_m}{n-m}; \quad \lambda C = \lambda\lambda d_m - \lambda\mathfrak{C}^m = \frac{n\lambda\lambda d_m - m\lambda\lambda d_n}{n-m};$$

d_m and d_n being found from the Table of data by means of the previously stated original formula,

$$\lambda L_x = \lambda d - \lambda^{-1}(\lambda(-\lambda g) + \lambda q . x),$$

which gave when x was 20, λd_x, that is $\lambda d_{20} = 3{\cdot}88631$, and when $x=60$, $\lambda d_{60} = 3{\cdot}79657$;

$$\therefore \quad C\mathfrak{C}^{20} = 3{\cdot}88631; \quad C\mathfrak{C}^{60} = 3{\cdot}79657.$$

Consequently

$$\lambda\mathfrak{C}^{40} = {\cdot}57939 - {\cdot}58954; \quad \therefore \lambda\mathfrak{C} = -\frac{\cdot 01015}{40} = -{\cdot}00025375 = \bar{1}{\cdot}99974625$$

and

$$\lambda C = {\cdot}589540 - 20\lambda\mathfrak{C} = {\cdot}589540 + {\cdot}005075 = {\cdot}594615.$$

Art. 5. But instead of proceeding at present with this formula, I will refer to my original formula, $L_x = d.\overline{gf}^x$, to show its value, and how g may be found by the vital rule of three by a method different from that in my paper of 1825, a mode I then pursued before I discovered by a general investigation that the above equation with d, g, q constant quantities did very approximatively express for a very long period the law of mortality.

The equation gives $\lambda L_x = \lambda d + \lambda g . g^x$, and therefore by taking x successively $=m$, $m+n$,

$m+2n$, we obtain

$$\lambda L_m=\lambda d+\lambda g.q^m,\ \lambda L_{m+n}=\lambda d+\lambda g.q^{m+n},\ \lambda L_{m+2n}=\lambda d+\lambda g.q^{m+2n},$$

whence we get

$$\lambda g.q^m\times(q^n-1)=\lambda L_{m+n}-\lambda L_m,\ \ \lambda g.q^m q^n.(q^n-1)=\lambda L_{m+2n}-\lambda L_{m+n};$$

and therefore, by division,

$$q^n=\frac{\lambda L_{m+2n}-\lambda L_{m+n}}{\lambda L_{m+n}-\lambda L_m};$$

and

$$\lambda g=\frac{\lambda(\lambda L_{m+n}-\lambda L_{m+2n})-\lambda(\lambda L_m-\lambda L_{m+n})}{n},$$

and we have

$$\lambda g=\frac{\lambda L_{m+n}-\lambda L_m}{q^m.(q^n-1)},$$

and λg in the application I am about to make will turn out to be a negative quantity, or in other words, g will be a positive fraction less than unity; and consequently to find g, as a negative number has no logarithm in the positive scale, if we are to proceed by logarithms, we must take the logarithm of the positive quantity $-\lambda g$, and say

$$\lambda(-\lambda g)=\lambda(\lambda L_m-\lambda L_{m+n})-\lambda q^m-\lambda(q^n-1);$$

and then g being found, we find d by the equation

$$\lambda d=\lambda L_m-\lambda g.q^m.$$

This is the way I have generally proceeded since I discovered the above approximative formula of mortality; but if we only consult this formula in order to find λd, which it contains, which is the only part of it necessary to find C, $\mathfrak{C}$ of the formula

$$\lambda L_x=C\mathfrak{C}^{\sigma^x}-\lambda^{-1}(\sigma^x.\lambda g_x.\overline{x-h}),$$

we need not take the trouble to find g or q of the preceding or old formula, and from the equations above put in the form

$$\lambda g.q^m=\lambda L_m-\lambda d,\ \ \lambda g.q^{m+n}=\lambda L_{m+n}-\lambda d,\ \ \lambda g.q^{m+2n}=\lambda L_{m+2n}-\lambda d,$$

multiplying the first and last together, we have

$$(\lambda g)^2q^{2m+2n}=(\lambda L_m-\lambda d)\times(\lambda L_{m+2n}-\lambda d);$$

and squaring the middle, we have also

$$(\lambda g)^2q^{2m+2n}=(\lambda L_{m+n}-\lambda d)^2;$$

these put equal to each other, give

$$(\lambda L_m-\lambda d)\times(\lambda L_{m+2n}-\lambda d)=(\lambda L_{m+n}-\lambda d)^2,$$

and therefore

$$\lambda L_m.\lambda L_{m+2n}-\lambda d(\lambda L_m+\lambda L_{m+2n})=(\lambda L_{m+n})^2-2\lambda L_{m+n}\lambda d;$$

and consequently

$$\lambda d=\frac{\lambda L_m.\lambda L_{m+2n}-\overline{\lambda L_{m+n}}^2}{\lambda L_m-2\lambda L_{m+n}+\lambda L_{m+2n}}.$$

Art. 6. The information which we have that the formula

$$\lambda L_x=\lambda d-\lambda^{-1}(\lambda(-\lambda g)+x\lambda q),$$

or its equal,

$$\lambda L_x = \lambda d + \lambda g . q^x,$$

will for a long series of years, with d, g, q constant, be a very near approximation to the values which the data afford, provided d, g, q be determined by the vital rule of three, by taking x successively equal m, $m+n$, $m+2n$, where m and $m+2n$ are somewhere near the limits of age through which the formula is meant to be applied, is a most valuable information; on account of the application of the above law to the most complicated intricacies which may be proposed, by means of what I consider rather a novel branch of mathematical investigation, which I term vital algorithm and analytical arithmetic of logarithms and anti-logarithms, in very complicated entanglements and disentanglements: for by the law of mortality we can get the value of L_x, the number living at the age x, or of L_{s+x}, or their logarithms in a series of powers of x with very converging coefficients, as will be shown further on, and thence we can obtain a series of the values of $L_{s+x} \times L_{s+x} \times L_{s+x}$ &c., however many lives there may be; or we may have the logarithms λL_{s+x}, λL_{s+x}, λL_{s+x} &c., to which if we add the logarithm of r^x, say x into the logarithm of r, r being the present value of unity due in one year certain, at the proposed rate of interest, and deduct from this sum the sum of $\lambda L_s + \lambda L_s + \lambda L_s$, &c., we shall have the logarithm of unity to be received in x years on the condition that all these proposed lives be in existence in x years time. And then finding the analytical expression for the anti-logarithm of this, expressed by a series $A_0 + A_1 x + A_2 x^2 + A_3 x^3 +$ &c., in which A_1, A_2, A_3, &c. express a series of very converging terms, we have the present value of unity to be received in x years, and then by a Table to be presented in this tract of powers of numbers, and the sum of these numbers from $x=1$ to any proposed required limit, we can by multiplying the coefficients A_1, A_2, A_3, &c., of the series A_1, A_2, A_3, &c., which will be mostly very convergent coefficients, if not always, find the value of all the annual payments between any one time and any other; many examples of the ease of this method comparatively with the seemingly insurmountable difficulties which appear on these subjects, I hope to have time to lay before the reader, and to discuss many other branches which may be of interest. But not to keep the reader in suspense, I will continue the subject referable to the formula $L_x = d . \overline{g^c}^{c^x}$, where from any proposed value of x to any far greater value, d, g, q may be considered as constants, though not actually so; and now I will show how nearly the formula

$$\lambda L_x = C\overline{c}^{c^x} - \lambda^{-1}(c^x . \lambda q_0 . \overline{x-h})$$

between the ages of 10 and 80 agrees with the Tables; and will then show how to find the constants of the equation from the commencement of life to extreme old age from the more complete formula

$$\lambda L_x = C\overline{c}^{c^x} + k_1 s^x + k_2 s^{2x} - \lambda^{-1}(c^x . \lambda q_0 . \overline{x-h}) + \mu r^x,$$

where from birth to extreme old age all the quantities except x are constant, but of interesting values, so that $k_1 s^x$ commences its significance at birth, and within less than one year decreases to total insignificance, but will never absolutely vanish; the term

$k\epsilon^{x}$ also commences of significance at birth, but sinks gradually, and sinks into insignificance at the age of 20, and then and after becomes of so small value that the terms sink into entire insignificance; the terms $C\mathfrak{C}^{x}$ and $\lambda^{-1}(\epsilon^{x}.\lambda q_{0}.\overline{x-h})$ are values of significance, from birth to extreme old age; the term $\mu\epsilon^{x}$ is insignificant till x is equal about 80, "but for analytical anticipation is of significance some years before," and then it slowly increases during the remainder of life*. The value $k_{1}\epsilon^{x}$ is particularly interesting, because in the Carlisle Table it shows a surprising agreement with the stated mortality of children from the age of birth till the age of 1 year.

Art. 7. But now continuing with respect to the original formula of near proximity to the law of mortality $L_{x}=d.\overline{g|}^{q^{x}}$, where d, g, and q may for a long period be considered as constant, their values being dependent on the three selections of ages, and putting it in the form $\lambda L_{x}=\lambda d+\lambda g.q^{x}$, and using a new and useful notation with respect to logarithms, by writing underneath a letter whose logarithm is to be expressed the prostrate small l, thus $\underset{\frown}{q}$, to denote the common logarithm of q, $\underset{\frown}{g}$ to denote the Napierian logarithm of q, and the prostrate l in the reverse position, thus $\underset{\smile}{q}$, to denote the number whose common logarithm is q, and $\underset{\smile}{g}$ the number whose Napierian logarithm is q, we have evidently from the equation $\lambda L_{x}=\lambda d+\lambda g.q^{x}$,

$$\underset{\frown}{L}_{a+x}=\underset{\frown}{d}+\underset{\frown}{g}.q^{a+x}=\underset{\frown}{d}+\underset{\frown}{g}.q^{a}\times(1+\underset{\smile}{g}.x+\tfrac{1}{2}.\overline{\underset{\smile}{g}}^{2}.x^{2}+\tfrac{1}{2.3}\overline{\underset{\smile}{g}}^{3}x^{3}+\tfrac{1}{2.3.4}.\overline{\underset{\smile}{g}}^{4}.x^{4},\ \&c.);$$

where the coefficients of x and its successive powers converge, in consequence of the smallness of the Napierian logarithm of q, which in the Carlisle mortality is about ·029, so that a very few terms of the series would be required, even if, for instance, the age a were 30, and we wished to know the value of λL_{30}. To use this theorem with advantage, we should be provided with a Table of the values of $\underset{\frown}{g}.q^{a}$ and of $\underset{\frown}{d}$ for every value of a, the youngest life of the three selected lives which are the foundation of the values of d, g, q, their values being different according to the ages of selection; and if λL_{a+x}, λL_{b+x}, λL_{c+x} &c. be, for the sake of example, as it was found by the above method, represented respectively by the converging series

$$A+{}^{1}Ax+{}^{2}Ax^{2}+{}^{3}Ax^{3},\ \&c.,\quad B+{}^{1}Bx+{}^{2}Bx^{2}+{}^{3}Bx^{3},\ \&c.,\quad C+{}^{1}Cx+{}^{2}Cx^{2}+{}^{3}Cx^{3},\ \&c.,$$

it is evident on taking $x=0$, that λL_{a}, λL_{b}, &c. will be represented by A, B, C, &c.; and as the logarithms of the chances of the persons of the present ages a, b, c, &c. living x years are respectively $\lambda\frac{L_{a+x}}{L_{a}}$, $\lambda\frac{L_{b+x}}{L_{b}}$, &c., the logarithm of the chance of a living x years, of b living in x years, of c, &c. will be respectively

$${}^{1}Ax+{}^{2}Ax^{2}+{}^{3}Ax^{3},\ \&c.,\quad {}^{1}Bx+{}^{2}Bx^{2}+{}^{3}Bx^{3},\ \&c.,\quad {}^{1}Cx+{}^{2}Cx^{2}+{}^{3}Cx^{3},\ \&c.;$$

and if r be the value of unity discounted for one year, using $\underset{\smile}{r}$ to express its common logarithm, we shall have the logarithm of the present value of unity to be received in

* But μ, though considered now to be constant, may show after x is 100 that it is not absolutely so, though data are wanting to show the nature of its variability.

x years, provided the persons of the present ages a, b, c, &c. are then all living, represented by

$$(\underline{r}+{}^{1}\mathrm{A}+{}^{1}\mathrm{B}+{}^{1}\mathrm{C}, \&c.)x+({}^{2}\mathrm{A}+{}^{2}\mathrm{B}+{}^{2}\mathrm{C}, \&c.)x^{2}+({}^{3}\mathrm{A}+{}^{3}\mathrm{B}+{}^{3}\mathrm{C}, \&c.)x^{3}, \&c.,$$

a very converging series; but in the use of the method it is necessary to have, instead of the common logarithm, the Napierian logarithm, and the Napierian logarithms of $\mathrm{I}_{+,+}$, $\mathrm{I}_{+,+}$, &c. would be expressed by

$$\underline{d}+\underline{g}.\underline{q}^{a}\times\left(1+\underline{g}.x+\underline{q}^{a}.\frac{x^{2}}{2}+\&c.\right),\quad \underline{d}+\underline{g}.\underline{q}^{b}\times\left(1+\underline{q}x+\underline{q}^{2}.\frac{x^{2}}{2}\ \&c.\right);$$

and if, for the sake of brevity, we represent $\underline{g}.(\underline{q}^{a}+\underline{q}^{b}+\underline{q}^{c}, \&c.)$, multiplied respectively by $\underline{q}$, $\frac{1}{2}\overline{\underline{q}}^{2}$, $\frac{1}{2.3}.\overline{\underline{q}}^{3}$, &c. by A_{1}, A_{2}, A_{3}, &c., we shall have the Napierian logarithm of the present value of unity to be received in x years, if the persons of the present ages a, b, c, &c. be living, represented by $\overline{\underline{r}+\mathrm{A}_{1}}.x+\mathrm{A}_{2}x^{2}+\mathrm{A}_{3}x^{3}+$ &c., a very converging series. And if the anti-Napierian logarithm of this expression be represented by $1+{}^{1}\mathrm{S}x+{}^{2}\mathrm{S}x^{2}+{}^{3}\mathrm{S}x^{3}$, &c., in which the coefficients of the different powers of x, namely, ${}^{1}\mathrm{S}$, ${}^{2}\mathrm{S}$, ${}^{3}\mathrm{S}$, form a very converging series, then on using the notation adopted in my paper of 1820, $\left.\begin{matrix}x\\1\\w\\z\end{matrix}\right|x^{n}$ to express the sum of all the values of x^{n}, from $x=w$ to $x=z$ inclusively of both by increasing x continually by unity, the present value of an annuity of 1 on the joint lives of persons now of the age of a, b, c, &c., the first payment to be made in w years, and the last in z years, will be represented by the series

$$\left.\begin{matrix}x\\1\\w\\z\end{matrix}\right|x^{0}+{}^{1}\mathrm{S}\left.\begin{matrix}x\\1\\w\\z\end{matrix}\right|x+{}^{2}\mathrm{S}\left.\begin{matrix}x\\1\\w\\z\end{matrix}\right|x^{2}+{}^{3}\mathrm{S}\left.\begin{matrix}x\\1\\w\\z\end{matrix}\right|x^{3}+\&c.,$$

which will be a converging series.

And if the annuity were intended to be entered on immediately, so that the first payment was to be in one year, and it was intended to continue as long as the joint lives continued, it would be written

$$\left.\begin{matrix}x\\1\\1\end{matrix}\right|x^{0}+{}^{1}\mathrm{S}\left.\begin{matrix}x\\1\\1\end{matrix}\right|x+{}^{2}\mathrm{S}\left.\begin{matrix}x\\1\\1\end{matrix}\right|x^{2}+{}^{3}\mathrm{S}\left.\begin{matrix}x\\1\\1\end{matrix}\right|x^{3}+\&c.;$$

and if this were to be an increasing annuity, continually increasing by the additions from year to year of the payment n, the xth payment would be $1+nx$, and the present value of that xth payment would be

$$\overline{1+nx}(1+{}^{1}\mathrm{S}x+{}^{2}\mathrm{S}x^{2}+{}^{3}\mathrm{S}x^{3}+{}^{4}\mathrm{S}x, \&c.)=1+({}^{1}\mathrm{S}+n)\times x+({}^{2}\mathrm{S}+n.\,{}^{1}\mathrm{S})\times x^{2}+({}^{3}\mathrm{S}+n.\,{}^{2}\mathrm{S})x^{3}+\&c.,$$

and therefore the present worth of such an increasing annuity on the joint lives would be represented by

$$\left.\begin{matrix}x\\1\\1\end{matrix}\right|x^{0}+({}^{1}\mathrm{S}+n)\left.\begin{matrix}x\\1\\1\end{matrix}\right|x+({}^{2}\mathrm{S}+n.\,{}^{1}\mathrm{S})\left.\begin{matrix}x\\1\\1\end{matrix}\right|x^{2}+({}^{3}\mathrm{S}+n.\,{}^{2}\mathrm{S})\left.\begin{matrix}x\\1\\1\end{matrix}\right|x^{3}+({}^{4}\mathrm{S}+n.\,{}^{3}\mathrm{S})\left.\begin{matrix}x\\1\\1\end{matrix}\right|x^{4}, \&c.;$$

and a similar mode may be adopted for more complicated regulations of the payments from year to year.

And to make this simple theorem easily available for the calculations, it will be convenient to have a Table of powers of numbers increasing regularly in arithmetical progression, with the common difference of 1, from 1 to 100, of the annexed form, which I call the Collecting Table, and which will be found of immense service. Such, for instance, as the following example; but a more extensive Table will be given further on.

x.	Sums of x.	x^2.	Sums of x^2.	x^3.	Sums of x^3.	x^4.	Sums of x^4.	To continue to $x=10$, or more.
1	1	1	1	1	1	1	1	&c.
2	3	4	5	8	9	16	17	&c.
3	6	9	14	27	36	81	98	&c.
4	10	16	30	64	100	256	354	&c.
5	15	25	55	125	225	625	979	&c.
6	21	36	91	216	441	1296	2273	&c.
7	28	49	140	343	784	2401	4676	&c.
8	36	64	204	512	1296	4096	8772	&c.
&c. to 100	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.

It is evident that if this Table be continued to great values of x, say to 100, or high powers of x, which may be necessary in some cases, we shall get very high numbers for the value of $\overset{x}{\underset{1}{\int}} x^n$; but these numbers, when multiplied by their coefficients, which will be very small, may not prevent the series from converging.

Art. 8. But the case renders a new arithmetical notation convenient, which will be explained further on, as only a few of the most significant figures will be required, and the remainder may be considered as noughts. This analysis does not only require the anti-Napierian logarithm to be taken of analytical expressions, but also the reversion of analytical expressions into Napierian logarithms to be found; for if there be a series $a+bx+cx^2+dx^3$, &c. when the coefficients a, b, c, &c. are a converging series, the Napierian logarithm of it is the Napierian logarithm of a

$$+\left(\frac{b}{a}x+\frac{c}{a}x^2+\frac{d}{a}x^3, \text{ \&c.}\right)-\frac{1}{2}\left(\frac{b}{a}x+\frac{c}{a}x^2+\frac{d}{a}x^3\right)^2+\frac{1}{3}\left(\frac{b}{a}x+\frac{c}{a}x^2+\frac{d}{a}x^3, \text{ \&c.}\right)^3 \text{ \&c.},$$

which may be represented by $\underset{\smile}{a}+{}^1bx+{}^1cx^2+{}^1dx^3+$ &c., and would give the values of ${}^1b$, ${}^1c$, ${}^1d$, &c., or rather proceed as follows.

Supposing the Napierian logarithm of

$$1+B_1x+B_2x^2+B_3x^3$$

were equal to

$$A_1x+A_2x^2+A_3x^3+\text{\&c.},$$

if these equations be put into fluxions, we shall obtain, after dividing by $\dot{x}$,

$$\frac{B_1+2B_2x+3B_3x^2+4B_4x^3 \text{ \&c.}}{1+B_1x+B_2x^2+B_3x^3 \text{ \&c.}}=A_1+2A_2x+3A_3x^2+4A_4x^3, \text{ \&c.},$$

and consequently

$$\left.\begin{array}{rrrrr} A_1+2A_2x & +3A_3x^2 & +4A_4x^3 & +\&c. \\ +\ A_1B_1x+2A_2B_1x^2+3A_3B_1x^3 & & & \&c. \\ +\ A_1B_2x^2+2A_2B_2x^3 & & & \&c. \\ +\ A_1B_3x^3 & & & \&c. \\ \&c. & & & \\ -B_1-2B_2x-3B_3x^2-4B_4x^3 & & & \&c. \end{array}\right\}=0.$$

Consequently

$$A_1=B_1;\ A_2=B_2-\tfrac{1}{2}A_1B_1;\ A_3=B_3-\tfrac{1}{3}(2A_2B_1+A_1B_2),$$

$$A_4=B_4-\tfrac{1}{4}\{3A_3B_1+2A_2B_2+A_1B_3\};\ A_5=B_5-\tfrac{1}{5}\{4A_4B_1+3A_3B_2+2A_2B_3+A_1B_4\},\ \&c.$$

for turning natural expressions into Napierian logarithms; and we also have for the reverse, namely, turning expressions of Napierian into anti-Napierian logarithms,

$$B_1=A_1;\ B_2=A_2+\tfrac{1}{2}A_1B_1;\ B_3=A_3+\tfrac{1}{3}(2A_2B_1+A_1B_2);\ B_4=A_4+\tfrac{1}{4}(3A_3B_1+2A_2B_2+A_1B_3),\ \&c.$$

Art. 9. And now supposing the Napierian logarithm of the chances of persons now of the ages α, β, γ, &c. being living in x years be turned into anti-Napierian logarithms by the theorem at the end of art. 8, that is to say, into analytical natural expressions corresponding to those Napierian logarithms, and that the value of those chances be respectively deducted from unity, they will represent the respective chances of their being extinct, and may be expressed by

$$p_\alpha x\times(1+{}^1p_\alpha x+{}^2p_\alpha x^2+{}^3p_\alpha x^3+\&c.);\ p_\beta x\times(1+{}^1p_\beta x+{}^2p_\beta x^2+\&c.);\ p_\gamma x(1+{}^1p_\gamma x+{}^2p_\gamma x^2+\&c.),\ \&c.,$$

and supposing the Napierian logarithms of these several multipliers

$$1+{}^1p_\alpha x+{}^2p_\alpha x^2+\&c.,\ 1+{}^1p_\beta x+{}^2p_\beta x^2,\ \&c.,$$

be taken by the same theorem of art. 8, and that there be n persons of the above said ages α, β, γ, &c., and that the sum of these Napierian logarithms thus taken be represented by ${}^1Px+{}^2Px^2+{}^3Px^3$, &c., and that the product of the n quantities p_α, p_β, p_γ, &c., be represented by P; then the present value of unity to be received in x years, provided all the persons before mentioned to be living are extinct, will be Px^n, multiplied by the anti-Napierian logarithm of $\underline{r+{}^1P}x+{}^2Px^2+{}^3Px^3$, &c.; and if the anti-Napierian logarithm be represented by $1+R_1x+R_2x^2+R_3x^3$, &c., the said value will be

$$P.x^n+P.R_1x^{n+1}+P.R_2x^{n+2}+\&c.$$

We may execute the required calculations in this manner instead of adopting rules laid down by authors of solving such questions by aid of Tables of the values of annuities on single lives and two joint lives, and obtaining by an inaccurate interpolation the values of annuities of a larger number of joint lives, as there may be required many values of annuities on three joint lives, on four joint lives, and on more joint lives, which

would not only require great labour, but leave very small confidence of an accurate result. And it is owing to this circumstance that, in addition to having a Table of the expressions of the Napierian logarithm, for every age, of the chance of persons of any age living x years, I propose also, for every age, to have the Napierian logarithm of the quotient of the expression which gives the chance of a person of any age being dead divided by its first term, namely, x multiplied by the coefficient it has in the value of the chance, as in many cases such a Table will introduce great facility.

There are much more intricate cases for calculation, which the law of mortality enables us to overcome; I allude to annuities and assurances depending on conditions of survivorships among the party who are involved in the annuity. Now I observe, from having the Napierian logarithm of the chance of each individual surviving x years, or having the Napierian logarithm of the expression after it is divided by its first term, of the chance of his being dead, and adding the coefficients of all the first terms together, all the second terms together, &c., and finding the anti-Napierian logarithm of the result, and multiplying this by all the first terms, which were directed to divide the various chances, we have the natural value of the chances compounded out of them.

Art. 10. Previously to proceeding, I venture to introduce a notation which I have found convenient with respect to vital algorithm, as the theory, and the application of it to important objects, introduces very large numbers, and also extremely small fractions, of which, in both cases, there is only a necessity to attend to a very few of the significant figures: thus, suppose we had the number 897654321, and that it answered for sufficient accuracy only to consider the number as 898000000, I would write it $898\textcircled{6}$; by the $\textcircled{6}$ I mean the six noughts which are not written down; and if we had the decimal fraction ·00000000763, in which eight noughts occur before the significant figure, I would write it $\textcircled{8}763$; and if these two quantities had to be multiplied together, I should write it $898\textcircled{6}\times\textcircled{8}763$; and $\textcircled{8}763$ consists of eleven places of decimals, and $898\times763=685174$; and this, if four significant figures were sufficient, I would write $6852\textcircled{2}$; to which add $\textcircled{6}$, we have $6852\textcircled{8}$; and adding to the left $\textcircled{11}$, as $\textcircled{8}763$ signifies ·00000000763, the product will stand $\textcircled{11}6852\textcircled{8}$, and would signify that three figures to the right are to be cut off as a decimal, and that the product is 6·852, because eleven places of decimals, including the first significant figure being multiplied by 10^{11}, leaves $11-8$ three places of decimals; and if we had to add 68·52 to $\textcircled{0}23$, it would be $\left.\begin{matrix}68\cdot52\\+\cdot023\end{matrix}\right\}=68\cdot543$; and so of other cases. The great use of this notation will appear in the construction of the Collecting Table, and its application to the analytical anti-Napierian logarithms. The algorithm of vital statistics introducing the necessity of intricate entanglements of common and Napierian logarithms and anti-Napierian logarithms, and reverses of those operations in analytical expression, I think it expedient to enter more particularly on the nature of logarithms than has been

done. In the first place, I observe that it is usual to consider the logarithm of a positive quantity and a negative quantity the same; thus, suppose g were a positive number greater than 1, its logarithm would be positive, and the logarithm of this logarithm might be written the logarithm of the logarithm of g; but if g were a positive number less than unity, its logarithm would be negative; and then $\lambda\lambda g$, if λ stood for the logarithm of, and $\lambda\lambda g$ for the logarithm of the logarithm of, would have no meaning in the positive scale of logarithms. This distinction I did not notice in the notation in my paper of 1825, though, properly, the notation ought to have been, as g was found to be a positive number less than 1, $\lambda(-\lambda g)$; but I did not neglect in my calculation to attend to the consequence; because, till the value of g is known as to its being less or greater than unity, there is a convenience, but only to be adopted with care, in writing $\lambda\lambda g$.

In this paper, as has been already explained, I use the prostrate small l, thus $\subset\!\!\!-\!\!\!\supset$, with the loop upwards, the first to denote the Napierian logarithm of, the second the common logarithm of; and with the loop downwards, thus $\supset\!\!\!-\!\!\!\subset$, to represent the anti-Napierian logarithm of, and the anti-common logarithm; and $\supset\!\!\!\subset$ would mean the common logarithm of the Napierian logarithm of, if they are placed horizontally in the line on the left of a character; thus $\supset\!\!\!\subset x$ would signify the common logarithm of the Napierian logarithm of x; but if placed below the character, which may be in some cases more convenient, thus $\underset{\supset\subset}{x}$, would signify the common logarithm of the Napierian logarithm of x, and $\underset{\subset\supset}{x}$ would signify the common logarithm of *minus* the Napierian logarithm of x.

Art. 11. It appears now time to show how to use the original formula $L_x = d.\overline{g|}^{x^2}$, in order to reduce it into a form for practice, and which may be written

$$\subset\!\!\!-L_x = \subset\!\!\!-d + \underset{\subset}{g}(1 + \underset{\subset}{q}.x + \frac{1}{2}\underset{\subset}{q^2}x^2 + \frac{1}{2.3}.\underset{\subset}{q^3}x^3 + \&c.),$$

where $\subset\!\!\!-$, whether put to the left of a character or below it, stands for the Napierian logarithm of, and $\underset{\subset}{q}$, $\frac{1}{2}\underset{\subset}{q^2}$, $\frac{1}{2.3}\underset{\subset}{q^3}$, &c., multiplied by $\underset{\subset}{g}$, stand for the coefficients of x, x^2, x^3, &c. in the development of $\subset\!\!\!-L_x$; and I add, with respect to any function $ax + bx^2 + cx^3$, &c., we would represent the anti-Napierian logarithm of it by $1 + b_1x + b_2x^2 + b_3x^3$, &c. The value of these coefficients will be different according as the values d, g, q are different, which values, from what has been stated, will differ for every selection used in the vital rule of three of the three lives, as, though they have for a long period in every selection the appearance of constancy, they are not absolutely constant; and I observe, that I use the same notation for the development of the value of $\subset\!\!\!-L_x$ in the more perfect formula which I have given, namely,

$$\subset\!\!\!-L_x = C\mathfrak{C}^{x^2} + k_1 s^x + k s^x - \subset\!\!\!-(\sigma^x.\underset{\subset}{q} \times \overline{x - h}) + \mu s^x;$$

but I should state the formula to which I here refer is

$$\supset\!\!\!-L_x = C\mathfrak{C}^{x^2} + k_1 s^x + k s^x - \supset\!\!\!-(\sigma^x \underset{\supset}{q}.\overline{x - h}) + \mu s^x.$$

But the formula I now state is more convenient for ultimate reduction; but the difference of the two being while $\multimap$ stands for the common logarithm of, and $\rightharpoonup$ for the anti-common logarithm of, in the formula referred to, the values of C, k, k, μ are not the same in this as in that, but would be, if multiplied by the Napierian logarithm of 10; and I will begin by showing the great use which even the original formula $L_x = d.\overline{g|^x}^{x^x}$ may be, though d, g, q are, instead of being absolute constants, quantities very slowly variable from one selection of three lives to another; though it is not equally valuable in point of accuracy to the improved formula, where all the quantities except x are constant from birth to extreme old age. And now, reverting to the formula $L_x = d.\overline{g|^x}^{x^x}$, and observing that in the Carlisle mortality the selection of the three ages may be distant from each other by even 30 years, being 10, 40, 70, we obtain a very efficiently-useful formula, although in some cases, though rarely, L_x given by the formula, and L_x of MILNE, may even differ two years; but still, the proportion of the chances of living to those ages given by each will be but a small per cent. of each other. When the selection is 10, 40, 70, we have $\backsim d$, the Napierian logarithm of $d = 8{\cdot}8833$; $\backsim q$ or $\underset{\sim}{q}$, the Napierian logarithm of $q = {\cdot}0377355$; $\underset{\sim}{q} = -{\cdot}0786136$; $\multimap(-\underset{\sim}{q}) = \bar{2}{\cdot}89605$; $\multimap\underset{\sim}{q} = \bar{2}{\cdot}57675$. Here the difference of ages in the three selected lives is successively 30 years; but as that difference may not give sufficient accuracy, I do not adopt it.

If $L_x = d.\overline{g|^x}^{x^x}$, with constant values of d, g, q, were true throughout life, we should have the logarithm of L_x = logarithm of d + logarithm of $g.q^x$, and

$$\lambda L_{a+x} = \text{logarithm of } d + \overline{\text{logarithm of } g|}.q^a.q^x,$$

as before observed; and the logarithm of the chance of a person of the age a being living in x years = the logarithm of $\frac{L_{a+x}}{L_a}$ = logarithm of $g.q^a(q^x-1)$; and similarly the logarithm of the chances of persons of the ages b, &c. living x years will be expressed. logarithm of $g.q^b(q^x-1)$, &c.; and consequently we shall have the logarithm of the chance of persons now of the ages a, b, c, e living x years

$$= \text{logarithm of } g\times(q^x-1)(q^a+q^b+q^c+q^e),$$

and also the logarithm of the chance of a person of the age p living x years = logarithm of $g\times(q^x-1).q^p$; and therefore if p be found so that $q^p = q^a+q^b+q^c+q^e$, or, which is the same thing, if p be the value of $\left(\frac{\text{logarithm of } (q^a+q^b+q^c+q^e)}{\text{logarithm of } q}\right)$, then will the chance of a person of the age p living x years, and the chance the four persons of the ages a, b, c, e being every one surviving in x years, be exactly the same. This circumstance had induced the learned Professor AUGUSTUS DE MORGAN, when commenting on the theorem $L_x = d.\overline{g|^x}^{x^x}$, which was given by me in the Philosophical Transactions in the year 1825, and the ingenious Mr. SPRAGUE, and others who appreciated it, and philosophically felt pleasure in bestowing praise where they thought merit was due for the discovery of a useful theorem, with good analytical judgment to observe.

that if the theorem were true through life, the value of annuities on any number of joint lives might be with ease obtained, which would be a most beautiful and useful property.

But, as I have shown in my paper of 1825, and here more fully illustrated, those values can only be considered as approximative constants for a limited period, though that period is very long, and whilst x increases from 0 to about 60, if a were $=10$. I have shown in that paper, in the Carlisle Table of Mortality of the late learned Mr. MILNE, that the theorem affords values for L_x at the different ages, say from 10 to 60, differing very triflingly from MILNE's Tables; and having in that paper expressly stated and shown that they were not absolutely constant, but depend on the ages of the three selected lives, it need not cause surprise that the theorem does not serve for that useful purpose; because p, determined by the equation $q^a+q^b+q^c+q^r=q^p$, would come out so much larger than the limits of the applicability of these first-found constants, that, if the method availed, it would be a contradiction to the assertion that the elements were variable, as any one may convince himself; but there are cases easily pointed out where the ages may be so taken as to afford a result by the theorem approximatively true. But this observation does not deprive the theorem $L_{a+x}=d.\overline{g|}^{a+x}$ of very great and serviceable value; but to make it extensively available it will require a Table for every age a of the constants d, g, q, varying from one age to the other; though the theorem

$$\text{-}L_x=C.\overline{c^x}+k\iota^x+k\varepsilon^x-\text{-}(e^x.\overline{x-h}.\underline{g})+\mu\nu^x$$

appears still more valuable, which has the same constants for every age, from birth to extreme old age, and which, from the age of about 10 to 80, will take the simpler form

$$\text{-}L_x=C\overline{c^x}-\text{-}(e^x.\overline{x-h}.\underline{g}),$$

in consequence of the portions, $k\iota^x$, $k\varepsilon^x$ and $\mu\nu^x$, between these limits being insignificantly small.

When the differences of the selected ages are only instead of 30 years assumed further on 20 years, that is, when difference $n=20$, and if m be respectively 10, 20, 30, 40, 50, 60, that is, if the youngest in the selection be respectively 10, 20, ... 60, we have, for finding d, g, q in the formula $L_x=d.\overline{g|}^{x}$, the following data:—

For the selection 10, 20, 30, we have

$\text{-}d=3{\cdot}88012$, $\text{-}q={\cdot}0132565$, $\text{-}(-\underline{g})=\bar{2}{\cdot}71185$, $\underline{g}=-{\cdot}051508$.

$\underline{d}=8{\cdot}9343$, $\underline{q}={\cdot}030526$, $\text{-}(-\underline{g})=\bar{1}{\cdot}07407$, $\underline{q}=-{\cdot}11860$.

For the selection 20, 40, 60, we have

$\text{-}d=3{\cdot}88137$, $\text{-}q={\cdot}012984$, $\text{-}(-\underline{g})=2{\cdot}72607$, $\underline{g}=-{\cdot}053211$.

$\underline{d}=8{\cdot}9374$, $\underline{q}={\cdot}027897$, $\text{-}(-\underline{g})=\bar{1}{\cdot}08829$, $\underline{q}=-{\cdot}12254$.

For the selection 30, 50, 70, we have

$\text{-}d=3{\cdot}8273$, $\text{-}q={\cdot}0192535$, $\text{-}(-\underline{g})=\bar{2}{\cdot}30241$, $\underline{g}=-{\cdot}020064$.

$\text{-}d=8{\cdot}8127$, $\text{-}q={\cdot}044334$, $\text{-}(-\underline{g})=\bar{2}{\cdot}66463$, $\underline{g}=-{\cdot}0462$.

For the selection 40, 60, 80, we have

$$\text{-}d=3{\cdot}75272,\quad \text{-}q=\cdot030345,\quad \text{-}(-\underline{q})=\bar{3}{\cdot}46094,\quad \underline{q}=-\cdot0028903.$$
$$\text{-}d=8{\cdot}6409,\quad \text{-}q=\cdot069872,\quad \text{-}(-\underline{q})=3{\cdot}82316,\quad \underline{q}=-\cdot0066552.$$

For the selection 50, 70, 90, we have

$$\text{-}d=3{\cdot}71469,\quad \text{-}q=\cdot0334825,\quad \text{-}(-\underline{q})=\bar{3}{\cdot}18036,\quad \underline{q}=-\cdot0015148.$$
$$\text{-}d=8{\cdot}55534,\quad \text{-}q=\cdot077096,\quad \text{-}(-\underline{q})=3{\cdot}54258,\quad \underline{q}=-\cdot0034880.$$

For the selection 60, 80, 100, we have

$$\text{-}d=3{\cdot}75272,\quad \text{-}q=\cdot0270605,\quad \text{-}(-\underline{q})=\bar{2}{\cdot}10984,\quad \underline{q}=-\cdot012878.$$
$$\text{-}d=8{\cdot}74186,\quad \text{-}q=\cdot062309,\quad \text{-}(-\underline{q})=\bar{2}{\cdot}47206.\quad \underline{q}=-\cdot029652.$$

The selections give, for the analytical expression of $\text{-}L_{a+x}$ corresponding to the several values of a below, the expressions below:—

a	$\text{-}L_{a+x}$.
10	$8{\cdot}9343-\{\cdot160927+\cdot004912x+\textcircled{4}75x^3+\textcircled{6}762x^3+\textcircled{8}58x^4+\textcircled{10}\ 35x^5$ &c.$\}$.
20	$8{\cdot}8127-\{\cdot22279+\cdot00662x+\textcircled{4}97995x^3+\textcircled{6}9924x^3+\textcircled{8}75x^4+\textcircled{10}44x^5+\textcircled{12}2x^6\}$.
30	$8{\cdot}8127-\{\cdot17468+\cdot0077443x+\textcircled{3}171678x^3+\textcircled{5}2537x^3+\textcircled{7}2812x^4+\textcircled{9}24x^5+\textcircled{11}2x^6\}$.
40	$8{\cdot}6409-\{\cdot10888+\cdot007608x+\textcircled{3}2658x^3+\textcircled{5}6191x^3+\textcircled{6}108x^4+\textcircled{8}12x^5+\textcircled{10}17x^6\}$.
50	$8{\cdot}5534-\{\cdot16470+\cdot012696x+\textcircled{3}48943x^3+\textcircled{4}1258x^3+\textcircled{6}2424x^4+\textcircled{8}3738x^5+\textcircled{10}48x^6\}$.
60	$8{\cdot}74186-\{\cdot54134+\cdot033930x+\textcircled{2}1051x^3+\textcircled{4}218826x^3+\textcircled{6}3396x^4+\textcircled{5}423x^5+\textcircled{10}44x^6\}$.

Where observe were a, 10, 20, 30, 40, 50, 60, and were x taken in each case $=0$, the above expression would be respectively the values of $\text{-}L_{10}$, $\text{-}L_{20}$, $\text{-}L_{30}$, $\text{-}L_{40}$, $\text{-}L_{50}$, $\text{-}L_{60}$, which would become $8{\cdot}9343-\cdot160927=8{\cdot}77337$; $8{\cdot}9374-\cdot22279=8{\cdot}71462$; $8{\cdot}8127-\cdot17468=8{\cdot}3887$; $8{\cdot}6409-\cdot10888=8{\cdot}5320$; $8{\cdot}5534-\cdot16470=8{\cdot}3887$; $8{\cdot}74186-\cdot54134=8{\cdot}20052$: the reader is already informed that the symbol, for instance, $\textcircled{8}3738$, signifies, $\cdot000000003738$.

Art. 12. And now returning to the assumed formula

$$\lambda L_x=C\mathfrak{C}^{x}-\lambda^{-1}\{e^{x}.\overline{x-h}.\lambda q_0\},$$

of which, in the case of the Carlisle mortality, I have already given the value of C, e, and putting $C\mathfrak{C}^{x}-\lambda L_x=M_x$, we shall have the equation

$$\lambda^{-1}\{e^{x}.\overline{x-h}.\lambda q_0\}=M_x,$$

and consequently

$$e^{x}.\overline{x-h}.\lambda q_0=\lambda M_x;$$

and using the vital rule of three, that is to say, selecting three values of x to find the three unknown quantities e, h, and q_0, say $x=m$, $x=m+n$, $x=m+2n$, we have the three equations,

$$e^{m}.(m-h).\lambda q_0=\lambda M_m,\quad e^{m+n}(m+n-h).\lambda q_0=\lambda M_{m+n},\quad e^{m+2n}(m+2n-h).\lambda q_0=\lambda M_{m+2n}.$$

Multiply the first and third of the equations together, and square the second, and we shall have by reduction, putting

$$\frac{\lambda M_m \times \lambda M_{m+2n}}{(\lambda M_{m+n})^2} = Q_m, \quad \frac{\overline{m-h}.\overline{m+2n-h}}{(m+n-h)^2} = Q_m,$$

that is,

$$\frac{\overline{(m+n-h}-n) \times \overline{(m+n-h}+n)}{\overline{m+n-h}^2} = Q_m,$$

that is,

$$\frac{\overline{m+n-h}^2 - n^2}{(m+n-h)^2} = Q_m;$$

consequently

$$\overline{m+n-h}^2 \times \overline{1-Q_m} = n^2,$$

and therefore

$$h = m+n+\frac{n}{\sqrt{1-Q_m}};$$

and m, n, λl_m, λl_{m+n}, λl_{m+2n} being taken from the Table for the given or assumed values of m and n, and C and $\mathfrak{C}$ having been found, we have the value of h; and from the equation

$$e^x . \overline{x-h} . \lambda q_0 = \lambda M_x$$

we have, by putting m and $m+n$ for x, the two equations

$$e^m \times \lambda q_0 \times \overline{m-h} = \lambda M_m$$

and

$$e^{m+n} . \lambda q_0 . \overline{m+n-h} = \lambda M_{m+n}$$

and consequently we have

$$e^n = \frac{\lambda M_{m+n} \times \overline{m-h}}{\lambda M_m \times \overline{m+n-h}},$$

and therefore

$$n\lambda e = \lambda(-\lambda M_{m+n}) + \lambda(h-m) - \lambda(-\lambda M_m) - \lambda(h-m-n),$$

because λM_m, &c. is negative, and therefore we cannot take its logarithm. Now having found h and e, we find q_0 from the equation

$$e^m . \overline{m-h} . \lambda q_0 = \lambda M_m,$$

which will give

$$\lambda\lambda q_0 = \lambda(-\lambda M_m) - \lambda(h-m) - m\lambda e;$$

and taking for m and n 20 and 30 respectively, which will give the data for the selection (by the vital rule of three) 20, 50, 80, we are to expect, if the theorem is an approximation to the law of mortality, that it will very nearly agree with the Tables of mortality for every age, from the age of 20 to 80, which, by the example I am about to give, will be found to be the case. And to proceed, from having found

$$\lambda C = \cdot 59461, \qquad \lambda \mathfrak{C} = \overline{1}\cdot 999746,$$

we have

$$\lambda C\mathfrak{C}^{30} = \cdot 589443; \qquad \lambda C\mathfrak{C}^{50} = \cdot 581882; \qquad \lambda C\mathfrak{C}^{80} = \cdot 57431;$$

$$\therefore \quad C\mathfrak{C}^{30} = 3\cdot 88631; \qquad C\mathfrak{C}^{50} = 3\cdot 81882; \qquad C\mathfrak{C}^{80} = 3\cdot 75249,$$

and therefore

$$M_{30}=C\mathfrak{C}^{30}-\lambda L_{30}=\left\{\begin{matrix}3\cdot88631\\-3\cdot78463\end{matrix}\right\}=\cdot10168 \qquad M_{50}=C\mathfrak{C}^{50}-\lambda L_{50}=\left\{\begin{matrix}3\cdot81882\\-3\cdot64316\end{matrix}\right\}=\cdot17566$$

$$M_{60}=C\mathfrak{C}^{60}-\lambda L_{60}=\left\{\begin{matrix}3\cdot75249\\-2\cdot97909\end{matrix}\right\}=\cdot77340$$

$$\left.\begin{matrix}\lambda M_{30}=\bar{1}\cdot007326\\=-\cdot99268\end{matrix}\right\} \quad \left.\begin{matrix}\lambda M_{50}=\bar{1}\cdot22467\\=-\cdot77533\end{matrix}\right\} \quad \left.\begin{matrix}\lambda M_{60}=\bar{1}\cdot04747\\-\cdot95253\end{matrix}\right\}$$

$$\lambda(-\lambda M_{30})=\bar{1}99680;\quad \lambda(-\lambda M_{50})=\bar{1}\cdot87814;\quad \lambda(-\lambda M_{60})=\bar{1}04747.$$

To find Q.

$$\begin{array}{r}\lambda(-\lambda M_{30})=\bar{1}\cdot99680\\ \lambda(-\lambda M_{60})=\bar{1}\cdot04747\\ \hline \bar{1}\cdot04427\\ -2\lambda(-\lambda M_{50})=\bar{1}\cdot75628\\ \hline \lambda Q=\bar{1}\cdot28799\\ Q=\cdot19408\\ 1-Q=\cdot80592\\ \lambda(1-Q)=\bar{1}\cdot90629\\ \frac{1}{2}\lambda(1-Q)=\bar{1}\cdot95336\\ \text{Its compt.}=\cdot04686\\ \lambda(n=30)=1\cdot47712\\ \hline \lambda\frac{30}{\sqrt{1-Q}}=1\cdot52398\\ \frac{30}{\sqrt{1-Q}}=33\cdot418\end{array}$$

$$h=m+n+\frac{n}{\sqrt{1-Q}}=83\cdot418$$

$$h-m=63\cdot418$$

$$h-m-n=33\cdot418$$

$$\lambda e=\frac{\lambda-\lambda M_{m+n}+\lambda\overline{h-m}-\lambda-\lambda M_m-\lambda\overline{h-m-n}}{30}$$

$$\begin{array}{rl}\lambda(-\lambda M_{m+n})= & \bar{1}\cdot87814\\ \lambda(h-m)\ 1\cdot80223 & \\ \hline & \bar{1}\cdot68037\\ & 1\cdot52081\\ \hline 30) & \cdot15956=\lambda e^{30}\\ \hline \lambda e= & \cdot00531866\\ \lambda e^{30}= & \cdot10637\end{array}$$

$$\begin{array}{rl}\lambda(-\lambda M_m)= & \bar{1}\cdot99680\\ \lambda(h-m-n)= & \bar{1}\cdot52401\\ \hline & \bar{1}\cdot52081\end{array}$$

$$\begin{array}{rl}\lambda\lambda q_0=\lambda(-\lambda M_{30}) & \\ -\lambda e^{30} & \\ -\lambda h-20) & \\ \lambda(-\lambda M_{30})= & \bar{1}\cdot99680\\ -\lambda e^{30}= & \bar{1}\cdot89363\\ -\lambda(h-20)= & \bar{2}\cdot19777\\ \hline \lambda\lambda q_0= & \bar{2}\cdot08820\\ \lambda q_0= & 0\cdot12252\end{array}$$

Formula.

$$\lambda L_x=C\mathfrak{C}^x-\lambda^{-1}\{e^{x-h}\cdot\lambda q_0\}$$

$$\lambda C=\cdot59461 \quad h=83\cdot412$$

$$C=3\cdot93197$$

$$\lambda e=\cdot0053186$$

$$\lambda\lambda q_0=\bar{2}\cdot08822$$

Proof of work (see the Theorem).

$$\lambda(-\lambda M_{30})=\left\{\begin{matrix}\lambda\lambda q_0 & =\bar{2}\cdot0882\\ \lambda63\cdot418 & =1\cdot80223\\ \lambda e^{30} & =\cdot10637\end{matrix}\right. \qquad \bar{1}\cdot99680$$

$$\lambda(-\lambda M_{50})=\left\{\begin{matrix}\lambda\lambda q & =\bar{2}\cdot0882\\ \lambda33\cdot418 & =1\cdot52398\\ \lambda e^{50} & =\cdot26593\end{matrix}\right. \qquad \bar{1}\cdot87811$$

$$\lambda(-\lambda M_{60})=\left\{\begin{matrix}\lambda\lambda q & =\bar{2}\cdot0882\\ \lambda3\cdot418 & =\cdot53377\\ \lambda e^{60} & =\cdot42548\end{matrix}\right. \qquad \bar{1}\cdot04745$$

x.	Carlisle. The formula gives λL_x.	Milne. λL_x.	x.	Carlisle. The formula gives λL_x.	Milne. λL_x.	x.	Carlisle. The formula gives λL_x.	Milne. λL_x.	x.	Carlisle. The formula gives λL_x.	Milne. λL_x.
10	3·81323	3·8102	28	3·75759	3·75952	46	3·67106	3·66811	64	3·48635	3·49734
11	3·81113	3·80823	29	3·75397	3·75557	47	3·66569	3·66162	65	3·47298	3·47972
12	3·80809	3·80618	30	3·75619	3·75143	48	3·65771	3·65523	66	3·45128	3·46150
13	3·80564	3·80400	31	3·74623	3·74702	49	3·65051	3·64915	67	3·43176	3·44264
14	3·80237	3·80175	32	3·74222	3·74257	50	3·64309	3·64316	68	3·41090	3·42292
15	3·80089	3·79934	33	3·73870	3·73815	51	3·63513	3·63729	69	3·38863	3·40226
16	3·79694	3·79664	34	3·73398	3·73376	52	3·62707	3·63104	70	3·36540	3·38039
17	3·79354	3·79373	35	3·72957	3·72933	53	3·61853	3·62439	71	3·33865	3·35776
18	3·79070	3·79071	36	3·72548	3·72485	54	3·60917	3·61731	72	3·32065	3·33102
19	3·78792	3·78767	37	3·71984	3·72024	55	3·59321	3·60991	73	3·27909	3·30038
20	3·78454	3·78462	38	3·71501	3·71550	56	3·59022	3·60206	74	3·24834	3·26505
21	3·78145	3·78154	39	3·71112	3·71063	57	3·58953	3·59373	75	3·21177	3·22401
22	3·77822	3·77851	40	3·70380	3·70544	58	3·56774	3·58456	76	3·20301	3·18041
23	3·77501	3·77546	41	3·69939	3·69975	59	3·55672	3·57392	77	3·12399	3·13322
24	3·77167	3·77240	42	3·69491	3·69373	60	3·54427	3·56146	78	3·08460	3·08386
25	3·76828	3·76930	43	3·68929	3·68744	61	3·53269	3·54667	79	3·06272	3·03383
26	3·76483	3·76612	44	3·68313	3·68106	62	3·51806	3·53084	80	2·97895	2·97909
27	3·76125	3·76290	45	3·67643	3·67459	63	3·50241	3·51428			

Art. 13. The very near coincidence of the result of the formula

$$\lambda L_x = C\mathfrak{C}^{x} - \lambda^{-1}\{e^{x}.\overline{x-h}.\lambda q_x\}$$

with Mr. MILNE'S Table, from the age 10 years to the age 80, that is to say, for seventy years in continuance, appears strongly demonstrative of the near proximity of the above formula to the law of mortality; and from the uniformity of the progression being evident, which uniformity in MILNE'S Table does not equally appear, gives reason for a preference for the number deduced from the law, to those of MILNE'S Table, for adoption. But it will appear that notwithstanding this agreement of the result of the formula whose constants C, $\mathfrak{C}$, e, h, q_x are obtained, the first two of them from the values of d in the two formulæ in my paper of 1825, which treats of the formula $L_x = d.\overline{g}|^{x}$, the one being obtained by the vital rule of three, by the selection of the three ages 20, 40, 60, the other by the selection 60, 80, 100, and the other three constants, namely e, h, q_x, by help of those constants C, $\mathfrak{C}$, by only three selected ages at thirty years' distance from each other, namely the ages 20, 50, 80; and though the vital rule of three is here constructed on a more recondite analysis than the former; still that uniformity and that interesting coincidence does not subsist with ages less than 10, nor with ages above 80, and especially for ages from birth to the age of a few months; because the more correct formula seems to require three additional terms discoverable by investigation. These I find to be of the form $k_1\varepsilon^{x}$, $k\varepsilon^{x}$, $\mu\nu^{x}$, where k, κ, k, ε, μ, ν are all constant quantities, and of such peculiarly interesting values, that I feel it proper to draw the reader's attention to their values, their effects, and the mode of the discovery of them. The effect of the first, $k_1\varepsilon^{x}$, commences at birth in its greatest value, but at the expiration of one month sinks to comparative insignificance, and before the end of one year leaves no appreciable signs of its existence; the second, $k\varepsilon^{x}$, arises in its effect with birth, but continually decreases in

effect till the age of about 21 in the Carlisle mortality, and then and after, through the remainder of life, becomes of total insignificance; and the third term does not come into appreciable effect for calculating the number of living till the age of about 80; though for anticipating the number of living which will result from some age to ages above 80, for the purpose of calculation, its effect cannot be overlooked when the age from which the anticipation is necessary is some years less than 80. And the methods of finding those constants which I have adopted will now be explained. Paying attention only to the additional function ks^x, the formula will stand

$$\lambda L_x = C\mathfrak{C}^{x'} + ks^x - M_x;$$

where M stands as above for $\lambda^{-1}\{e^{x.\overline{x-h}}.\lambda q_0\}$; and putting $\lambda L_x + M_x - C\mathfrak{C}^{x'} = K_x$, for the sake of brevity, and taking, in order to have two equations, in order to find the two constants k, s, two values of x, namely $x=1$, $x=2$, we shall, from MILNE'S Tables having λL_1, λL_2, and from the known values of $C\mathfrak{C}=3{\cdot}92971$, $C\mathfrak{C}^2=3{\cdot}9271$, and the values of M_1 and M_2, have the value K_1 and K_2; and as we have $ks=K_1$, $ks^2=K_2$, we have

$$s=\frac{K_2}{K_1} \text{ and } k=\frac{K_1}{s}=\frac{K_1^2}{K_2},$$

that is,

$$\lambda s=\lambda K_2-\lambda K_1=\bar{1}{\cdot}79811, \text{ and } \lambda k=\bar{1}{\cdot}16855,\ s={\cdot}62822,\ k={\cdot}14742.$$

These values of k and s being now known, we introduce the term $_fk_fs^x$, and we shall have

$$\lambda L_x = C\mathfrak{C}^{x'} + {}_fk_fs^x + ks^x - M_x = K_x - M_x + {}_fk_fs^x;$$

and consequently

$${}_fk_fs^x = \lambda L_x + M_x - K_x;$$

put this $={}_fK_x$ for the sake of brevity, and we shall have ${}_fk_fs^x={}_fK_x$; and in order to have two equations for the purpose of finding $_fk$ and $_fs$, take for x the two ages of $x=0$, that is, the age of birth, and $x=\frac{1}{12}$, that is, the age of one month, and we have the two equations

$${}_fk={}_fK_0 \text{ as } s^0=1, \text{ and } {}_fk_fs^{\frac{1}{12}}={}_fK_{\frac{1}{12}},$$

and we obtain

$${}_fk={\cdot}02266, \quad \lambda_fk=\bar{2}{\cdot}35526, \quad \lambda_fs^{\frac{1}{12}}=\bar{1}{\cdot}27720, \quad \lambda_fs=\bar{9}{\cdot}3264\text{*}.$$

It now remains to find $\mu_{\theta,\nu}$ of the expression $\mu\nu^x$; which only is of appreciable value when $_fk_fs^x$, ks^x become perfectly of inappreciably small consideration; that is, in the case

* In deducing the above values of $_fk$, s, there were some slips of the pen: for instance, taking $k={\cdot}14042$ for $k={\cdot}14742$, so that $_fk$ was taken too large, namely ·02266 by ·007, and should be taken therefore $={\cdot}01566$; this change will require a change in the value of $_fs$, which is obtained by making some slight alteration in MILNE'S date for $L_{\frac{1}{12}}$, that is to say, for the age of one month, and that change I have made in my calculation to an increase of eight years, making $L_{\frac{1}{12}}=9475$ instead of 9467, which is as likely to be correct as the other. Some small alteration I thought I found necessary in order not to get into imaginary quantities, and my calculation gives $\lambda_fs^{\frac{1}{12}}=\bar{1}{\cdot}43767$, $\lambda_fs=\bar{7}{\cdot}25104$, and of course the formula gives $\lambda L_{\frac{1}{12}}=9475$ to be adopted, though for the first months of age the last may be retained.

where the equations may be considered

$$\lambda L_x = C\xi^{\sigma} - \lambda^{-1}\{\sigma^{\cdot}\overline{x-h}.\lambda q_0\} + \mu\nu^{\sigma};$$

and therefore, putting $\lambda L_x + M_x - C\xi^{\sigma} = Q_x$, we shall have $\mu\nu^{\sigma} = Q_x$.

The operations for λs, λk, λg, λf, and $\lambda\mu$, $\lambda\nu$, the logarithms of the constants in the general formula

$$\lambda L_x = C\xi^{\sigma} + f_1 s^{\sigma} + k s^{\sigma} - \lambda^{-1}\{\sigma^{\cdot}\overline{x-h}.\lambda q_0\} + \mu\nu^{\sigma},$$

are as follow:—

For λs, λk,

$\lambda k_1 = \lambda K_1 = \bar{2}\cdot96666$, $\lambda k s^{\sigma} = \lambda K_x = \bar{2}\cdot76477$

$-\lambda k_2 = \bar{2}\cdot96666$

$\lambda s = \bar{1}\cdot79811$

$\lambda k_2 = \bar{2}\cdot96666$

$\lambda k_2 - \lambda s = \lambda k = \bar{1}\cdot16855$

$k = \cdot14742$

For λf, λg,

$\lambda L_0 = 4\cdot00000$	$C = 3\cdot93197$	$\lambda L_{\frac{1}{12}} = 3\cdot97621$	$C\xi^{\frac{1}{12}} = 3\cdot93179$
$M_0 = \cdot09505$	$k = \cdot14042$	$M_{\frac{1}{12}} = \cdot09505$	$k s^{\frac{1}{12}} = \cdot13518$
$4\cdot09505$	$4\cdot07239$	$4\cdot07126$	$4\cdot06697$
$-4\cdot07239$	$4\cdot06697$		
$f = \cdot02264$	$f g^{\frac{1}{12}} = \cdot00429$	$\lambda f_1 = \bar{3}\cdot63246$	
$\lambda f = \bar{2}\cdot35526$		$-\lambda f = \bar{2}\cdot35526$	
		$\lambda g^{\frac{1}{12}} = \bar{1}\cdot2777$	
		$\lambda g = \bar{9}\cdot3264$	
		$\lambda f = \bar{2}\cdot35526$	

For $\lambda\mu$, $\lambda\nu$.

$\lambda\mu + 90\lambda\nu = \lambda Q_{90}$ } $\lambda\mu + 100\lambda\nu = \lambda Q_{100}$ }	$\lambda L_{90} = 2\cdot15229$	$\lambda L_{100} = \cdot95425$	$\lambda Q_{100} = \cdot33421 = \lambda\mu\nu^{100}$	$\lambda\mu\nu^{100} = \cdot33421$
	$M_{90} = 1\cdot74908$	$M_{100} = 4\cdot91349$	$\lambda Q_{90} = \bar{1}\cdot23042 = \lambda\mu\nu^{90}$	$\lambda\nu^{100} = 11\cdot0379$
	$3\cdot90137$	$5\cdot86774$	$1\cdot10379 = \lambda\nu^{10}$	$\lambda\mu = \overline{11}\cdot29631$
	$C\xi^{90} = 3\cdot73138$	$C\xi^{100} = 3\cdot70336$	$\lambda\nu = \cdot110379$	
	$Q_{90} = \cdot16999$	$Q_{100} = 2\cdot15888$		

And we have

$C = 3\cdot931968$; $\xi = \cdot99942$; $\lambda C = \cdot59461$, $\lambda\xi = \bar{1}\cdot99974617$;

$k = \cdot4742$, $\lambda k = \bar{1}\cdot16855$; $\lambda s = \bar{1}\cdot79811$; $f = \cdot015522$, $\lambda f = \bar{2}\cdot1824$;

$e = 1\cdot01247$, $\lambda e = \cdot005318666$; $q_0 = 1\cdot0286$, $\lambda q_0 = \cdot012252$, $\lambda\lambda q_0 = \bar{2}\cdot08822$;

$\lambda_1 g = 7\cdot\bar{2}5104$, $\lambda_1 g^{\frac{1}{12}} = \bar{1}\cdot43767$;

$h = 83\cdot418$; $\mu = (10)1978$, $\lambda\mu = \overline{11}\cdot29631$; $\nu = 1\cdot2894$, $\lambda\nu = \cdot110379$;

which will give an easy means of regularly finding the value of L_x, the number of persons living out of the number L_u, the number taken for the birth, as a base, for every year of age to 100, or beyond that if the accuracy can be depended on beyond 100; and, what will be interesting to the reader in favour of the formula, it will give the number agreeing with MILNE'S Table living for the monthly portions of the deaths of children below one year of age with a very satisfactory agreement, considering that perfect accuracy in MILNE'S Table cannot be expected to exist, owing to the scanty means he could have had for that purpose. In a paper I presented for consideration to the fourth section of the International Statistical Congress, already referred to, held during the week commencing on 16th July last, "On the one uniform Law of Human Mortality from the age of Birth to extreme old age, and on the Law of Sickness," which was honoured by its publication among its reports, I only offered hints, without the abstruse mathematical portions being brought forward of this paper which I am venturing to offer to this Society; which I felt was partly a duty I had to perform; namely, to add my small services to the Congress; especially in consequence of the insufficient state of my health ever since and before I attempted to search into my former published and unpublished papers on the interesting subject of Vital Statistics, and useful Application of the Results, with a view to improve, and add matter of interest to the subject; which I flatter myself I had treated on with some approbation from the scientific public; and being doubtful if I should be able to offer even the paper I had so far completed for the approbation of the Society, I therefore feel satisfied that my having given these slight hints will not be a cause to render this attempt to bring these few pages before the Society unacceptable, especially as, since the hints were written, I think I have discovered a greatly improved form of the formula of mortality, and a more satisfactory one. The formula of mortality which in these hints was given was

$$\lambda L_x = \text{constant} + k\theta^x - k\beta^x . x - nq^x - P_x,\ P_x \text{ being} = \theta . (w)^{w^x (x-u)},$$

where all the values on the right-hand side of the equation are constant except x, from birth to extreme old age, including θ, w, and u; and θ very nearly I said unity, which value it was taken; but I stated I thought it could not be exactly unity, the reason for which doubt was that P_x having been put for the common logarithm of the number whose common logarithm is θ, raised to the power $w^{w^x (x-u)}$, if θ were exactly 1, the number whose common logarithm is θ would be exactly 10, and the equation, of which the aforesaid equation is the logarithm, would be

$$L_x = \text{constant} \times A^{w^x} B^{w^x . x} \times C^{w^x} \times D^{P_x};$$

A, B, C, D being constants, and $\lambda^{-1} P_x = 10^{-w^{w^x (x-u)}}$, and it appeared to me that nature could not be governed by the conventional notation of the decimal arithmetic.

Art. 14. In the paper presented to the International Congress, I gave a Table, resulting from the above formula, of persons living to the extent of life, from birth to the age of 100, and for the age of 1, 2, 3, 6, 9, and 12 months, which appeared to

me very satisfactory; but I do not repeat that Table here, as I have given a Table from what I consider the improved formula: but I gave in that paper the values for the four cases, namely, Carlisle, Northampton, Sweden, and De Parcieux, of the aforesaid valuable constants of the formula just stated, and there alluded to; in this I mean to subjoin the results for tables of mortality of constants for the last three places, which are satisfactory in my opinion. But I am not able to say if, before this goes to press, I shall be able to give the results which an investigation of the constants in the formula, which I consider an improvement of the other, will give for the last three mortalities.

Art. 15. The term 'expectation of life' in common use is not applied to that which I should call the orthodox expectation of life, and therefore, not venturing to discard the usual adoption of it, I will introduce the term orthodox expectation of life for that term of years to which it is an equal chance whether a person of a given age shall reach or not; and I observe, if the Napierian logarithm of the chance of a person of a given age living x years beyond that age be represented by ${}^1Ax+{}^2Ax^2+{}^3Ax^3+$&c., which will be a negative quantity, this must be put $=-\cdot 691472$, namely, Napierian logarithm of $\frac{1}{2}$. And the value of x which will result from that equation is that which I call the orthodox expectation of life of that person; and if there be any number of joint lives, and the sum of the Napierian logarithms of the chances of each separately living x years be represented by ${}^1Ax+{}^2Ax^2+{}^3Ax^3+$&c., if this be put $=-\cdot 691472$, namely, the Napierian logarithm of $\frac{1}{2}$, the value of x, which this equation will give, will be the period beyond the present to which the joint existence of those joint lives has an equal chance of attaining or not attaining; and as the coefficients ${}^1A$, ${}^2A$, ${}^3A$ are very converging, a very few terms will be sufficient, perhaps merely the first term; and if there be two separate combinations of joint lives, which I will call the A combination and the B combination, and the Napierian logarithm of the chance of the A combination lasting x years be represented by ${}^1Ax+{}^2Ax^2+{}^3Ax^3+$&c., and that of the B combination lasting x years be represented by ${}^1Bx+{}^2Bx^2+{}^3Bx^3$, &c., then if these two be equal to each other, and x comes out positive and not beyond the limit of the accuracy of the theorem, x will be the term to which it is an equal chance of one combination in particular surviving or not surviving the other; but should x come out negative, that term of years does not exist within the limits at least of accuracy of the theorem.

Art. 16. If there be two sets of lives, which I will call the A combination and the B combination, and the separate chances of their existing x years be respectively represented by

$$1+{}^1Ax+{}^2Ax^2+{}^3Ax^3+\text{&c., and } 1+{}^1Bx+{}^2Bx^2+{}^3Bx^3+\text{&c.}$$

respectively, then the chance that the B combination shall fail during the continuance of the A combination, and between the periods $t=$ a given quantity to $t=x$, will be the fluent of

$$(1+{}^1Ax+{}^2Ax^2+{}^3Ax^3\ \text{&c.})\times -\text{fluxion of } ({}^1Bx+{}^2Bx^2+{}^3Bx^3,\ \text{&c.})$$

between those limits, because the fluxion of the discontinuance is *minus* the fluxion of

the continuance, that is to say, that chance is equal to

$$-({}^1\mathrm{B}.\overline{x-t}+\tfrac{1}{2}{}^1\mathrm{K}.\overline{x^2-t^2}+\tfrac{1}{3}{}^3\mathrm{K}.\overline{x^3-t^3},\ \&c.),$$

where $${}^1\mathrm{K}=2\,{}^2\mathrm{B}+{}^1\mathrm{A}\,{}^1\mathrm{B};\quad {}^2\mathrm{K}=3\,{}^3\mathrm{B}+2\,{}^1\mathrm{A}\,{}^2\mathrm{B}+{}^2\mathrm{A}\,{}^1\mathrm{B};$$

$${}^3\mathrm{K}=4\,{}^4\mathrm{B}+3\,{}^1\mathrm{A}.{}^3\mathrm{B}+2\,{}^2\mathrm{A}.{}^2\mathrm{B}+{}^3\mathrm{A}\,{}^1\mathrm{B};\quad {}^4\mathrm{K}=5\,{}^5\mathrm{B}+4\,{}^1\mathrm{A}.{}^4\mathrm{B}+3\,{}^2\mathrm{A}.{}^3\mathrm{B}+2\,{}^3\mathrm{A}.{}^2\mathrm{B}+{}^4\mathrm{A}.{}^1\mathrm{B};$$

where the law of continuation is evident. And the chance that the combination A failed previously to the failure of the combination B, is the fluent of

$$({}^1\mathrm{A}\dot{x}+{}^2\mathrm{A}\dot{x}^2+{}^3\mathrm{A}\dot{x}^3,\ \&c.)({}^1\mathrm{B}\dot{x}+2\,{}^2\mathrm{B}x\dot{x}+3\,{}^3\mathrm{B}x^2\dot{x},\ \&c.),$$

which evidently is the excess of the chance of the combination of B failing independently of any connecting A, above the chance of the combination B failing whilst the combination A exists, and thus we may proceed to successive additional cases of conditional contingencies with respect to more combinations. Now it is observable that in consequence of the great convergency of the coefficients ${}^1\mathrm{A}$, ${}^2\mathrm{A}$, ${}^3\mathrm{A}$, &c., and of ${}^1\mathrm{B}$, ${}^2\mathrm{B}$, ${}^3\mathrm{B}$, &c., and of the small values of ${}^1\mathrm{A}$ and ${}^1\mathrm{B}$, except very shortly after birth, which result from what has been previously stated, if all of them, except ${}^1\mathrm{A}$, ${}^1\mathrm{B}$, be considered as nothing, unless x be very great, the first value will be very nearly expressed by the first term $-{}^1\mathrm{B}.\overline{x-t}$, because ${}^1\mathrm{A}.{}^1\mathrm{B}$ is small of the second degree; and the second value in a similar way, for a long period in which it shall occur that both combinations fail, is very nearly $\frac{1}{2}\mathrm{A},\mathrm{B},.\overline{x^2-t^2}$, giving an equal chance which shall have failed first; and this is a generalization of Mr. Morgan's hypothesis of two lives only, which enabled him to solve questions respecting the lives of three persons A, B, C, contingent on the life or death of A, provided B and C be both dead, involving contingencies of survivorship between them. But that this law cannot accurately exist for any possible continuous law of mortality, I have proved in my paper of 1825, unless of the form $\mathrm{L}_a=e'-e''.\epsilon^a$, where e', e'', ϵ are constant quantities at pleasure, and a the age, and which, in an extreme case of ϵ differing infinitely little from unity, is reducible into the form

$$\mathrm{L}_a=g'-g''a,\ \text{if } g'=e'-e'',\ \text{and } g''=e''\epsilon,$$

e' and e'' being infinitely large, differing from each other by a finite quantity, and ϵ infinitely small, and in consequence g'' a finite quantity. But with respect to the two combinations generally—(and thus is found the chance of one combination failing during the continuance of the other combination, or of its failing after the continuance of the other combination, and this is evidently considerably more general than the cases of Messrs. Morgan, Baily, and Milne, in which there are only three lives concerned,)—I observe that they will give useful practical solutions however many lives are concerned, and however complicated the conditions of survivorships may be between them, whereas it may be a question if the solutions of those gentlemen, whose memory I respect, are at all practical; especially when there are only tables of single and two joint lives at hand. I therefore consider that the solutions I had given to those three-lived questions, and to questions of more lives, in my Tract of 1820, to which I refer the reader, which were to be derived from calculated Tables of annuities, are worthy of the attention of

scientific inquirers. For instance, the first of them and the following, in which, instead of there being only three lives concerned, there are any number: the first question of those gentlemen is to find the value of the assurance of the life of A provided in the lifetime of B and C; and my solution, expressed in the notation of that paper, is as follows:—p representing one year, if the tables are calculated for annual payments; n the period from which the assurance is to commence; m the period to which the assurance is to continue; r the present value of unity due in one year certain at the rate of interest involved in the question; then the value calculated from the Table of annuities for every unity assured, the present ages of A, B, C being a, b, c respectively, is

$$\frac{L_{a-p,\,b-\frac{1}{2}p,\,c-\frac{1}{2}p}}{L_{a,\,b,\,c}} \times \overset{r}{\underset{n}{\overset{p}{\big]}}}_{m} a-p, b-\tfrac{1}{2}p, c-\tfrac{1}{2}p - \frac{L_{b-\frac{1}{2}p,\,c-\frac{1}{2}p}}{L_{b,\,c}} \times \overset{r}{\underset{n}{\overset{p}{\big]}}}_{m} a, b-\tfrac{1}{2}p, c-\tfrac{1}{2}p;$$

an extremely small and insignificant quantity being neglected.

Art. 17. This, with Tables calculated for two joint lives, and the annuities for those interpolated from them, instead of being impracticable, as the other solutions are, will be found perfectly simple and easy. And in my second Tract, namely that of 1825, I gave Tables for calculating the values of annuities at any likely rate of mortality to be adopted, and at any rate of interest, and applicable when the lives are subject to different rates of mortality, and the same theorem for the value of assurance, *mutatis mutandis*, however many lives B, C, D, E, &c. are proposed to be jointly living at the death of A. And with regard to art. 7 of the paper of 1825, I will only in this place direct the reader to my solution to the question, which is one of the more intricate nature of those given, and solved by those gentlemen. The question being the value of the assurance on the first death of A and B, which shall be second or third of the deaths which shall happen of the three lives A, B, C, my solution to this question in my first paper is simply

$$\tfrac{1}{2}\cdot \overset{r}{\underset{n}{\overset{p}{\big]}}}_{m} b+\tfrac{1}{2}\cdot \overset{r}{\underset{n}{\overset{p}{\big]}}}_{m} a+\tfrac{1}{2} \overset{r}{\underset{n}{\overset{p}{\big]}}}_{m} b,\, c+\tfrac{1}{2} \overset{r}{\underset{n}{\overset{p}{\big]}}}_{m} a,\, c- \overset{r}{\underset{n}{\overset{p}{\big]}}}_{m} a,\, b,\, c;$$

that is, half the sum of the assurances of B, of A, of B, C, of A, C, less the assurance of the joint lives A, B, C, all of which are easily computed. This solution, when I obtained it, seemed to me so enormously superior to those which those learned gentlemen had given to it, that I found it necessary to show my readers that they depended on the same principles, by reducing the portions of my solutions to the portions of the earlier solutions; and the remaining questions of that section are of a similar superiority to the earlier solutions.

Art. 18. But in making this remark I wish to express myself indebted to those gentlemen for their labours, whose names remain honoured in the memory of scientific persons who have cultivated different branches of science, and to state I allude to the comparison that my readers may follow an improved path if they meet with one, and to remind those who may follow with benefit for science others who have trodden down the aspe-

rities of the path to repay with gratitude the benefit they have received, and not in self-conceit to forget that a dwarf on a giant's shoulders may see a more extended prospect of beauties than the giant could; and hoping to be excused for this digression, I will proceed by observing that the best way of solving intricate questions relative to the value of life interest, may not at all be, as is usual, to have recourse to the value of life annuities. With respect to the expression given above, namely the fluent of

$$(1+{}^1Ax+{}^2Ax^2+{}^3Ax^3, \&c.) \times - \text{ fluxion of } (1+{}^1Bx+{}^2Bx^2, \&c.),$$

and the one which follows, I observe that if in the two sets of combinations of lives all the lives of the combination A were required to be living, and a specified number of specified persons of the combination B were also required to be living, and the assurance is to be on the failure of the remaining portion of joint lives in the B combination, provided all the lives of the A combination and all the lives specified of the B combination were also living, then the solution would be the same, with the exception that all those specified lives of the B combination must first be transferred to the A combination; this must be evident; but it is mentioned that in case with respect to the B combination only a certain number specified, without specification of which lives of the B combination, are to be living, the question may be solved with this difference, that the sum of all the values must be taken which will occur by the various modes of withdrawing the specified number from the B combinations; but there are cases where a less laborious mode may apply.

Art. 19. But returning to the originally expressed combinations A and B, suppose it were required to find the value of an insurance of unity on the failure of the combination B, provided it happened during the continuance of the combination A. Let the expression $1+{}^1Ax+{}^2Ax^2+{}^3Ax^3$, &c., instead of representing, as before, the anti-Napierian logarithm of the sum of all the Napierian logarithms of the chances of existence of the different lives separately in the combination A, now represent the anti-Napierian logarithm of that sum, having the coefficient of the first power of x increased by the Napierian logarithm of the present value of unity due in one year at the rate of interest to be involved in the calculation, then the fluent of the expression

$$(1+{}^1Ax+{}^2Ax^2+{}^3Ax^3, \&c.)\times(-{}^1B\dot{x}-2\,.\,{}^2Bx\dot{x}-3\,.\,{}^3Bx^2\dot{x}, \&c.),$$

instead of being the chance of a failure of the combination B between the given period and x during the continuance of the combination A, will be the present value of unity payable on that event taking place, and so of others. For instance, if only three lives of the present ages a, b, c were concerned, then the value of the assurance on the life A, if it happened in the lifetime of B and C, which is the first of art. 6 in my paper of 1820, would be solved thus: supposing the present ages of A, B, C to be a, b, c, and the Napierian logarithms of

$$\frac{L_{a+x}}{L_a}, \frac{L_{b+x}}{L_b}, \frac{L_{c+x}}{L_c}$$

to be respectively

$$A_1x+A_2x^2+A_3x^3+ \&c., \quad B_1x+B_2x^2+B_3x^3 \&c., \quad C_1x+C_2x^2+C_3x^3+ \&c.,$$

and the Napierian logarithm of the present worth of unity certain in one year to be $\overline{\varrho}$, find the anti-Napierian logarithm of

$$\overline{\varrho+A_1}.x+A_2x^2+A_3x^3+\&c.,$$

and the anti-Napierian logarithm of

$$\overline{B_1+C_1}.x+\overline{B_2+C_2}.x^2+\overline{B_3+C_3}.x^3+\&c.,$$

which will be easily done by the method explained above, and will only require a few terms in consequence of the coefficients of x in the above expressions being small; and calling the first analytical anti-logarithm

$$1+{}^1Ax+{}^2Ax^2+{}^3Ax^3\ \&c.,$$

and the second

$$1+{}^1Bx+{}^2Bx^2+{}^3Bx^3\ \&c.,$$

and putting the fluent of *minus* the fluxion of the first

$$-({}^1A\dot{x}+2.{}^2Ax\dot{x}+3.{}^3Ax^2\dot{x},\ \&c.),\times(1+{}^1Bx+{}^2Bx^2+\&c.)={}^1K\dot{x}+{}^2Kx\dot{x}+{}^3Kx^2\dot{x},\ \&c.,$$

then the value of the assurance for the term commencing in t years and ending in x years will be

$${}^1K\overline{x-t}+\tfrac{1}{2}.{}^2K\overline{x^2-t^2}+\tfrac{1}{3}.{}^3K.\overline{x^3-t^3},\ \&c.,$$

and will not require many terms, and the equation is thus solved without the Tables of the values of annuities; and in a similar way are other questions of this species; and the solution in the more compounded cases of art. 7 of my first paper, such as example 1, taken from Messrs. Baily, or Milne, or Mr. Morgan's original paper, which is the foundation of them, containing an intermediate conditional contingence of survivorship, and which in my paper above alluded to contains the double fluent reducible to a single fluent multiplied by a variable quantity, and a single fluent added, and the solution in much more intricate cases, and containing any number of lives, and even a long string of intermediate contingencies which would involve double, triple, quadruple, &c. fluents, &c., may be effected. As a simple example, suppose there were three separate combinations of lives, which I will call A, B, C combinations of lives, and it be required to find the value of an assurance on the failure of the C combination after the failure of the B combination, the A combination having failed previously, provided that failure should happen during certain given periods. First, find by the method above the analytical value of the contingency that the B combination shall fail after a given time, the combination A having failed previously; and suppose this to be

$$K+{}^1Kx+{}^2Kx^2+\&c.;$$

and having found the anti-Napierian logarithm of

$$1+\frac{{}^1K}{K}x+\frac{{}^2K}{K}x^2+\frac{{}^3K}{K}x^3,\ \&c.,$$

and having found the Napierian logarithm of the separated chances of every life in the combination B being living, and added the sum to the Napierian logarithm of

$$1+\frac{{}^1K}{K}x+\frac{{}^2K}{K}x^2,\ \&c.,$$

and having found the anti-Napierian logarithm of this sum, and multiplied this by K,

call this

$$H+{}^1Hx+{}^2Hx^2+{}^3Hx^3, \&c.,$$

then on multiplying *minus* the fluxion of this by the anti-Napierian logarithm of the sum of all logarithms of the separate chances of each life in the combination C being living, increased by ϱx (previously ϱ being the Napierian logarithm of the value of unity certain due in one year at the rate of interest to be involved in the calculation), the fluent of this will give the value required, and will not require many terms of the resulting series, which I represent by

$${}^1I(x-t)+{}^2I(x^2-t^2)+{}^3I(x^3-t^3), \&c.,$$

t being the period of commencing of the assurance.

Art. 20. And much more intricate cases may occur in valuations of property offered to the public and to reversionary companies. But instead of in this place dwelling on the intricacies which may occur, I will refer to art. 5 of my paper of 1820 with respect to calculations referring to the formulæ of the convenient notation there used,

$$\overset{r}{\underset{m}{\overset{p}{\underset{n}{|}}}}\, a, b, c, \&c., \qquad \overset{r}{\underset{m}{\overset{p}{\underset{n}{|}}}}\, a, b, c, \&c., a', b', c', \&c.,$$

in which I have developed the various combinations and their connexions of annuities of 1, 2, 3, &c. joint lives which would come into the calculation, which I derived by the development of the expression

$$(xE_{a,n}+D_{a,n})\times(xE_{b,n}+D_{b,n})\times(xE_{c,n}+D_{c,n})\times \&c.=1,$$

when $x=1$, where $E_{a,n}$, $E_{b,n}$, &c. denote the chances of persons of the ages a, b, &c. being living in n years, and $D_{a,n}$, $D_{b,n}$ the chances of their being then dead; and in consequence

$$E_{a,n}+D_{a,n}=1,\ E_{b,n}+D_{b,n}=1, \&c.,$$

the x attached to the E being supposed $=1$, but only introduced, as x, to point out, by its exponent in the development, the various combinations of persons on which the annuities are to be calculated, if, previously to the development, $D_{a,n}$, $D_{b,n}$ be expunged by the equations

$$D_{a,n}=1-E_{a,n},\ D_{b,n}=1-E_{b,n}, \&c.$$

But had we introduced y, also a representative of 1, and written the expression

$$xE_{a,n}+yD_{a,n}\times xE_{b,n}+yD_{b,n}\times \&c.=1,$$

and not expunged D, the exponents of x and y in the developments would represent the combinations and connexions of the various cases which would occur of persons living and persons dead in connexion with each other in the time n; so that if we chose to have Tables constructed on a certain number of the persons being dead of the whole number, we might have a variety of modes of calculating the value of life contingencies, instead of being dependent on the value of joint lives; but I do not mention this by any means as a recommendation, but quite the contrary. But if Tables of the reversions of annuities

enjoyed by any number of joint lives were calculated, they would in many cases be more convenient than the value of annuities on joint lives; as, for instance, if we had to find the reversion of an annuity on three lives now aged a, b, c, we should only have to search the Table of the reversion on three joint lives, and we should have the value at once: but if the calculation were to be made by methods above, and we represent the anti-Napierian logarithm of the present expression, as we did that of the former expression, by ${}^1Sx+{}^2Sx^2+{}^3Sx^3$, &c., the present value of the annuity of which the first payment is to be in the time w, and the last in the time z, will be represented by the expression

$$P\,\overset{\overset{x}{\overline{1}}}{{}^{w}_{z}|}\,x^{s}+P^1S\,\overset{\overset{x}{\overline{1}}}{{}^{w}_{z}|}\,x^{s+1}+P^2S\,\overset{\overset{x}{\overline{1}}}{{}^{w}_{z}|}\,x^{s+2}+P^3S\,\overset{\overset{x}{\overline{1}}}{{}^{w}_{z}|}\,x^{s+3},\ \&c.,$$

where it may be observed that if the condition of the deaths, say of α, β, γ, did not enter the question, this formula ought to be the same as the first, which would require P to be 1, which may appear a paradox; but the paradox is solved if we consider the condition would be the same if they entered in the question or not, if they could not die; that is to say, if p_α, p_β, &c., and consequently P be $=1$. Then, by means of the Collecting Table, the value of the above series will be given.

Art. 21. The valuation by the old methods of annuities depending on such complications, by the aid of Tables of only two joint lives, as the value of the annuities on three, four, five, &c. joint lives can only be obtained from them by very troublesome interpolations of very inaccurate results, evidently calls for an improved method. For instance, if the annuity depended on the joint existence of three lives of the present ages of a, b, c after the existence of all the lives d, e, f, the labour and insufficiency in point of accuracy of that mode of valuation will immediately appear; because, the chance of each of them individually being living in x years being for the sake of ease represented by a, b, c, d, e, f respectively, by multiplying out at length the chance of the conditions required being fulfilled, that chance will be then represented by

$$abc\times\overline{1-d}\times\overline{1-e}\times\overline{1-f}=abc-abce-abcd-abcf+abced+abcfe+abcfd-abcdef.$$

It appears that by the old methods, with only Tables of two joint lives, we should have to interpolate the value of one of three annuities of three joint lives, three of four different annuities of four joint lives, three of five different annuities of five joint lives, and one annuity of six joint lives, and the results of every one of these interpolations would be inaccurate.

Art. 22. On special risk, considered with respect to single lives, and on many lives in connexion; and the valuation of certain contingencies; and of property depending on those contingencies and under influential connexion; I am not aware whether or not any one has gone before me on this important subject; but I have not, in my official practice in the science of assurance, lost sight of its existence. And to draw the reader's attention at once to the subject, I will suppose we had the two problems to consider—the first, to assure a sum on the extinction of the coexistence of two coexistent lives A, B, commonly called two joint lives, that is, to be payable at the death of the first of the two

of them which may die; and the other problem, to assure the like sum on the death of A in particular, provided it happens in the lifetime of B.

Then if A and B were of equal age, and both subject to the same law of mortality, without any speciality, the value of the assurance in the first case would be just double of that in the second case, for however long or however short the assurance might be. But if A and B were both sailors, continually together in the same ship, the common risk for each by risk of shipwreck, &c. would equally attach to each; and if the assurance were only for a few hours, say, for instance, for one voyage from Dover to Calais, so that the risk for time were insignificant with respect to the common uninfluenced risk; there then would be only to be considered the risk for each not to escape the consequence of the wreck; but the chances would now come into play, of all on board being lost, of none being lost, or of a portion, and what portion of them being lost; and that the portion lost were the two A and B, or only one of them, and which one that might be, which might depend on the powers of each for swimming; but should the chance be that the wreck, if it happened, would cause the death of both A and B, then the assurance for the period of the intended voyage on the death of one of them only, of both, or of one in particular named, of this in the lifetime of the other, would be all the same. This is but one case of specialities, and specialities of influences of risks. And perhaps there are very few cases of assurances and valuations connected with conjoint lives, if any, or even of single, with reference to affections which may arise from climate, localities, epidemics, endemics, ancestorial influences, &c., and also epochs, which are not affected by speciality; but though the subject may be very interesting, I have not in this paper entered largely on it, but will only now touch on a portion of it which may be found by my readers well worthy of their attention. Suppose by the common law of mortality, uninfluenced by any speciality, out of the number of the age a, represented by L_a, there would be living in x years the number L_{a+x}; but that they each for any period become subject to an extra risk of death during an infinitely small time, $\dot{t}=\alpha\dot{x}$, where α may, as the case may be, be either a constant quantity or a function of x; then it is evident L_{a+x} will not be the number of them living in x years; and to find what that number would be, suppose it represented by M_x; then its fluxion $\dot{M}_x$, on the supposition that the lives of the then existing number were not deteriorated by the pre-existence of those specialities to that period, would evidently be

$$=M_x\frac{\dot{L}_{a+x}}{L_{a+x}}-\alpha\dot{x}M_x.$$

But it is to be observed that this supposition is not necessarily tenable, but it would be so in case the circumstance of speciality was transient after the moment of its operation; such, for instance, if it were a wreck at sea, leaving no effects; and therefore I shall at present only consider the case when the hypothesis is tenable. And resuming the equation, we have

$$\frac{\dot{M}_x}{M_x}=\frac{\dot{L}_{a+x}}{L_{a+x}}-\alpha\dot{x};$$

and supposing α to be constant, that is to say, supposing the circumstance of speciality to cause a constant instantaneity of effect, we shall have

$$c\frac{M_x}{M_0}=c\frac{L_{a+x}}{L_a}-\alpha x;$$

and therefore if $\aleph$ represent the number whose Napierian logarithm is 1, that is, if $\aleph$ be put for $c-1$, then, because $M_0=L_a$, we have

$$M_{a+x}=L_{a+x}\times\aleph^{-\alpha x}.$$

If α were very small, which would be the case if the speciality depended on the risk of death by shipwreck, which might be but 1 or 2 per cent. per annum to a sailor either always at sea or occasionally at sea, then the above expression would give M_{a+x}, the number out of L_a living who would be living in x years

$$=L_{a+x}\times\overline{1-\alpha.x}.$$

Art. 23. No. 1. This Table of Napierian Logarithms of persons living at every Age according to MILNE'S Carlisle Mortality will be found useful; I have therefore given it.

x Age.		x Age.		x Age.	
0	9·21034	34	8·59739	68	7·88156
1	9·04322	35	8·58709	69	7·83400
2	8·95854	36	8·57678	70	7·78344
3	8·89206	37	8·56617	71	7·73061
4	8·85338	38	8·55526	72	7·66996
5	8·82434	39	8·54403	73	7·59940
6	8·80627	40	8·53208	74	7·51806
7	8·79392	41	8·51899	75	7·42357
8	8·78508	42	8·50512	76	7·32317
9	8·77848	43	8·49064	77	7·21450
10	8·77338	44	8·47595	78	7·10085
11	8·76889	45	8·46105	79	6·98564
12	8·76405	46	8·44613	80	6·85961
13	8·75904	47	8·43120	81	6·72982
14	8·75385	48	8·41649	82	6·62274
15	8·74830	49	8·40479	83	6·43455
16	8·74219	50	8·38868	84	6·27019
17	8·73536	51	8·37526	85	6·09807
18	8·72847	52	8·36077	86	5·90536
19	8·72144	53	8·34555	87	5·69036
20	8·71440	54	8·32918	88	5·44674
21	8·70738	55	8·31214	89	5·19850
22	8·70025	56	8·29405	90	4·95583
23	8·69333	57	8·27487	91	4·65396
24	8·68626	58	8·25375	92	4·31749
25	8·67914	59	8·22924	93	3·98809
26	8·67180	60	8·20056	94	3·68888
27	8·66441	61	8·16650	95	3·40129
28	8·65661	62	8·13006	96	3·13549
29	8·64788	63	8·09193	97	2·89037
30	8·63799	64	8·05293	98	2·63906
31	8·62784	65	8·01235	99	2·39790
32	8·61758	66	7·97039	100	2·19722
33	8·60749	67	7·92696		

Art. 23. No. 2. Power-collecting Table referred to at p. 540.

x and x^2.	$8x$.	x^3.	$8x^2$.	x^3.	$8x^3$.	x^4.	$8x^4$.	x^5.	$8x^5$.
1	1	1	1	1	1	1	1	1	1
2	3	4	5	8	9	16	17	32	33
3	6	9	14	27	36	81	98	243	276
4	10	16	30	64	100	256	354	1024	1300
5	15	25	55	125	225	625	979	3125	4425
6	21	36	91	216	441	1296	2275	7776	12201
7	28	49	140	343	784	2401	4676	16807	29008
8	36	64	204	512	1296	4096	8772	32768	61776
9	45	81	285	729	2025	6561	15333	59049	12083 (1)
10	55	100	385	1000	3025	10000	25333	10000 (1)	22085 (1)
11	66	121	506	1331	4356	14641	39974	16105 (1)	38188 (1)
12	78	144	650	1728	6084	20736	60710	24883 (1)	63071 (1)
13	91	169	819	2197	8281	28561	89271	37129 (1)	10020 (2)
14	105	196	1015	2744	11025	38416	12769 (1)	53782 (1)	15398 (2)
15	120	225	1240	3375	14400	50625	17831 (1)	75938 (1)	22992 (2)
16	136	256	1496	4096	18496	65536	24385 (1)	10476 (2)	33468 (2)
17	153	289	1785	4913	23409	83521	32737 (1)	14199 (2)	47667 (2)
18	171	324	2109	5832	29241	10498 (1)	43235 (1)	18896 (2)	66568 (2)
19	190	361	2470	6859	36100	13032 (1)	56267 (1)	27461 (2)	91324 (2)
20	210	400	2870	8000	44100	16000 (1)	72267 (1)	32000 (2)	12333 (3)
21	231	441	3311	9261	53361	19448 (1)	91715 (1)	40841 (2)	16417 (3)
22	253	484	3795	10648	64009	23426 (1)	11514 (2)	51536 (2)	21571 (3)
23	276	529	4324	12167	76176	27984 (1)	14312 (2)	64363 (2)	28007 (3)
24	300	576	4900	13824	90000	33178 (1)	17630 (2)	79627 (2)	35969 (3)
25	325	625	5525	15625	10563 (1)	39063 (1)	21536 (2)	97658 (2)	45735 (3)
26	351	676	6201	17576	12329 (1)	45698 (1)	26106 (2)	11881 (3)	57616 (3)
27	378	729	6930	19683	14288 (1)	53144 (1)	31421 (2)	14349 (3)	71965 (3)
28	406	784	7714	21952	16484 (1)	61466 (1)	37567 (2)	17210 (3)	89175 (3)
29	435	841	8555	24389	18923 (1)	70728 (1)	44640 (2)	20511 (3)	10969 (4)
30	465	900	9455	27000	21623 (1)	81000 (1)	52740 (2)	24300 (3)	13399 (4)

TABLE (continued).

x and x^0.	δx.	x^2.	δx^2.	x^3.	δx^3.	x^4.	δx^4.	x^5.	δx^5.
31	496	961	10416	29791	24602 (1)	92352 (1)	61975 (2)	28629 (3)	16262 (4)
32	528	1024	11440	32768	27878 (1)	10486 (2)	72461 (2)	33555 (3)	19617 (4)
33	561	1089	12529	35937	31472 (1)	11859 (2)	84320 (2)	39135 (3)	23530 (4)
34	595	1156	13685	39304	35403 (1)	13363 (2)	97684 (2)	45435 (3)	28074 (4)
35	630	1225	14910	42875	39690 (1)	15006 (2)	11269 (3)	52522 (3)	33326 (4)
36	666	1296	16206	46656	44356 (1)	16796 (2)	12949 (3)	60466 (3)	39373 (4)
37	703	1369	17575	50653	49421 (1)	18742 (2)	14823 (3)	69344 (3)	46307 (4)
38	740	1444	19109	54872	54908 (1)	20851 (2)	16909 (3)	79235 (3)	54231 (4)
39	780	1521	20540	59319	60840 (1)	23134 (2)	19223 (3)	90224 (3)	63253 (4)
40	820	1600	21140	64000	67240 (1)	25600 (2)	21782 (3)	10240 (4)	73493 (4)
41	861	1681	23821	68921	74132 (1)	28258 (2)	24607 (3)	11586 (4)	85079 (4)
42	903	1764	25585	74088	81541 (1)	31117 (2)	27719 (3)	13069 (4)	98144 (4)
43	946	1849	27434	79507	89492 (1)	34188 (2)	31138 (3)	14701 (4)	11285 (5)
44	990	1936	29370	85184	98010 (1)	37481 (2)	34886 (3)	16492 (4)	11934 (5)
45	1035	2025	31395	91125	10712 (2)	41006 (2)	38986 (3)	18452 (4)	14780 (5)
46	1081	2116	33511	97336	11686 (2)	44775 (2)	43464 (3)	20596 (4)	16840 (5)
47	1128	2209	35720	103823	12724 (2)	48797 (2)	48345 (3)	22935 (4)	19133 (5)
48	1176	2304	38024	110592	13830 (2)	53084 (2)	53652 (3)	25480 (4)	21681 (5)
49	1225	2401	40425	117649	15006 (2)	57648 (2)	59467 (3)	28248 (4)	24506 (5)
50	1275	2500	42925	125000	16526 (2)	62500 (2)	65667 (3)	31250 (4)	27631 (5)
51	1326	2601	45526	132651	17583 (2)	67652 (2)	72432 (3)	34502 (4)	31081 (5)
52	1378	2704	48230	140608	18989 (2)	73116 (2)	79743 (3)	38021 (4)	34883 (5)
53	1431	2809	51030	148877	20478 (2)	78905 (2)	87654 (3)	41820 (4)	39065 (5)
54	1485	2916	53955	157464	22052 (2)	85031 (2)	96137 (3)	45917 (4)	43656 (5)
55	1540	3025	56980	166375	23716 (2)	91506 (2)	10529 (4)	50328 (4)	48690 (5)
56	1596	3136	60116	175616	25472 (2)	98345 (2)	11512 (4)	55073 (4)	54197 (5)
57	1653	3249	63365	185193	27324 (2)	10556 (3)	12568 (4)	60179 (4)	62214 (5)
58	1711	3364	66729	195112	29275 (2)	11316 (3)	13700 (4)	65626 (4)	66778 (5)
59	1770	3481	70210	205379	31329 (2)	12117 (3)	14911 (4)	71490 (4)	73927 (5)
60	1830	3600	73810	216000	33489 (2)	12960 (3)	16207 (4)	77600 (4)	81703 (5)

Art. 24. In applying the formula $\subset L_x$, the Napierian logarithm of the number of persons living at the age x,

$$=C\mathfrak{C}^x+k\mathfrak{s}^x+k\mathfrak{s}^x-\subset(\sigma^x.\overline{x-h}.\underset{\sim}{q_0})+\mu\nu^x,$$

to the useful purposes to which it is serviceable, it stands in need of reduction into another, for which might be represented an expression like

$$A+Bx+Cx^2+Dx^3, \text{ \&c.};$$

but such a form might not have the converging property requisite for the purpose required if x were a large number: but if a represent a certain age, and the age to which the law were meant to apply were represented by $a+x$ instead of x, then the formula would stand

$$\subset L_{a+x}=C\mathfrak{C}^{a+x}+k\mathfrak{s}^{a+x}+k\mathfrak{s}^{a+x}-\subset(\sigma^{a+x}\times\overline{a+x-h}\times\subset q_0)+\mu\nu^{a+x},$$

and would admit of the terms of each member to stand in the above form of x^0, x, x^2, with given converging coefficients, and each of the members of the last equation would be convertible into a converging series for most or all the values which x would be required to have; and therefore the right side of the last equation, which would consist of all the developments together, would form a series, as above, which would be convenient, and might be expressed by

$$A_0+A_1x+A_2x^2+A_3x^3+ \text{ \&c.};$$

and the Napierian logarithm of the number of persons living at the age a would be represented by the first term A_0 of that series; and the Napierian logarithm of $\frac{L_{a+x}}{L_a}$, that is, of the chance of a person now of the age a being living in x years, would be

$$=A_1x+A_2x^2+A_3x^3+\text{\&c.},$$

because $\mathfrak{C}^{a+x}$ is

$$=\mathfrak{C}^a\times\mathfrak{C}^x=\mathfrak{C}^a\times(1+\underset{\sim}{\mathfrak{c}}.x+\frac{1}{2}\underset{\sim}{\overline{\mathfrak{c}}^2}x^2+\frac{1}{2.3}\underset{\sim}{\overline{\mathfrak{c}}^3}x^3+\frac{1}{2.3.4}\underset{\sim}{\overline{\mathfrak{c}}^4}x^4+\text{\&c.}),$$

because $\underset{\sim}{\mathfrak{c}}$, the Napierian $\mathfrak{C}$, is very small, and a very few terms of the above series will be sufficient. Similarly, $\mathfrak{s}^x$ will be

$$\mathfrak{s}^a\times(1+\underset{\sim}{s}.x+\frac{1}{2}\overline{s}^2x^2+\frac{1}{2.3}\overline{s}^3x^3, \text{ \&c.});$$

and if x is equal to or greater than one, would be of perfectly insignificant value, and omissible.

I may observe that

$$\mathfrak{s}^{a+x}=\mathfrak{s}^a\times\left(1+\underset{\sim}{s}.x+\frac{1}{2}\underset{\sim}{\overline{s}^2}x^2+\frac{1}{2.3}\underset{\sim}{\overline{s}^3}x^3, \text{ \&c.}\right)$$

is a very convergent series; and if a be equal to 20 in the Carlisle mortality, or above that value, would be of insignificant value, and the term which depends on it omissible; in like manner the term $\mu.\nu^{a+x}$, depending on ν^{a+x}, which is

$$=\nu^a\times\left(1+\underset{\sim}{\nu}.x+\frac{1}{2}\overline{\nu}^2x+\frac{1}{2.3}\overline{\nu}^3x^3, \text{ \&c.}\right),$$

will be converging, and will be insignificant whilst $a+x$ is less than 80, but will be significant when $a+x$ is greater than 80.

It remains now to consider the important term $\subset\{e^{a+x}\times\overline{a+x-h}\subset q_0\}$, which for abbreviation I call M_{a+x}; and putting $h-a=w$, we get

$$M_{a+x}=\subset\left\{e^{a}.\underset{\subset}{q_0}\times\overline{a+x-h}\times\left(1+\underset{\subset}{e}x+\frac{1}{2}\overrightarrow{\underset{\subset}{e}}^{2}x^{2}+\frac{1}{2.3}\overrightarrow{\underset{\subset}{e}}^{3}x^{3}+\&c.\right)\right\}$$

$$=\subset\left\{e^{a}.\underset{\subset}{q_0}\times\left(-w-w.\underset{\subset}{e}x-\frac{1}{2}w.\overrightarrow{\underset{\subset}{e}}^{2}.x^{2}-\frac{1}{2.3}w.\overrightarrow{\underset{\subset}{e}}^{3}.x^{3}-\frac{1}{2.3.4}\overrightarrow{\underset{\subset}{e}}^{4}x^{4},\ \&c.\right.\right.$$

$$\left.\left.+x+\underset{\subset}{e}x^{2}+\frac{1}{2}\overrightarrow{\underset{\subset}{e}}^{2}x^{3}+\frac{1}{2.3}\overrightarrow{\underset{\subset}{e}}^{3}x^{4}+\&c.\right)\right\};$$

consequently, if V_a be put $=\subset-e^{a}\underset{\subset}{q_0}w$, that is, equal to the anti-Napierian logarithm of $-e^{a}.\underset{\subset}{q_0}w$, and we put

$$A_1=e^{a}\underset{\subset}{q_0}\times\overline{1-\underset{\subset}{we}};\qquad A_2=e^{a}.\underset{\subset}{q_0}\times\overline{1-\frac{1}{2}\underset{\subset}{we}}\times\underset{\subset}{e};$$

$$A_3=\frac{1}{2}e^{a}.\underset{\subset}{q_0}\times\left(1-\frac{1}{3}\underset{\subset}{we}\right).\overrightarrow{\underset{\subset}{e}}^{2};\qquad A_4=\frac{1}{2.3}e^{a}.\underset{\subset}{q_0}.\overline{1-\frac{1}{4}\underset{\subset}{we}}.\overrightarrow{\underset{\subset}{e}}^{3};\ \&c.,$$

we shall have

$$M_{a+x}=\subset\{-e^{a}.\underset{\subset}{q_0}w+A_1x+A_2x^{2}+A_3x^{3}+A_4x^{4}+\&c.\}$$

$$=V_a\times\subset\{A_1x+A_2x^{2}+A_3x^{3}+\&c.\};$$

or making use of the theorem for finding the anti-Napierian logarithm above given, and taking $A_0=0$, we have

$$M_{a+x}=V_a\times(1+B_1x+B_2x^{2}+B_3x^{3}+\&c.);$$

and consequently we have, for instance, in the Carlisle mortality, the Napierian logarithm of

$$L_{a+x}={}^{1}A_a+{}^{1}A_1x+{}^{1}A_2x^{2}+{}^{1}A_3x^{3},\ \&c.,$$

if

$${}^{1}A_a=C\mathfrak{C}^{a}+k\mathfrak{f}^{a}+k\mathfrak{s}^{a}-V_a+\mu\mathfrak{r}^{a};$$

$${}^{1}A_1=C\mathfrak{C}^{a}.\underset{\subset}{\mathfrak{C}}+k\mathfrak{f}^{a}.\underset{\subset}{\mathfrak{f}}+k\mathfrak{s}^{a}.\underset{\subset}{\mathfrak{s}}-V_aB_1+\mu\mathfrak{r}^{a}.\underset{\subset}{\mathfrak{r}};$$

$${}^{1}A_2=\frac{1}{2}C\mathfrak{C}^{a}.\overrightarrow{\underset{\subset}{\mathfrak{C}}}^{2}+\frac{1}{2}k\mathfrak{f}^{a}.\overrightarrow{\underset{\subset}{\mathfrak{f}}}^{2}+\frac{1}{2}k\mathfrak{s}^{a}.\overrightarrow{\underset{\subset}{\mathfrak{s}}}^{2}-V_aB_2+\frac{1}{2}\mu\mathfrak{r}^{a}\overrightarrow{\underset{\subset}{\mathfrak{r}}}^{2};$$

$${}^{1}A_3=\frac{1}{2.3}C\mathfrak{C}^{a}.\overrightarrow{\underset{\subset}{\mathfrak{C}}}^{3}+\frac{1}{2.3}k\mathfrak{f}^{a}.\overrightarrow{\underset{\subset}{\mathfrak{f}}}^{3}+\frac{1}{2.3}k\mathfrak{s}^{a}.\overrightarrow{\underset{\subset}{\mathfrak{s}}}^{3}-V_a+\mu\mathfrak{r}^{a}\overrightarrow{\underset{\subset}{\mathfrak{r}}}^{3};$$

$${}^{1}A_4=\&c.\qquad A_5=\&c.\qquad \&c.$$

But here it is to be observed, that if a be equal to one year or more, the term affected with $\mathfrak{f}$ is insignificant; and whilst $a+x$ is less than 80, in the Carlisle mortality, for instance, the term affected with $\mathfrak{r}$ is insignificant; and when a is as great as 20, the term affected with $\mathfrak{s}$ is insignificant; which observation shows that though the values of ${}^{1}A_a$, ${}^{1}A_1$, ${}^{1}A_2$ appear intricate, only some of the terms for any value of a come into play in them, and that the values have much more simplicity than is apparent, without taking that circumstance into consideration. And, for instance, when $a+x$ is between 20 and 80, or even between about 10 and 80, a being no tless than 10, in the Carlisle Table,

there will only be in the values of ${}^1A_0$, ${}^1A_1$, ${}^1A_2$, &c. the terms affected with $\mathfrak{C}$ and e; and when a is as great as 80, there will be only the terms affected with $\mathfrak{C}$, e, and ν, as those affected with μ and ε will be of total insignificance. And meanwhile it will be easy to have a Table calculated for every age from birth, even including the first months after birth, which will give all the values of ${}^1A_0$, ${}^1A_1$, ${}^1A_2$, ${}^1A_3$ by bare inspection. And to illustrate this observation, I will give the value of $-\mathfrak{c}\mathrm{M}_{a+x}$, according to the Carlisle mortality, for the respective values of a, 30, 40, 50, as follows:—

$$-\mathfrak{c}\mathrm{M}_{30+x}=2\cdot17621-\begin{cases}\cdot014087x\\ +\cdot0003357x^2\\ +\cdot000\ 0033924x^3\\ +\textcircled{8}\,95944x^3\end{cases}\qquad -\mathfrak{c}\mathrm{M}_{40+x}=\overline{1}\cdot99926-\begin{cases}\cdot021562x\\ +\cdot00041513x^2\\ +\textcircled{5}\,2839x^3\\ \textcircled{7}\,12245x^4\end{cases}$$

&c. &c.

$$-\mathfrak{c}\mathrm{M}_{50+x}=1\cdot73924-\begin{cases}\cdot030745x\\ +\cdot00050696x^2\\ +\textcircled{5}\,26777x^3\\ +\textcircled{7}\,1395x^4\end{cases}$$

&c.

Art. 25. But from what has been stated, the result of the analysis of any problem may give the present value due the xth year's payment by an expression of the form of

$$A_0x^p+A_1x^{p+1}+A_2x^{p+2}+A_3x^{p+3},\ \&c.,$$

where A_0, A_1, A_2 will be determined by the methods explained by the conditions of the problem. If we wish to have the present value of the sum due to every yearly payment between the values m and n of x, we have, according to our notation, to find the sum of the series $A_0\overset{x}{\underset{1}{\overline{\underset{n}{\underset{m}{|}}}}}x^p+A_1\cdot\overset{x}{\underset{1}{\overline{\underset{n}{\underset{m}{|}}}}}x^{p+1}+A_2\cdot\overset{x}{\underset{1}{\overline{\underset{n}{\underset{m}{|}}}}}x^{p+2}+A_3\cdot\overset{x}{\underset{1}{\overline{\underset{n}{\underset{m}{|}}}}}x^{p+3}$, where $\overset{x}{\underset{1}{\overline{\underset{n}{\underset{m}{|}}}}}$ signifies, when applied to the term x for instance, the sum $m^p+\overline{m+1}^p+\overline{m+2}^p$ up to n^p inclusively of m and n; and I therefore give a Table (see p. 543,) which I call a Collecting Table of Powers, to facilitate the summation.

This Table for the collection of powers given does not go to the extent which may be required in complicated cases, though it may be sufficient, for instance, for finding the value of an annuity on several joint lives, or in cases not very much more intricate; but when the powers required of x are much higher than this Table comprehends, it will not be difficult, but, on the contrary, very easy to find the sum of these high powers of the series 1, 2, 3, &c. from the well-known theorem that $\frac{1}{2}\overset{x}{\underset{1}{\overline{|}}}x^p=1^p+2^p+3^p+$ &c. x^p by

the well-known theorem that the sum is equal to

$$\frac{x^{p+1}}{p+1}+\frac{1}{2}x^p+\frac{1}{2}\cdot\frac{p}{2.3}x^{p-1}-\frac{1}{6}\cdot\frac{p.\overline{p-1}.\overline{p-2}}{2.3.4.5}\cdot x^{p-3}+\frac{1}{6}\cdot\frac{p.\overline{p-1}.\overline{p-2}.\overline{p-3}.\overline{p-4}}{2.3.4.5.6.7}x^{p-5},\ \&c.,$$

as very few of the terms will be sufficient, perhaps only the first term, or the two first terms, or the three first terms, in which several cases the theorem will stand $1^p+2^p+3^p+\&c.\ldots.x^p$, respectively equal to

$$\frac{x^{p+1}}{p+1};\ x^p\times\left(\frac{x}{p+1}+\frac{1}{2}\right);\ x^{p-1}\times\left(\frac{x^2}{p+1}+\frac{x}{2}+\frac{1}{2}\cdot\frac{p}{2.3}\right).$$

And to explain why high powers may be required, I may first mention that the yearly payments which the question may require to be made may not all require to be of the same value, but of values depending on the time in which they are to be made, or in other words, some function of x: if, for example, the payments were uniformly to increase as x increased from one to two, to three, &c., and if the corresponding payments were to be one pound, two pounds, three pounds, &c., then each term of the series $A+Bx+Cx^2$, &c. which would express the function of the value required for every value of x from 1 to x, to be summed, would have to be multiplied by x, and so be changed by that multiplier into $Ax+Bx^2+Cx^3+$ &c.; and the conditions relative to the yearly payments may be such as to introduce multiplications of much higher powers of x. And in complicated questions, even when the complication is not great, as for instance in the case which may not unlikely occur, to find the present value of the reversion of an annuity to be granted to the joint lives A, B after the longest life of five other given lives, C, D, E, F, G; were this to be attempted by the method given by authors by Tables of joint lives, even though we might have Tables of the value of any number of joint lives for every age (which we have not, and in consequence of not having such Tables very insufficient interpolations are to be had recourse to), the labour would be very great.

We should have, simple as the question appears by its enunciation, to search for the value of thirty-two annuities, of which one would be on two joint lives, six on three joint lives, ten on four joint lives, ten on five joint lives, five on six joint lives, and one on seven joint lives; but if the question were to be solved by the method explained, we should only have to find, by inspection of the Collecting Table, if it extended to those high powers, severally the sums of the several progressions

$$1^5+2^5+3^5\ \&c.;\ 1^6+2^6+3^6\ \&c.;\ 1^7+2^7+3^7\ \&c.\ \&c.$$

severally multiplied by coefficients, say A_1, A_2, A_3, &c., found without much trouble from the conditions of the question, such values, A_1, A_2, A_3, &c., being of series converging so swiftly that a few terms which the multiplication produced would be sufficient.

And in the operation we should find the notation I have proposed useful, as for instance to write $\overset{(5)}{\cdot}234$, $789\overset{(6)}{\cdot}$, for ·00000234, 789000000, as numbers of such description will come into operation.

As the application of the theorem requires an easy mode of finding an analytical expression for the anti-logarithm of an analytical expression, and of giving the analytical

expression of a logarithm of an analytical expression, and as there are or may be in some applications of the theorems many entanglements of logarithms in analytical expressions, and anti-analytical expressions of logarithms, I will now give different methods of operation for those purposes. I will commence by stating that if the analytical logarithmic expression is that of the common logarithm, then this must be converted to that of the Napierian logarithm by the multiplication of the coefficients by the Napierian logarithm of 10, that is, by the reciprocal of ·4342944, &c.; but a very few terms of this decimal will be required. Let the Napierian logarithm be represented by

$$A_0+A_1x+A_2x^2+A_3x^3+\&c.,$$

and by the notation I have adopted because $\backsim A_0$ will express the anti-Napierian logarithm of A_0, that is to say, the natural number of which A_0 is the anti-Napierian logarithm, and the expression, namely the anti-Napierian logarithm of

$$(A_0+A_1x+A_2x^2+\&c.),$$

will stand

$$\backsim A_0\times\backsim(A_1x+A_2x^2+A_3x^3+\&c.);$$

but

$$\backsim(A_1x+A_2x^2+A_3x^3+\&c.),$$

or the anti-Napierian logarithm of

$$(A_1x+A_2x^2+\&c.),$$

is

$$=1+A_1x+A_2x^2+\&c.+\frac{1}{2}\overline{A_1x+A_2x^2+\&c.}^2+\frac{1}{2.3}\overline{A_1x+\&c.}^3\ \&c.$$

$$\left.\begin{aligned}=1+A_1x+A_2x^2+A_3x^3+\ & A_4x^4\\ \tfrac{1}{2}A_1^2x^2+A_1A_2x^3+\ & A_1A_3x^4\\ +\ & \tfrac{1}{2}A_2^2x^4\\ +\tfrac{1}{2.3}A_1^3x^3+\tfrac{1}{2}A_1^2A_2x^4\\ +\tfrac{1}{2.3.4}A_1^4x^4\end{aligned}\right\}\ \&c.;$$

and consequently if the anti-Napierian logarithm of $A_0+A_1x+A_2x^2+\&c.$ be expressed by $B_0+B_1x+B_2x^2+B_3x^3$, &c., we shall have the following equations between B_0, B_1, B_2, &c., and A_0, A_1, A_2, &c.:

$$B_0=\backsim A_0;\ B_1=B_0A_1;\ B_2=B_0\left(A_2+\tfrac{1}{2}A_1^2\right);$$

$$B_3=B_0\times\left(A_3+A_1A_2+\frac{1}{2.3}A_1^3\right);\ B_4=B_0\left(A_4+A_1A_3+\frac{1}{2}A_2^2+\frac{1}{2}A_1^2A_2+\frac{1}{2.3.4}A_1^4\right),\ \&c.;$$

so that as B_0, B_1, B_2, &c. are found by these equations from A_1, A_2, A_3, &c., and on the contrary, A_0, A_1, A_2, &c. are found from B_0, B_1, B_2, &c. from the equations

$$A_0=\backsim B_0,\ A_1=\frac{1}{B_0}\times B_1,\ A_2=\frac{1}{B_0}\times B_1-\frac{1}{2}A_1^2;$$

$$A_3=\frac{1}{B_0}\times B_3-A_1A_2-\frac{1}{2.3}A_1^3;\ A_4=\frac{1}{B_0}\times B_4-A_1A_3-\frac{1}{2}A_2^2-\frac{1}{2}A_1^2A_2-\frac{1}{2.3.4}A_1^4,\ \&c.;$$

and where, as will generally be the case, A_1, A_2, A_3, &c., or B_1, B_2, B_3, &c. are a series of terms which are very convergent, a very few terms of the said coefficients will be found necessary, which is a very advantageous circumstance for our purpose.

Another mode of finding the anti-Napierian logarithm of the expression

$$A_1x+A_2x^2+A_3x^3+\&c.$$

is to consider it

$$=c+A_1xx+A_2x^2x+A_3x^3x,\ \&c.=\left(1+A_1x+\frac{1}{2}A_1^2x^2+\frac{1}{2.3}A_1^3x^3+\&c.\right)$$
$$\times\overline{1+A_2x^2+\frac{1}{2}A_2^2x^4+\&c.}\times\overline{1+A_3x^3+\frac{1}{2}A_3^2x^6+\&c.}\times\&c.;$$

this being multiplied out at length would give what might be considered an easier way than the last for effecting our purpose, if many terms were thought requisite. But another mode of finding the anti-Napierian logarithm of $A_1x+A_2x^2+\&c.$, which would be the anti-Napierian logarithm of $A_0+A_1x+A_2x^2+\&c.$ if $A_0=0$, and therefore $B_0=1$, would be by result of the equations of $A_1x+A_2x^2+\&c.=$ the Napierian logarithm of $1+B_1x+B_2x^2+\&c.$, by putting the equation in fluxions, which would give

$$A_1+2A_2x+3A_3x^2+\&c.=\frac{B_1+2B_2x+3B_3x^2+\&c.}{1+B_1x+B_2x^2+\&c.};$$

this will give the following equation for finding B_1, B_2, &c.,

$$\left.\begin{array}{rrrr} A_1+2A_2x & +3A_3x^2 & +4A_4x^3 & +\&c. \\ +A_1B_1x & +2A_2B_1x^2 & +3A_3B_1x^3 & +\&c. \\ & +A_1B_2x^2 & +2A_2B_2x^3 & +\&c. \\ & & +A_1B_3x^3 & +\&c. \\ -B_1-2B_2x & -3B_3x^2 & -4B_4x^3 & +\&c. \end{array}\right\}=0;$$

and consequently

$$B_1=A_1;\ B_2=A_2+\tfrac{1}{2}A_1B_1;\ B_3=A_3+\tfrac{2}{3}A_2B_1+\tfrac{1}{3}A_1B_2;$$
$$B_4=A_4+\tfrac{3}{4}A_3B_1+\tfrac{3}{4}A_2B_2+\tfrac{1}{4}A_1B_3;\ B_5=A_5+\tfrac{4}{5}A_4B_1+\tfrac{3}{5}A_3B_2+\tfrac{2}{5}A_2B_3+\tfrac{1}{5}A_1B_4,\ \&c.,$$

where B_1, B_2, B_3, &c. are found respectively one after the other by a very simple law; and this method may also, if many terms be thought requisite, be preferable to the first method, and the equation reversed would give

$$A_1=B_1,\ A_2=B_2-\tfrac{1}{2}A_1B_1,\ A_3=B_3-\tfrac{2}{3}A_2B_1-\tfrac{1}{3}A_1B_2,\ A_4=B_4-\tfrac{3}{4}A_3B_1-\tfrac{3}{4}A_2B_2-\tfrac{1}{4}A_1B_3,\ \&c.$$

N.B. The calculations which these sheets required were laborious, and necessitated the very frequent, in fact the continual use of a table of Logarithms, and the searching for the Anti-logarithms; and I feel pleasure in stating that I found the last edition of Anti-logarithms, published by Mr. H. E. Filipowsky, a very serviceable assistance, and I have no doubt that calculators in the same path must be glad to possess these Tables.

Art. 26. The following Table, to show the result of the formula $\lambda L_x = C\mathfrak{C}^x + k\rho^x + ks^x - M_x + \mu\nu^x$, is calculated for Carlisle Mortality.

Age x =	1	2	3	4	5	6	7	8	9	10
λL_x formula.	3·92747	3·89062	3·86653	3·85053	3·83975	3·83148	3·82612	3·82162	3·81774	3·81323
λL_x Milne ...	3·92742	3·89092	3·86177	3·84497	3·83231	3·82461	3·81915	3·81531	3·81245	3·81023
L_x formula...	8461	7773	7274	7088	6914	6784	6701	6631	6573	6504
L_x Milne ...	8461	7779	7274	6998	6797	6676	6594	6536	6493	6460

Age x =	11	12	13	14	15	16	17	18	19	20
λL_x formula.	3·81113	3·80809	3·80504	3·80237	3·80057	3·79694	3·79354	3·79070	3·78792	3·78453
λL_x Milne ...	3·80823	3·80618	3·80400	3·80175	3·79934	3·79664	3·79372	3·79071	3·78767	3·78462
L_x formula...	6473	6399	6383	6334	6299	6260	6219	6176	6136	6088
L_x Milne ...	6431	6400	6368	6335	6300	6261	6219	6176	6133	6090

Age x =	21	22	23	24	25	26	27	28	29	30
λL_x formula.	3·78145	3·77822	3·77501	3·77167	3·76828	3·76483	3·76125	3·75759	3·75397	3·75019
λL_x Milne ...	3·78154	3·77851	3·77546	3·77240	3·76930	3·76612	3·76290	3·75952	3·75557	3·75143
L_x formula...	6046	6001	5957	5911	5863	5819	5771	5722	5675	5626
L_x Milne ...	6047	6005	5963	5921	5879	5836	5793	5748	5698	5642

Age x =	31	32	33	34	35	36	37	38	39	40
λL_x formula.	3·74623	3·74222	3·73870	3·73398	3·72957	3·72548	3·71984	3·71501	3·71112	3·70380
λL_x Milne ...	3·74702	3·74257	3·73815	3·73376	3·72933	3·72485	3·72024	3·71550	3·71063	3·70544
L_x formula...	5575	5524	5442	5420	5353	5315	5246	5188	5142	5056
L_x Milne ...	5585	5528	5472	5417	5362	5307	5251	5194	5136	5075

Age x =	41	42	43	44	45	46	47	48	49	50
λL_x formula.	3·69939	3·69491	3·68929	3·68313	3·67643	3·67106	3·66569	3·65771	3·65051	3·64309
λL_x Milne ...	3·69975	3·69373	3·68744	3·68106	3·67459	3·66811	3·66162	3·65523	3·64914	3·64316
L_x formula...	5009	4954	4890	4821	4747	4689	4631	4547	4472	4396
L_x Milne ...	5009	4940	4869	4798	4727	4657	4588	4521	4458	4397

Age x =	51	52	53	54	55	56	57	58	59	60
λL_x formula.	3·63513	3·62707	3·61853	3·60917	3·59321	3·59022	3·57953	3·56774	3·55672	3·54427
λL_x Milne ...	3·63729	3·63104	3·62439	3·61731	3·60991	3·60206	3·59373	3·58454	3·57392	3·56146
L_x formula...	4316	4237	4155	4066	3919	3892	3798	3696	3603	3502
L_x Milne ...	4338	4276	4211	4143	4073	4000	3924	3842	3749	3643

Age x =	61	62	63	64	65	66	67	68	69	70
λL_x formula.	3·53269	3·51806	3·50242	3·48635	3·47298	3·45128	3·43176	3·41090	3·38863	3·36540
λL_x Milne ...	3·54667	3·53084	3·51428	3·49734	3·47972	3·46150	3·44264	3·42292	3·40226	3·38039
L_x formula...	3410	3297	3180	3045	2972	2833	2702	2576	2447	2320
L_x Milne ...	3521	3395	3268	3143	3018	2894	2771	2648	2525	2401

Age x =	71	72	73	74	75	76	77	78	79	80
λL_x formula.	3·33864	3·32065	3·27909	3·24834	3·21177	3·20301	3·12579	3·08460	3·06270	2·97895
λL_x Milne ...	3·35736	3·33102	3·30038	3·26505	3·22401	3·18041	3·13322	3·08386	3·03383	2·97909
L_x formula...	2181	2092	1901	1771	1628	1596	1336	1215	1155	
L_x Milne ...	2277	2143	1997	1841	1675	1515	1359	1213	1081	

Age x =	81	82	83	84	85	86	87	88	89	90
λL_x formula.	2·93253	2·87351	2·80613	2·73383	2·64479	2·56753	2·47561	2·37526	2·26730	2·15239
λL_x Milne ...	2·92273	2·86034	2·79449	2·72326	2·64838	2·56476	2·47129	2·36549	2·25768	2·15229

Age x =	91	92	93	94	95	96	97	98	99	100
λL_x formula.	2·03631	1·89820	1·76150	1·59630	1·47969	1·33147	1·16301	1·09266	1·00205	·95396
λL_x Milne ...	2·02119	1·87508	1·73239	1·60206	1·47712	1·36173	1·25527	1·14613	1·04139	·95424

x=	0 or Birth.	1 Month.	2 Months.	3 Months.	6 Months.	9 Months.	12 Months.	Average deaths per month.
λL_x formula.	400000	3·97622	3·96755	3·96182	3·94753	3·93494		In first month 0
λL_x Milne ...	400000	3·97622	3·97621	3·96501	3·95279	3·94027		Between 1 and 2 .....
L_x formula...	10000	9467	9279	9158	9158	8609		Between 2 and 3 .....
L_x Milne ...	10000	9467	9313	9226	9226	8715		Between 3 and 6 .....
								Between 6 and 9 .....

Art. 27. In a paper I had the honour of presenting to the International Statistical Congress, in consequence of the flattering invitation I had received to offer my assistance towards the scientific objects in view, I presented a sketch of some investigations I had been making since the publication of my paper of 1820 and that of 1825, relative to the subject of mortality, and on some investigations I had since made on the law of sickness and invalidity subjects, which I was prevented, in consequence of the absence of sufficient health, to finish for presentation to the Royal Society, as a supplement to those two papers which were honoured by a place in the Society's Transactions. But in my paper I presented to the Congress, I did not show how I obtained the formulæ of the one uniform law of mortality I presented, from birth to extreme old age, reserving that communication for the honour I intended myself to present it to this Society, should my health sufficiently recover to render it possible to me, though I presented the formula which follows, $\lambda L_x = \text{constant} + kc^x - fg^x - nq^x - P_x$, in which λL_x represents the common logarithm of the number of persons living at the age x, $P_x = \theta . w^{e^{x-u}}$, $k, e, f, g, n, q, \theta, w, \varpi, u$ constant quantities from birth to the extremity of old age, to be found from statistical tables given of the actual persons living at every age in places to which the formula is meant to apply; and these constants I found for Carlisle mortality, Northampton mortality, De Parcieux mortality, and Sweden, male and female mixed mortality, from the examination of published statistical tables; and I computed for these four stated rates of mortality from this formula by means of these constants the number of living at every age, and arranged the results opposite the statements, which show a remarkably satisfactory agreement between the formula and the statement which it is intended to represent throughout, from commencement to every year of age. And in the Carlisle Table, where there are data for comparison, for the first months after birth, where there appears great irregularity in the deaths, even the close approximate agreement seems very interesting; and the value of the constants there given are in the formula resulting when θ is taken $=1$, which, though it may not be its exact value, is very nearly so.

	u.	λk.	λc.	λf.	λg.	λn.	λq.	λw.	$\lambda \varpi$.	Constant.
Carlisle	90·37	$\bar{1}$·2310	$\bar{1}$·76774	$\bar{1}$·59375	4·34652	$\bar{2}$·75526	0126	$\bar{1}$·98952	$\bar{1}$·50837	3·8631
Northampton ...	90·131	$\bar{1}$·43172	$\bar{1}$·72758	No data.	No data.	$\bar{1}$·11526	011213	$\bar{1}$·9954	$\bar{1}$·15125	3·92650
Sweden	96·137	$\bar{1}$·23562	$\bar{1}$·7918	No data.	No data.	$\bar{2}$·87042	01296	$\bar{1}$·99608	$\bar{1}$·03727	3·87142
De Parcieux ...	86·21	$\bar{2}$·99	$\bar{1}$·8415	No data.	No data.	$\bar{1}$·3323	006005	$\bar{1}$·99293	$\bar{1}$·2250	3·19130

I gave the results for the Carlisle mortality to the Congress, which were extremely satisfactory, but I did not give the results for the Northampton mortality and for the Sweden, and De Parcieux's mortality, which (though with the exception that for the few first months of age these statistical tables give no data) I find, I think, equally satisfactory; but not having sent the results to the Congress, I presume that I am authorized, without infringing on the rule of this Society not to publish what has been already

published, to present, as I have done here, the results for the Northampton mortality; and it was my wish also to insert the satisfactory results I found for the other two, but for want of time I am prevented, though I think they would be found interesting. The aforesaid formula bears a very different form from the formula of this paper, $\neg l_x = C\xi^{x'} + k\xi^{x} + kx' - \neg M_x + \mu\nu'$, where $M_x = \sigma^{x}.\overline{h-x}.q-$, &c., the constants here not being represented by the same letters as in the former; these two differ interestingly in appearance, and so much so as to lead to surprise that the result of each, for the Carlisle mortality, gives such very near approximation to the Table of observation. I was therefore, as both formulæ have particular features of interest, induced to examine by analysis their analytical analogy.

Northampton Mortality, from the formula sent to the Congress.

(From the Northampton formula; the results being from the formula which was printed in the Reports of the International Statistical Congress, which results were not given.)

Age x =	0	1	2	3	4	5	6	7	8
λL_x formula .	4·06633	3·93704	3·86626	3·82676	3·80290	3·78988	3·78053	3·77361	3·76797
λL_x Morgan .	4·06633	3·93702	3·86231	3·83129	3·80929	3·79581	3·78283	3·77269	3·76545

Age x =	9	10	11	12	13	14	15	16	17
λL_x formula .	3·76295	3·75820	3·75355	3·74889	3·74418	3·73936	3·73444	3·72941	3·72425
λL_x Morgan .	3·75833	3·75397	3·74992	3·74609	3·74218	3·73822	3·73424	3·73022	3·72591

Age x =	18	19	20	21	22	23	24	25	26
λL_x formula .	3·71897	3·71353	3·70797	3·70225	3·69638	3·68737	3·68119	3·67785	3·67135
λL_x Morgan .	3·72115	3·71592	3·71029	3·70415	3·69767	3·69108	3·68440	3·67761	3·67071

Age x =	27	28	29	30	31	32	33	34	35
λL_x formula .	3·66468	3·65433	3·65080	3·64359	3·63619	3·62859	3·62073	3·61281	3·60460
λL_x Morgan .	3·66370	3·65658	3·64933	3·64197	3·63448	3·62685	3·61909	3·61119	3·60314

Age x =	36	37	38	39	40	41	42	43	44
λL_x formula .	3·59618	3·58755	3·57868	3·56958	3·56025	3·55067	3·54084	3·53075	3·52040
λL_x Morgan .	3·59494	3·58664	3·57807	3·56937	3·56500	3·55132	3·54180	3·53191	3·52192

Age x =	45	46	47	48	49	50	51	52	53
λL_x formula .	3·50977	3·49888	3·48770	3·47628	3·46446	3·45196	3·43943	3·42653	3·41327
λL_x Morgan .	3·51162	3·50106	3·49024	3·47014	3·46775	3·45591	3·44348	3·43640	3·41697

Age x =	54	55	56	57	58	59	60	61	62
λL_x formula .	3·39960	3·38543	3·37098	3·35595	3·34042	3·32433	3·30763	3·29028	3·27221
λL_x Morgan .	3·40312	3·38881	3·37401	3·35870	3·34282	3·32634	3·30920	3·29134	3·27277

Age x =	63	64	65	66	67	68	69	70	71
λL_x formula .	3·25425	3·23361	3·21286	3·19111	3·16812	3·14363	3·11771	3·09040	3·06097
λL_x Morgan .	3·25258	3·22350	3·21272	3·19089	3·16791	3·14364	3·11793	3·09061	3·06145

Age x =	72	73	74	75	76	77	78	79	80
λL_x formula .	3·02944	2·99557	2·95889	2·91834	2·87596	2·82861	2·77758	2·72252	2·63755
λL_x Morgan .	3·03019	2·99651	2·95999	2·92012	2·87622	2·82930	2·77960	2·72754	2·67117

Age x =	81	82	83	84	85	86	87	88	89
λL_x formula .	2·59479	2·51156	2·43295	2·35326	2·24725	2·15001	2·03269	1·90233	1·76807
λL_x Morgan .	2·60832	2·53908	2·46090	2·36922	2·26951	2·16137	2·04532	1·91908	1·79239

Age x =	90	91	92	93	94	95	96		
λL_x formula .	1·60203	1·44327	1·25940	1·05593	1·04019	60181	34315		
λL_x Morgan .	1·66276	1·53479	1·38021	1·20402	·95424	60206			

This Table was not given in the paper written for the (late) International Congress, but was calculated for the present, from the formula $\lambda L_x = u + kx^r - k_1 x^s - nq^x - P_x$; P_x being put for $\delta(w)^{x^s \cdot \overline{x-w}}$, all the qualities except x being constant from birth.

Art. 28. It remains now slightly to touch on the law of sickness, though the subject is already mentioned in the paper I presented to the Congress, in consequence of a similarity, as far as I have had data to discover, with the law of mortality. I gave the formula, but not the investigation of that formula; but I gave Tables of comparison with the sickness stated to have occurred in different societies and in different places, which appear to be extremely satisfactory as to the approximate agreement with the stated results. The formula I gave is as follows:—If S^x be the number of weeks of sickness due to a person of x in the Society, the log of $S_x = A + BC^x$, where A, B, C are apparent constants for a long period, to be found by the vital rule of three, from the stated sickness prevailing at three selected ages; and here the apparent constants are evidently not truly constant, as their values will slightly change with the selection. The formula $S_x = A.\overline{B|}^{c^x}$ has the same form as $L_x = D.\overline{g|}^{x^s}$, the theorem of mortality, and, as in that theorem, the elements A, B, C, though apparently constant for a less time, depend on three selected ages for the determining their values, and are only apparently constant, but afford for a long time near approximations to the amount of sickness which occurs in different societies. The values of A, B, C of the formula, as determined from statements of societies, I gave in my paper presented to the International Congress, with Tables showing the near agreement with the statements of those societies. I quote from that paper as follows:—

Selected ages, 25, 45, 65, from Mr. ANSELL, all England.

Selected ages, 25, 45, 65, Town:

$\lambda A = \overline{1}\cdot72778$; $\lambda B = \overline{2}\cdot6772$; $\lambda C = \cdot020433$.

Selected ages, 25, 45, 65, Scotland:

$\lambda A = \overline{1}\cdot66237$; $\lambda B = \overline{2}\cdot41351$; $\lambda C = \cdot02433$.

Selected ages, 25, 45, 65, ANSELL, Town:

$\lambda A = \overline{1}\cdot65112$; $\lambda B = \overline{1}\cdot04297$; $\lambda C = \cdot02818$.

Selected ages, 35, 50, 65, ANSELL, Town:

$\lambda A = \overline{1}\cdot92937$; $\lambda B = \overline{1}\cdot70942$; $\lambda C = \cdot0165265$

Selected ages, 25, 45, 65, City district:

$\lambda A = \overline{1}\cdot84121$; $\lambda B = \overline{2}\cdot47741$; $\lambda C = \cdot009206$.

Selected ages, 25, 45, 65, City:

$\lambda A = \overline{1}\cdot84121$; $\lambda B = \overline{1}\cdot1553$; $\lambda C = \cdot0301685$.

Selected ages, 30, 40, 50, Rural:

$\lambda A = \overline{1}\cdot85616$; $\lambda B = \overline{2}\cdot02951$; $\lambda C = \cdot0301685$.

There seems to be a distinction between the law of mortality and that of sickness, inasmuch as g, in the law of mortality, is a positive fraction less than unity, whereas the similar term B in the law of sickness is greater than unity.

Art. 29. Here I must draw the reader's very particular attention to the formula

$$\text{-}\!\!\supset L_x = C\mathfrak{C}^x + k_1 s^x + k s^x - M_x + \mu \nu^x, \text{ where } \text{-}\!\!\supset M_x = e^x . \overline{x-h} . q \text{-}\!\!\supset,$$

where for analytical anticipation it is put in the form

$$\subset\!\!\text{-} L_{u+x} = C\mathfrak{C}^{u+x} + k_1 s^{u+x} + k s^{u+x} - M_{u+x} + \mu \nu^{u+x};$$

and the two portions $k s^{u+x}$ and $\mu \nu^{u+x}$ are developed into series proceeding by the powers of x, because those series for several of the first terms, when x is at all large, are so divergent that in that case they become of no practical service; and in fact, though, if a sufficient number of terms be used, they are ultimately convergent and would lead to the true value, they present so formidable an obstacle as to cause, if those series be used, a complete refusal of the aid which was expected from them to render the formula analytically anticipatory, at least at those periods when these functions have a prevailing influence over anticipation; and if this difficulty had not been overcome by the adoption of a subterfuge, which I almost despaired of finding, a great part of the highly important analytically anticipating powers of the formula would be destroyed. The series alluded to are exhibited as follows:—

$$k s^{u+x} = k s^u \times \left(1 + \underset{\subset}{s} x + \frac{1}{2} \underset{\subset}{\overline{s}^2} x^2 + \frac{1}{2.3} \cdot \underset{\subset}{\overline{s}^3} x^3, \text{ \&c.}\right),$$

and

$$\mu \nu^{u+x} = \mu \nu^u \times \left(1 + \underset{\subset}{\nu} x + \frac{1}{2} \underset{\subset}{\overline{\nu}^2} x^2 + \frac{1}{2.3} \cdot \underset{\subset}{\overline{\nu}^3} x^3, \text{ \&c.}\right).$$

And alluding to the first of these, I observe that in the formula above, when we refer it to the Carlisle mortality, $\text{-}\!\!\supset s = \overline{1}\cdot 79811$; $\subset\!\!\text{-} s = -\cdot 46427$; $\text{-}\!\!\supset k = \overline{1}\cdot 16855$; $k = \cdot 14742$: if we wished to anticipate only for ten years forward, $\underset{\subset}{s} x$ would be, say about $-\frac{9}{4}$, and the series $1 + \underset{\subset}{s} x + \frac{1}{2} \underset{\subset}{\overline{s}^2} x^2 +$ &c. would be

$$1 - \frac{9}{2} + \frac{1}{2} \cdot \left|\frac{\overline{9}^2}{2}\right| - \frac{1}{2.3} \cdot \left|\frac{\overline{9}^3}{2}\right| + \frac{1}{2.3.4} \cdot \left|\frac{\overline{9}^4}{2}\right| - \text{ \&c.} = 1 - \frac{9}{2} + \frac{9}{2} \cdot \frac{9}{4} - \frac{9}{2} \cdot \frac{9}{4} \cdot \frac{9}{6} + \frac{9}{2} \cdot \frac{9}{4} \cdot \frac{9}{6} \cdot \frac{9}{8} - \text{ \&c.};$$

so that we should not only have to go to the fifth power of x before the series began to converge, but to go to much higher powers, whereas the real value of $k s^{10}$ sought is but of the small value of about ·001411, of very small value, which may be considered even of insignificant value with respect to the total value of $\subset\!\!\text{-} L_{u+x}$, notwithstanding its perplexing annoyance. But if we wished to anticipate for twenty years, although if a is even equal to 0, the value of the function $k s^{u+x}$ is of such total insignificance with respect to the other portion of $\subset\!\!\text{-} L_{u+x}$, being no greater than about ·000014, whilst the value of $\subset\!\!\text{-} L_{u+x} = 3\cdot 81023$, yet in the development of $k s^{u+x}$ into the series above, the number of diverging terms alternately positive and negative of greater value before the series even

began to converge would render the development of perfect impracticability, though ultimately we could arrive as near as we chose to the insignificant value ·000014; and I found that with such an annoyance it was no use to wrestle, no more than it would have been for David to wrestle with Goliath. I therefore was at the pains of finding a subterfuge by an interpolation which discards all the terms of a series alternately greatly positive and greatly negative, which so nearly destroyed each other's effect when we arrived to very nearly the true but very insignificant value ·000014. I have reason to think good mathematicians would think such a subterfuge was unattainable, but what follows will show the contrary.

Art. 30. Calculating the value ks^0, ks^5, ks^{10}, ks^{20}, we find the value to form the very swiftly converging series, ·147424, ·014424, ·001441, ·000188, ·00008, and beyond this term perfectly insignificant; so that, if we wished to express the value of s^{5x}, where x is a whole number, by the common method of interpolation, we might easily do it thus:—

$$1+\Delta_1 x+\Delta_2 .x.\frac{x-1}{2}+\Delta_3 .x.\frac{x-1}{2}\cdot\frac{x-2}{3}+\Delta_4 .x.\frac{x-1}{2}\cdot\frac{x-2}{3}\cdot\frac{x-3}{4};$$

beyond which term the differences are of perfect insignificance, Δ_1, Δ_2, Δ_3, Δ_4 being the first of the several orders of differences of the value ks^0, ks^5, ks^{10}, &c.; and it is evident that Δ_1, Δ_2, Δ_3, Δ_4 are respectively $k(s^5-1)$, $k(s^5-1)^2$, $k(s^5-1)^3$, &c., which we may adopt for the sake of proof, instead of taking the actual difference. And as the form in which we obtain the value of ks^x requires for our purpose to be expressed by a series such as $I_0+I_1x+I_2x^2+I_3x^3+I_4x^4$, if the above value of ks^x be expanded, it will give

$$I_0=k,\quad I_1=\Delta_1-\frac{1}{2}\Delta_2+\frac{1}{3}\Delta_3-\frac{1}{4}\Delta_4,\quad I_2=\Delta_2-\frac{1}{2}\Delta_3+\frac{11}{24}\Delta_4,\quad I_3=\frac{1}{6}\Delta_3-\frac{1}{4}\Delta_4,\quad I_4=\frac{1}{24}\Delta_4,$$

where, as already observed, $\Delta_1=k(s^5-1)$, $\Delta_2=k(s^5-1)^2$, &c.; and we find from the above values, I_0, I_1, I_2, I_3, I_4; and as ks^{5x} is therefore $=I_0+I_1x+I_2x^2+I_3x^3+I_4x^4$, if in the room of $5x$ we write x, we shall have the equation

$$ks^x=I_0+\frac{1}{5}I_1x+\frac{1}{5^2}I_2x^2+\frac{1}{5^3}I_3x^3+\frac{1}{5^4}I_4x^4=J_0+J_1x+J_2x^2+J_3x^3+J_4x^4,$$

if $J_0=I_0$, $J_1=\frac{1}{5}I_1$, $J_2=\frac{1}{5^2}I_2$, $J_3=\frac{1}{5^3}I_3$, $J_4=\frac{1}{5^4}I_4$, and we have

$$I_0=\cdot147422,\quad I_1=-\cdot253484,\quad I_2=+\cdot158862,\quad I_3=-\cdot042453,\quad I_4=+\cdot004068,$$
$$J_0=I_0,\qquad J_1=-\cdot050696,\quad J_2=\cdot0063545,\quad J_3=-\textcircled{4}33962,\quad J_4=\textcircled{5}65088,$$
$$\lambda-J_1=\overline{2}\cdot70497,\quad \lambda J_2=\overline{3}\cdot80308,\quad \lambda-J_3=\overline{5}\cdot53100,\quad \lambda J_4=\overline{6}\cdot81350.$$

And with respect to μs^{a+x}, the least value of a may be taken when $a=60$, as μs^a is so extremely insignificant if a be less than 60. I will consider the value of μs^{60+x} first, and consider afterwards μs^{60+5+x}, if a exceeds 60 by b; and I observe that μs^{60}, μs^{60+5}, μs^{60+10}, μs^{60+15}, μs^{60+20} form a regular series; and here I stop, though this series is of a different character from the series ks^0, ks^{5x}, ks^{10x}, &c., because the terms of the latter series dwindle to such insignificance, but those of the other, on the contrary, continually increase.

There may be cases, indeed, when we should not stop; but though we have not data for ages beyond 100, the coefficient μ in the expression $\mu\nu^{s+x}$, though taken as sufficiently expressed by the constant value till $a+x$ does exceed 100, is evidently not absolutely constant; for if so, a person would be represented to be capable to live for ever; and I therefore satisfy myself by omitting to introduce a term for which I have no data, by limiting my term of anticipation to the period when $a+x$ becomes $=100$.

I now observe that for the Carlisle mortality (where $\lambda\mu=\overline{1}1{\cdot}29927$, $\lambda\nu=\cdot110349$), $\mu\nu^{50}$, $\mu\nu^{50+5}$, $\mu\nu^{50+10}$, $\mu\nu^{50+15}$, $\mu\nu^{50+20}$ give the following series of terms:—

(4)83303, (3)10572, (2)37662, 013417.

And adopting the same mode of interpretation as in the former case, and representing $\mu\nu^{50+x}$ by

$$'I_0+'I_1x+'I_2x^2+'I_3x^3+'I_4x^4,$$

and putting

$$'I_0='J_0,\ \tfrac{1}{5}'I_1='J_1,\ \tfrac{1}{5^2}'I_2='J_2,\ \tfrac{1}{5^3}'I_3='J_3,\ \tfrac{1}{5^4}'I='J_4,$$

we have

$'I_0=\cdot000083303$, $'I_1=-\cdot00040315$, $'I_2=+\cdot00121095$, $'I_3=-\cdot00065977$, $'I_4=+\cdot0001489$,

$\quad=-'J_1=\overline{5}{\cdot}98816$, $\quad='J_2=\overline{5}{\cdot}68517$, $\quad=-'J_3=\overline{6}{\cdot}72248$, $\quad='J_4=\overline{6}{\cdot}23825$,

$'J_0=\cdot000083303$, $'J_1=-(4)97313$, $'J_2=(4)484363$, $'J_3=-(5)52753$, $'J_4=(6)23825$.

But here it is necessary to observe, that though these series for finding kv^{s+x}, $\mu\nu^{s+x}$ are exactly true with respect to any value of x, if it be divisible by 5 (limited with respect to $\mu\nu^{s+x}$ to the case observed above, $a+x$ not being greater than 100), still they are not identical with that expression, as they may differ both in plus and in minus with them, if x, when divided by 5, should leave a remainder, say of 1, 2, 3, 4, which are four cases of more exact identity; but as the variance from identity is quite insignificant with respect to the value of L_{a+x}, as will be shown further on, it is perfectly allowable to use this method without paying the slightest attention to the more absolute identity of the expression; but to proceed to prove this assertion. I represent kv^x by

$$J_0+J_1x+J_2x^2+J_3x^3+J_4x^4+W_x,$$

and also represent $\mu\nu^{50+x}$ by

$$'J_0+'J_1x+'J_2x^2+'J_3x^3+'J_4x^4+'W_x,$$

and I am to show that W_x and $'W_x$ are either equal to nothing, or are insignificant with respect to L_{a+x}, whether they shall turn out to be positive or negative. Now I observe in the case where x is divisible exactly by 5, the above investigation shows that W_x and $'W_x$ are both absolutely equal 0; and for the rest I will take x, by way of example, successively 11, 12, 13, 14, in which x divided by 5 leaves either 1, 2, 3, or 4, and where the anticipation is respectively for 11, 12, 13, 14 years.

By direct calculation . . $k\varepsilon^{11}$= ·000887, $k\varepsilon^{12}$= ·000537, $k\varepsilon^{13}$= ·000350, $k\varepsilon^{14}$= ·000220,
By the anticipating formula $k\varepsilon^{11}$= ·00106, $k\varepsilon^{12}$= ·00129, $k\varepsilon^{13}$= ·00355, $k\varepsilon^{14}$=−·00353,
W_{11}=−·00018, W_{12}=−·00075, W_{13}=−·00320, W_{14}=+·00375,

where x is not divisible by 5; but in cases where x is divisible by 5, W_x is always equal to 0, and among the cases the three negative values are

$$\left.\begin{matrix}-\cdot00018\\-\cdot00075\\-\cdot00320\end{matrix}\right\}=-\cdot00435.$$

Their sum differs very little from the one positive case ·00375, and they are all perfectly insignificant compared to the total values of λL_{11}=3·808 &c., λL_{12}=3·806 &c., λL_{13}=3·804 &c., λL_{14}=3·803 &c.; but when I say perfectly insignificant, I mean with respect to the chance of living,—though that difference would, in a small degree, cause the number answering, say to the age of 11, to make it apply to an age very triflingly differing from the exact age of 11, but this is not of the slightest importance with respect to valuing chances of anticipation. It may be observed that when $k\varepsilon^x$ is very small, as in the case above enumerated, we see that the anticipated values of $k\varepsilon^x$ may have not only a large proportion to each other, but may even be of contrary signs, as $k\varepsilon^x$ cannot be negative, though by the anticipating formula in the case of k it comes out negative; but these are cases where the value is of no importance in consequence of its smallness; and for long before the age of 20, the effect of $k\varepsilon^x$ in the anticipation can be omitted.

And now with respect to the formula which is not as imperatively required as the other,

$$\mu\nu^{a+x}=(\text{when } a=60)\ {}'J_a+{}'J_1x+{}'J_2x^2+{}'J_3x^3+{}'J_4x^4+{}'W_x,$$

let the anticipation also be successively for 11, 12, 13, 14 years. I purposely take years not divisible by 5, because, as I have said above, when x=5, and $a+x$ does not exceed 100, ${}'W_x$ is invariably =0, the variability of μ not coming into play; we have

The real values of $\mu\nu^{71}$=·0013630, $\mu\nu^{72}$=·0017573, $\mu\nu^{73}$=·0022656, $\mu\nu^{74}$= ·0029217
Anticipating formula $\mu\nu^{71}$=·0013264, $\mu\nu^{72}$=·0017106, $\mu\nu^{73}$=·0022601, $\mu\nu^{74}$= ·0029430
${}'W_{71}$=·0000366, ${}'W_{72}$=·0000467, W_{73}=·0000055, W_{74}=−·0000213

The mode above described of getting rid of the annoyance of the non-convergency of the developed expression $k\varepsilon^x$ is particularly worthy of attention, and is entitled to be noticed by a name which I call the Interpolation by selected terms taken *per saltum*, as it offers another and an easier mode of calculating the three analytical anticipating Tables mentioned, which are so efficient in practical valuations; and calculating those Tables independent of each other when, as in the mode pointed out first, required the calculation of the value of $-L_{a+x}$ and from that the value of L_{a+x}; and lastly, the value

of $\frac{L_{a+x}}{L_a}$, say in the form $'B_1x+'B_2x+'B_3x^3$, and then the value $=(1+\frac{'B_2}{'B_1}x+\frac{'B_3}{'B_1}x^2+\&c.)$. And not only does this mode furnish the means of finding those Tables from the developed function of mortality, but, if I mistake not, it furnishes a means off-hand, from any Table of mortality formed only from statistical information, and without requiring the formulæ of mortality above pointed out, to furnish efficient analytical participating tables.

It was my wish not only to conclude this paper with the three Tables above-mentioned, which have been calculated, but which I have been prevented from examining from causes to which mortality is liable, but also to add some interesting matter. But I hope to be able to add a supplement to this paper, and to be permitted to publish it in the Royal Society's Transactions, to illustrate the practical adaptation of the analysis, with some other matters of vital statistics and invalidism.

XXIV. *On* Tschirnhausen's *Transformation.* *By* Arthur Cayley, *Esq.*, *F.R.S.*

Received November 7,—Read December 5, 1861.

The memoir of M. Hermite, "Sur quelques théorèmes d'algèbre et la résolution de l'équation du quatrième degré,"* contains a very important theorem in relation to Tschirnhausen's Transformation of an equation $f(x)=0$ into another of the same degree in y, by means of the substitution $y=\phi x$, where ϕx is a rational and integral function of x. In fact, considering for greater simplicity a quartic equation,

$$(a, b, c, d, e)(x, 1)^4=0,$$

M. Hermite gives to the equation $y=\phi x$ the following form,

$$y=a\mathrm{T}+(ax+b)\mathrm{B}+(ax^2+4bx+6c)\mathrm{C}+(ax^3+4bx^2+6cx+4d)\mathrm{D}$$

(I write B, C, D in the place of his T_0, T_1, T_2), and he shows that the transformed equation in y has the following property: viz., every function of the coefficients which, expressed as a function of a, b, c, d, e, T, B, C, D, does not contain T, is an *invariant*, that is, an invariant of the two quantics

$$(a, b, c, d, e)(\mathrm{X}, \mathrm{Y})^4, \ (\mathrm{B}, \mathrm{C}, \mathrm{D})(\mathrm{Y}, -\mathrm{X})^2.$$

This comes to saying that if T be so determined that in the equation for y the coefficient of the second term (y^3) shall vanish, the other coefficients will be invariants; or if, in the function of y which is equated to zero, we consider y as an absolute constant, the function of y will be an invariant of the two quantics. It is easy to find the value of T; this is in fact given by the equation

$$0=a\mathrm{T}+3b\mathrm{B}+3c\mathrm{C}+d\mathrm{D};$$

and we have thence for the value of y,

$$y=(ax+b)\mathrm{B}+(ax^2+4bx+3c)\mathrm{C}+(ax^3+4bx^2+6cx+3d)\mathrm{D};$$

so that for this value of y the function of y which equated to zero gives the transformed equation will be an invariant of the two quantics. It is proper to notice that in the last-mentioned expression for y, all the coefficients except those of the term in x^0, or $b\mathrm{B}+3c\mathrm{C}+3d\mathrm{D}$, are those of the binomial $(1, 1)^4$, whereas the excepted coefficients are those of the binomial $(1, 1)^3$; this suffices to show what the expression for y is in the general case.

I have in the two papers, "Note sur la Transformation de Tschirnhausen,"† and "Deuxième Note sur la Transformation de Tschirnhausen,"† obtained the transformed

* Comptes Rendus, t. xlvi. p. 961 (1858). † Crelle, t. lviii. pp. 259 and 263 (1861).

equations for the cubic and quartic equations; and by means of a grant from the Government Grant Fund, I have been enabled to procure the calculation, by Messrs. DAVIS and OTTER, under my superintendence, of the transformed equation for the quintic equation. The several results are given in the present memoir; and for greater completeness, I reproduce the demonstration which I have given in the former of the above-mentioned two Notes, of the general property, that the function of y is an invariant. At the end of the memoir I consider the problem of the reduction of the general quintic equation to Mr. JERRARD's form $x^5+ax+b=0$.

Considering for simplicity the foregoing two equations

$$(a, b, c, d, e)(x, 1)^4=0,$$
$$y=(ax+b)\mathrm{B}+(ax^3+4bx+3c)\mathrm{C}+(ax^3+4bx^3+6cx+3d)\mathrm{D};$$

let the second of these be represented by $y=\mathrm{V}$, the transformed equation in y is

$$(y-\mathrm{V}_1)(y-\mathrm{V}_2)(y-\mathrm{V}_3)(y-\mathrm{V}_4)=0,$$

where V_1, V_2, V_3, V_4 are what V becomes upon substituting therein for x the roots x_1, x_2, x_3, x_4 of the quartic equation respectively. Considering y as a constant, the conditions to be satisfied in order that the function in y may be an invariant are that this function shall be reduced to zero by each of the two operators

$$a\partial_a+2b\partial_b+3c\partial_c+4d\partial_e-(\mathrm{D}\partial_\mathrm{C}+2\mathrm{C}\partial_\mathrm{B}),$$
$$4b\partial_a+3c\partial_b+2d\partial_c+e\partial_d-(2\mathrm{C}\partial_\mathrm{D}+\mathrm{B}\partial_\mathrm{C}):$$

These conditions will be satisfied if each of the factors $y-\mathrm{V}_1$, &c. has the property in question; that is, if $y-\mathrm{V}$, or (what is the same thing) if V, supposing that x denotes therein a root of the quartic equation, is reduced to zero by each of the two operators. Considering the first operator, which for shortness I represent by

$$\Delta-(\mathrm{D}\partial_\mathrm{C}+2\mathrm{C}\partial_\mathrm{B}),$$

in order to obtain $\Delta\mathrm{V}$ we require the value of Δx. To find it, operating with Δ on the quartic equation, we have

$$(a, b, c, d)(x, 1)^3\Delta x+(a, b, c, d)(x, 1)^3=0,$$

or $\Delta x=-1$. In $\Delta\mathrm{V}$, the part which depends on the variation of Δx then is

$$-a\mathrm{B}+(-2ax-4b)\mathrm{C}+(-3ax^3-8bx-6c)\mathrm{D},$$

and the other part of $\Delta\mathrm{V}$ is at once found to be

$$+a\mathrm{B}+(4ax+6b)\mathrm{C}+(4ax^3+12bx+9c)\mathrm{D};$$

whence, adding,

$$\Delta\mathrm{V}=2(ax+b)\mathrm{C}+(ax^3+4bx+3c)\mathrm{D},$$

and this is precisely equal to

$$(\mathrm{D}\partial_\mathrm{C}+2\mathrm{C}\partial_\mathrm{B})\mathrm{V};$$

so that V is reduced to zero by the operator $\Delta-(\mathrm{D}\partial_\mathrm{C}+2\mathrm{C}\partial_\mathrm{B})$.

Similarly, if the second operator is represented by

$$\nabla - (2C\partial_D + B\partial_C),$$

then we have

$$(a, b, c, d, e)(x, 1)^4 \nabla x + x(b, c, d, e)(x, 1)^3 = 0,$$

which by means of the equation

$$(a, b, c, d, e)(x, 1)^4 = 0$$

is reduced to $\nabla x = x^2$. Hence in ∇V the part depending on the variation of x is

$$ax^2B + (2ax^3 + 4bx^2)C + (3ax^4 + 8bx^3 + 6cx^2)D,$$

and the other part of ∇V is at once found to be

$$(4bx + 3c)B + (4bx^3 + 12cx + 6d)C + (4bx^3 + 12cx^2 + 12dx + 3e)D;$$

and, adding, the coefficient of D vanishes on account of the quartic equation, and we have

$$\nabla V = (ax^3 + 4bx + 3c)B + 2(ax^3 + 4bx^2 + 6cx + 3d)C,$$

which is precisely equal to

$$(2C\partial_D + B\partial_C)V,$$

so that V is reduced to zero by the operator

$$\nabla - (2C\partial_D + B\partial_C),$$

which completes the demonstration; and the demonstration in the general case is precisely similar.

In the case of the cubic equation we have

$$(a, b, c, d)(x, 1)^3 = 0,$$
$$y = (ax + b)B + (ax^3 + 3bx + 2c)C;$$

and writing the second equation in the form

$$(y - bB - 2cC) + x(\ -aB - 3bC) + x^2(\qquad -aC) = 0,$$

multiplying by x and reducing by the cubic equation, we have

$$dC \qquad\qquad + x(y - bB + \ cC) + x^2(\ -\ aB) = 0,$$

and repeating the process,

$$dB \qquad\qquad + x(\ 3cB + \ dC) + x^2(y + 2bB + cC) = 0;$$

or, what is the same thing, we have the system of equations

$$\left\{\begin{array}{ccccc} y - bB - 2cC, & - & aB - 3bC, & & -aC \\ & & dC, & y - bB + cC, & -a\ B \\ dB & , & & 3cB + dC, & y + 2bB + cC \end{array}\right\}(1, x, x^2) = 0,$$

and the resulting equation in y is of course that formed by equating to zero the determinant formed out of the matrix in this equation. The developed expression is

$$(1, 0, \mathfrak{C}, \mathfrak{D})(y, 1)^3 = 0,$$

where

$\frac{1}{3}\mathbb{C}=$

	B^3		BC		C^3
ac	$+1$	ad	$+1$	bd	$+1$
b^2	-1	be	-1	c^2	-1
	± 1		± 1		± 1

$\mathbb{D}=$

	B^3		B^2C		BC^2		C^3
a^2d	$+1$	abd	$+3$	acd	-3	ad^2	-1
abc	-3	ac^2	-6	b^2d	$+6$	bcd	$+3$
b^3	$+2$	b^2c	$+3$	bc^2	-3	c^3	-2
	± 3		± 6		± 6		± 3

The sum of the coefficients in each column should here and elsewhere in the present memoir be equal to zero, and I have by way of verification annexed to each column the sums ($\pm$ a number) of the positive and negative coefficients. The coefficients $\mathbb{C}$, $\mathbb{D}$, and therefore the function in y, are invariants of the two forms,

$$(a, b, c, d\,(X, Y)^3, \quad (B, C\,(Y, -X);$$

or in the present case, where there are only two coefficients B, C, the coefficients $\mathbb{C}$, $\mathbb{D}$, and therefore also the function in y, are covariants of the single form $(a, b, c, d\,(B, C)^3$, considering therein (B, C) as the facients.

It may be remarked that in the present case, assuming the invariance of the function in y, we may obtain the transformed equation in a very simple manner by writing in the first instance $C=0$, this gives

$$(a, b, c, d\,(x, 1)^3=0,$$

$$y=(ax+b)B,$$

and thence

$$\frac{1}{a}(a, b, c, d\,(y-bB, aB)^3=0;$$

or developing,

$$y^3+3y(ac-b^2)B^2+(a^2d-3abc+2b^3)B^3=0,$$

where the expressions for the coefficients are to be completed by the consideration that these coefficients are covariants of the form $(a, b, c, d\,(B, C)^3$. But it is only in the case in hand of a cubic equation that the transformed equation can be obtained in this manner.

In the case of a quartic equation, we have

$$(a, b, c, d, e\,(x, 1)^4=0,$$

$$y=(ax+b)B+(ax^2+4bx+3c)C+(ax^3+4bx^2+6cx+3d)D,$$

and these give the system of equations

$$\begin{vmatrix} (y-b\mathrm{B}-3c\mathrm{C}-3d\mathrm{D}, & -a\mathrm{B}-4b\mathrm{C}-6c\mathrm{D}, & -a\mathrm{C}-4b\mathrm{D}, & -a\mathrm{D} \\ c\mathrm{D}, & y-b\mathrm{B}-3c\mathrm{C}+d\mathrm{D}, & -a\mathrm{B}-4b\mathrm{C}, & -a\mathrm{C} \\ c\mathrm{C}, & 4d\mathrm{C}+c\mathrm{D}, & y-b\mathrm{B}+3c\mathrm{C}+d\mathrm{D}, & -a\mathrm{B} \\ c\mathrm{B}, & 4d\mathrm{B}+c\mathrm{C}, & 6c\mathrm{B}+4d\mathrm{C}+c\mathrm{D}, & y+3b\mathrm{B}+3c\mathrm{C}+d\mathrm{D} \end{vmatrix} \Big)(1, x, x^2, x^3)=0,$$

and the transformed equation is therefore found by equating to zero the determinant formed out of the matrix contained in this equation.

The developed result, which was obtained by a different process in the 'Deuxième Note' above referred to, is

$$(1, 0, \mathbb{C}, \mathbb{B}, \mathbb{G})(y, 1)^4=0,$$

where

$\frac{1}{2}\mathbb{C}=$

	B^2		BC		BD^2	C^2		CD		D^2
ae b^2	$+3$ -3	ad bc	$+6$ -6	ae bd c^2	$+2$ -2 .	$+1$ $+8$ -9	be cd	$+6$ -6	ce d^2	$+3$ -3
	± 3		± 6		± 2	± 9		± 6		± 3

$\frac{1}{4}\mathbb{B}=$

	B^3		$\mathrm{B}^2\mathrm{C}$		$\mathrm{B}^2\mathrm{D}$	BC^2		BCD	C^3		BD^2	$\mathrm{C}^2\mathrm{D}$		CD^2		D^3
a^2d abc b^3	$+1$ -3 $+2$	a^2e abd ac^2 b^2c	$+1$ $+2$ -9 $+6$	abe acd b^2d bc^2	$+1$ -3 $+2$	$+4$ -12 $+8$	ac^2 ad^2 b^2e bcd c^3	-6 $+6$	-4 $+4$	ade bcd bd^2 c^2d	-1 $+3$ -2	-4 $+12$ -8	ae^2 bde c^2e cd^2	-1 -2 $+9$ -6	be^2 cde d^3	-1 $+3$ -2
	± 3		± 9		± 3	± 12		± 6	± 4		± 3	± 12		± 9		± 3

and

$\mathbb{G}=$

	B^4		$\mathrm{B}^3\mathrm{C}$		$\mathrm{B}^3\mathrm{D}$	$\mathrm{B}^2\mathrm{C}^2$		$\mathrm{B}^2\mathrm{CD}$	BC^3		$\mathrm{B}^2\mathrm{D}^2$	$\mathrm{BC}^2\mathrm{D}$	C^4
a^3e a^2bd a^2c^2 ab^2c b^4	$+1$ -4 ∞ $+6$ -3	a^2be a^2cd ab^2d abc^2 b^2c	$+8$ -12 -20 $+36$ -12	a^2ce a^2d^2 ab^2e $abcd$ ac^3 b^3d b^2c^2	$+12$ -12 -8 $+12$ ∞ -4 .	-6 ∞ $+30$ -48 $+54$ -48 $+18$	a^3de $abce$ abd^2 ac^2d b^3e b^2cd bc^3	∞ $+60$ -72 $+36$ -36 $+12$.	-4 -12 $+16$ $+36$ $+48$ -192 $+108$	a^3e^2 $abde$ ac^2e acd^2 b^2ce b^2d^2 bc^2d c^4	$+2$ -16 $+36$ -18 -18 $+14$ . .	-4 $+20$ $+36$ ∞ ∞ -160 $+108$.	$+1$ -16 -18 $+48$ $+48$ ∞ -144 $+81$
	± 7		± 44		± 24	± 102		± 108	± 208		± 52	± 164	± 178

	BCD^2	$\mathrm{C}^2\mathrm{D}$		BD^2	$\mathrm{C}^2\mathrm{D}^2$		CD^3		D^4
abe^2 $acde$ ad^2 b^2de bc^2e bcd^2 c^2d	∞ $+60$ -36 -72 $+36$ $+12$.	-4 -12 $+48$ $+16$ $+36$ -192 $+108$	ace^2 ad^2e b^2e^2 $bcde$ bd^2 c^2e c^2d^2	$+12$ -8 -12 $+12$ -4 . .	-6 $+30$ ∞ -48 -48 $+54$ $+18$	ade^2 bce^2 bd^2e c^2de cd^3	$+8$ -12 -20 $+36$ -12	ae^3 bde^2 c^2e^2 cd^2e d^4	$+1$ -4 ∞ $+6$ -3
	± 108	± 208		± 24	± 102		± 44		± 7

I write

$$U' = aB^3 + 4bBC + c(2BD + 4C^3) + 4dCD + eD^3,$$

$$H' = (ac - b^2)B^3 + 2(ad - bc)BC + (ae - 2bd + c^2)BD + 4(bd - c^2)C^3 + 2(be - cd)CD + (ce - d^2)D^3;$$

and I represent by Φ' the expression which has just been found for $\frac{1}{4}\mathfrak{B}$. These functions, U', H', Φ', are invariants of the two forms

$$(a, b, c, d, e)(X, Y)^4, \quad (B, C, D)(Y, -X)^3;$$

we have, moreover, the invariants

$$ae - 4bd + 3c^2, \quad ace - ad^2 - b^2e + 2bcd - c^3,$$

which I represent as usual by I, J, and the invariant $BD - C^2$, which I represent by Θ'. This being so, we have

$$\mathfrak{C} = 6H' - 2I\Theta',$$

$$\mathfrak{B} = 4\Phi',$$

$$\mathfrak{C} = IU'^2 - 3H'^2 + I^2\Theta'^2 + 12J\Theta'U' + 2I\Theta'H',$$

the last of which may be verified as follows:—viz. writing $a = e = 1$, $b = d = 0$, $c = \theta$, it becomes

$$\begin{aligned}
&(1 + 3\theta^2)\{B^3 + \theta(2BD + 4C^3) + D^3\}^2 \\
&- 3\{\theta B^3 + (1 + \theta^2)BD - 4\theta C^3 + \theta D^3\}^2 \\
&+ (1 + 3\theta^2)^2(BD - C^2)^2 \\
&+ 12(\theta - \theta^2)(BD - C^2)\{\theta B^3 + (1 + \theta^2)BD - 4\theta C^3 + \theta D^3\} \\
= \; & B^4 \\
&+ B^3C \quad (12\theta) \\
&+ B^2C^2 \quad (-6\theta + 54\theta^2) \\
&+ B^2D^2 \quad (2 + 36\theta^2) \\
&+ BC^2D \quad (-4 + 36\theta^2) \\
&+ B^4 \quad (1 - 18\theta^2 + 81\theta^4) \\
&+ C^2D^2 \quad (-6\theta + 54\theta^2) \\
&+ C^2D \quad (12\theta) \\
&+ D^4,
\end{aligned}$$

which is an identical equation.

The expression for the invariant I (quadrinvariant) of the function $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E})(y, 1)^4$ is $\mathfrak{E} + 3(\frac{1}{6}\mathfrak{C})^2$, or $\mathfrak{E} + 3(H' - \frac{1}{3}I\Theta')^2$, viz. it is

$$\begin{aligned}
&IU'^2 - 3H'^2 + I^2\Theta'^2 + 12J\Theta'U' + 2I\Theta'H' \\
&\quad + 3H'^2 + \tfrac{1}{3}I^2\Theta'^2 \qquad - 2I\Theta'H',
\end{aligned}$$

or, finally, it is

$$IU'^2 + \tfrac{4}{3}I^2\Theta'^2 + 12J\Theta'U',$$

which is equal to

$$\frac{1}{I}[(IU'+6J'\Theta')^2+\tfrac{4}{3}(I^3-27J^2)\Theta'^2];$$

so that the condition in order that this invariant may be equal to zero is

$$IU'+[6J\pm 2\sqrt{-\tfrac{1}{3}(I^3-27J^2)}]\Theta'=0,$$

which agrees with a result of M. HERMITE's.

There should, I think, be an identical equation of the form

$$JU'^2-IU'^2H'+4H'^3+M\Theta'^2=-\Phi'^2,$$

which would serve to express the square of the invariant Φ' in terms of the other invariants U', H', Θ', I, J; but assuming that such an equation exists, the form of the factor M remains to be ascertained. The invariant J (cubinvariant) of the form $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}\,|\,y, 1)^4$ contains Φ'^2, and it would be necessary to make use of the identical equation just referred to in order to reduce it to its simplest form; and (this being so) I have not sought for the expression of the cubinvariant of $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}\,|\,y, 1)^4$.

For the quintic we have the equations

$$\begin{aligned}&(a, b, c, d, e, f\,|\,x, 1)^5=0,\\ y=\;&(ax+b)\mathrm{B}\\ &+(ax^2+5bx+4c)\mathrm{C}\\ &+(ax^3+5bx^2+10cx+6d)\mathrm{D}\\ &+(ax^4+5bx^3+10cx^2+10dx+4e)\mathrm{E},\end{aligned}$$

and this leads to the system of equations

$$\left|\begin{array}{cccc}
y-b\mathrm{B}-4c\mathrm{C}-6d\mathrm{D}-4e\mathrm{E}, & -a\mathrm{B}-5b\mathrm{C}-10c\mathrm{D}-10d\mathrm{E}, & -a\mathrm{C}-5b\mathrm{D}-10c\mathrm{E}, & -a\mathrm{D}-5b\mathrm{E},\\
f\mathrm{E}, & y-b\mathrm{B}-4c\mathrm{C}-6d\mathrm{D}+e\mathrm{E}, & -a\mathrm{B}-5b\mathrm{C}-10c\mathrm{D}, & -a\mathrm{C}-5b\mathrm{D},\\
f\mathrm{D}, & 5e\mathrm{D}+f\mathrm{E}, & y-b\mathrm{B}-4c\mathrm{C}+4d\mathrm{D}+e\mathrm{E}, & -a\mathrm{B}-5b\mathrm{C},\\
f\mathrm{C}, & 5e\mathrm{C}+f\mathrm{D}, & 10d\mathrm{C}+5e\mathrm{D}+f\mathrm{E}, & y-b\mathrm{B}+6c\mathrm{C}+4d\mathrm{D}+e\mathrm{E},\\
f\mathrm{B}, & 5e\mathrm{B}+f\mathrm{C}, & 10d\mathrm{B}+5e\mathrm{C}+f\mathrm{D}, & 10c\mathrm{B}+10d\mathrm{C}+5e\mathrm{D}+f\mathrm{E},
\end{array}\right.
\begin{array}{c}
-a\mathrm{E}\\
-a\mathrm{D}\\
-a\mathrm{C}\\
-a\mathrm{B}\\
y+4b\mathrm{B}+6c\mathrm{C}+4d\mathrm{D}+e\mathrm{E}
\end{array}\Big|(1, x, x^2, x^3, x^4)=0,$$

and the transformed equation is obtained by equating to zero the determinant formed out of the matrix contained in this equation.

The determinant in question was calculated by the formula

	Div.
$\square = -12.345$	1
$+13.245$	2
-14.235	3
$+15.234$	4
-23.145	5
$+24.135$	6
-25.134	7
-34.125	8
$+35.124$	9
-45.123	10,

where the duadic symbols refer to the first and fifth columns, viz. 12 is the determinant formed out of the lines 1 and 2 of these columns, and so for the other like symbols; and the triadic symbols refer to the second, third, and fourth columns, viz. 345 is the determinant formed out of the lines 3, 4, 5 of these columns, and so for the other like symbols.

The ten divisions were separately calculated. It is to be noticed that these divisions other than 4 and 6 correspond to each other in pairs, while each of the divisions 4 and 6 corresponds to itself, as thus:

Div. 1, -10
2, -9
3, -7
5, -8
4, -4
6, -6,

viz. if in the place of

$$y;\ a, b, c, d, e, f;\ \text{B, C, D, E,}$$

we write

$$-y;\ f, e, d, c, b, a;\ \text{E, D, C, B,}$$

then division 1 becomes division 10 with its sign reversed, and so for divisions 2 and 9, 3 and 7, 5 and 8; while each of the divisions 4 and 6 is unaltered, except that the sign is reversed. But the corresponding divisions were each of them calculated, and the property in question was used as a verification. Another very convenient verification, which was employed for the several divisions, was obtained by putting

$$a=b=c=d=e=f=\text{B}=\text{C}=\text{D}=\text{E}=1,$$

upon which supposition the determinant becomes

$$\begin{vmatrix} y-15, & -26, & -16, & -6, & -1 \\ 1, & y-10, & -16, & -6, & -1 \\ 1, & 6, & y, & -6, & -1 \\ 1, & 6, & 16, & y+10, & -1 \\ 1, & 6, & 16, & 16, & y+15 \end{vmatrix}$$

and the values of the ten divisions respectively are

y^4,	y^3,	y^2,	y,	1	
		6,	−288,	+ 4608, −24576	1
		16,	−576,	+ 6144, −16384	2
		26,	−544,	+ 3584, −24576	3
1,	0,	−96,	0,	−28672, 0	4
				0, 0	5
				0, 0	6
		26,	+544,	+ 3584, +24576	7
				0, 0	8
		16,	+576,	+ 6144, +16384	9
		6,	+288,	+ 4608, +24576	10
1,	0,	0,	0,	0, 0	

A verification similar to this was in fact employed at each step of the calculation of a division: viz. in forming a product such as $(\lambda X+\mu Y+\&c.)(\lambda' X+\mu' Y+\&c.)$, where λ, μ, &c., λ', μ', .. &c. are numerical coefficients, and X, Y, &c. are monomial products of a, b, c, d, e, f and B, C, D, E, the sum of the numerical coefficients of the product is $(\lambda+\mu+\&c.)(\lambda'+\mu'+\&c.)$.

It was of course necessary to employ such verifications, as a test of the correctness of the several divisions, before proceeding to collect them together, but the collection itself affords an exceedingly good ultimate verification. The following is an exemplification: the terms in y which involve the product BCDE are obtained by the collection of the corresponding terms in the ten divisions, as follows:

			1	2	3	4	5	6	7	8	9	10	
y	BCDE.	a^2f^2	+ 1	− 2	− 2	+ 1	+ 1	− 2	− 2	+ 1	− 2	+ 1	
	+ 30	$abef$	+ 49	− 16	− 20	+ 4	− 25	+50	− 20	− 25	− 16	+ 49	
	+ 980	$acdf$	+ 80	+200	+184	− 148	+100		+184	+100	+200	+ 80	
	− 280	ace^2	+ 80	.	+ 80	− 440							
	− 180	ad^2e	.	− 60	− 60	− 60							
	− 280	b^2df	.	.		− 440	.		+ 80	.	.	+ 80	
	− 825	b^2e^2	.	.		− 825							
	− 180	bc^2f	.	.		− 60	.		− 60	.	− 60		
	+ 740	$bcde$	.	.		+ 740							
		bd^3											
		c^3e											
		c^2d^2											
	±1750=0		+ 210	+122	+182	−1228	+ 76	+48	+182	+ 76	+122	+210	=±1228

where it may be remarked that the greater part, but not all, of the component coefficients are divisible by 5. I soon observed in the process of summing the ten divisions that all the resulting coefficients should be divisible by 5 (the only exception is as to the terms in y^5 which contain B^5, C^5, D^5, and E^5 respectively), and the circumstance that they are so in each particular instance is as far as it goes a verification, which, however, only applies to those of the component coefficients which are not themselves divisible by 5. But it was known *à priori* (I will presently show how this is so) that the sum of the resulting coefficients should be equal to zero, and that they are so in fact is a verification as to *all* the coefficients. The foregoing specimen term BCDE is one which remains unaltered when B, C, D, E are changed into E, D, C, B; and on making the further change a, b, c, d, e, f into f, e, d, c, b, a, the coefficient of BCDE remains, as it should do, unaltered; this is a verification of the coefficients of the terms ace^3, b^3df; ad^2e, bc^2f, which are respectively interchanged by the substitution in question, but not of the other terms a^2f^2, $abef$, $acdf$, b^2e^2, $bcdf$, which are respectively unaltered by the substitution. I did *not* employ what would have been another convenient verification of the several divisions, viz. the comparison of their values on putting therein $a=b=c=d=e=f=1$, with the corresponding values as calculated independently from the determinant

$$\begin{vmatrix} y-B-4C-6D-E, & -B-5C-10D-10E, & -C-5D-10E, & -D-5E, & -E \\ E, & y-B-4C-6D+E, & -B-5C-10D, & -C-5D, & -D \\ D, & 5D+E, & y-B-4C+4D+E, & -B-5C, & -C \\ C, & 5C+D, & 10C+5D+E, & y-B+6C+4D+E, & -B \\ B, & 5B+C, & 10B+5C+D, & 10B+10C+5D+E, & y+4B+6C+4D+E \end{vmatrix}$$

The calculation of the ten divisions of this determinant would however itself have been a work of some labour.

The last-mentioned determinant is $=y^5$; in fact, equating it to zero, we have the transformed equation corresponding to the system of equations

$$(1,1,1,1,1,1)(x,1)^5=0,$$

$$y=(x+1)B+(x^3+5x+4)C+(x^3+5x^2+10x+6)D+(x^4+5x^3+10x^2+10x+4)E.$$

But the first of these equations is $(x+1)^5=0$, and the second is

$$y=(x+1)\{B+(x+4)C+(x^2+4x+6)D+(x^3+4x^2+6x+4)E\},$$

so that for each of the five equal roots $x=-1$, we have $y=0$, or the transformed equation in y is $y^5=0$.

And since upon writing $a=b=c=d=e=f=1$ the transformed equation becomes $y^5=0$, it is clear that in the coefficient of any monomial product of B, C, D, E, the sum of the numerical coefficients of the several monomial products of a, b, c, d, e, f must be $=0$, which is the property above referred to as affording a verification of the calculated expression of the transformed equation.

The final result is that the equations

$$(a, b, c, d, e, f\,(x, 1)^5=0,$$

$$y= (ax + b)\mathrm{B} + (ax^3+5bx + 4c)\mathrm{C} + (ax^3+5bx^3+10cx + 6d)\mathrm{D} + (ax^4+5bx^3+10cx^2+10dx+4e)\mathrm{E}$$

give for the transformed equation in y

$$(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}\,(y, 1)^5=0,$$

where

$\frac{1}{5}\mathfrak{C}=$

	B²		BC		BD	C²		BE	CD		CE	D²		DE		E²
ac	+2	ad	+6	ae	+4	+2	af	+1	+1	bf	+4	+2	cf	+6	df	+2
b^2	−2	bc	−6	bd	−4	+10	be	−1	+15	ce	−4	+10	de	−6	e^2	−2
				c^2	.	−12	cd	.	−16	d^2	.	−12				
	±2		±6		±4	±12		±1	±16		±4	±12		±6		±2

$\frac{1}{5}\mathfrak{D}=$

	B³		B²C		B²D	BC²		B²E	BCD	C³		BCE	BD²	C²D
a^2d	+2	a^2e	+4	a^2f	+1	+1	abf	+1	+10	+3	acf	∞	+6	+4
abc	−6	abd	+2	abe	+3	+19	ace	−4	−8	−4	ade	−12	−22	−46
b^3	+4	ac^2	−24	acd	∞	−52	ad^2	∞	−48	−20	b^2f	+8	+4	+20
		b^2c	+18	b^2d	−4	+20	b^2e	+3	+30	+25	bce	+4	+20	+70
				bc^2	.	+12	bcd	.	+16	−20	bd^2	.	−8	−80
							c^3	.	.	+16	c^2d			+32
	±6		±24		±4	±52		±4	±56	±44		±12	±30	±126

	BDE	C²E	CD²		BE²	CDE	D³		CE²	D²E		DE²		E³
adf	∞	−6	−4	aef	−1	−10	−3	af^2	−1	−1	bf^2	−4	cf^2	−2
ae^2	−8	−4	−20	bdf	+4	+8	+4	bef	−3	−19	cef	−2	def	+6
bcf	+12	+22	+46	be^2	−3	−30	−25	cdf	∞	+52	d^2f	+24	e^3	−4
bde	−4	−20	−70	c^2f		+48	+20	ce^2	+4	−20	de^2	−18		
c^2e		+8	+80	cde		−16	+20	d^2e		−12				
cd^2			−32	d^3			−16							
	±12	±30	±126		±4	±56	±44		±4	±52		±24		±6

$\frac{1}{2}\mathbb{C} =$

	B^4		B^3C		B^2D	B^2C^2		B^2E	B^2CD	BC^3		B^2CE	B^2D^2	BC^2D	C^4
a^3e	+1	a^3f	+1	a^2bf	+2	+9	a^2cf	+6	+10	−2	a^2df	+12	−6	∞	−4
a^2bd	−4	a^2be	+7	a^2ce	+20	−22	a^2de	−6	−18	−10	a^2e^2	−12	+10	−20	+5
a^2c^2	∞	a^2cd	−16	a^2d^2	−24	∞	ab^2f	−5	+17	+29	$abef$	+28	+38	+48	+6
ab^2c	+6	ab^2d	−22	ab^2e	−18	+31	$abce$	+8	+96	−92	$abde$	−36	−86	−8	−50
b^4	−3	abc^2	+48	$abcd$	+32	−56	abd^2	∞	−144	+40	ace^2	+32	+100	+72	−64
		b^3c	−18	ac^3	∞	+96	ac^2d	∞	+128	+128	acd^2	∞	−24	+144	+160
				b^3d	−12	−70	b^3e	−3	−105	+75	b^3f	−28	−14	+60	+25
				b^2c^2		+12	b^2cd	.	+16	−360	b^2ce	+4	−70	−240	+50
							bc^3	.	.	+192	b^2d^2	.	+52	−440	∞
											bc^2d	.	.	+384	−320
											c^4	.	.	.	+192
	±7		±56		±54	±148		±14	±267	±464		±76	±200	±708	±438

	B^3DE	BC^2E	BCD^2	C^3D		B^2E^2	$BCDE$	BD^3	C^2E	C^2D^2		BCE^2	BD^2E	C^2DE	CD^3
a^2ef	+5	−5	−7	+1	a^2f^2	+1	−1	−1	−1	+1	abf^2	+5	−5	−7	+1
$abdf$	−8	+92	−8	−36	$abef$	−5	+6	−4	−4	−25	$acef$	−8	+92	−8	−36
abe^2	−29	−67	+15	−25	$acdf$	+20	+196	+28	+28	−80	ad^2f	+60	+12	+72	−20
ac^2f	+60	+12	+72	−20	ace^2	−14	−56	+100	−28	+10	ade^2	−42	−54	+78	+190
$acde$	−12	+36	+196	+28	ad^2e	∞	−36	−68	+60	+336	b^2ef	−29	−67	+15	−25
ad^3	∞	∞	−72	+240	b^2df	−14	−56	−28	+100	+10	$bcdf$	−12	+36	+196	+28
b^2cf	−42	−54	+78	+190	b^2e^2	+12	−165	−75	−75	∞	bce^2	+26	−110	−510	−50
b^2de	+26	−110	−510	−50	bc^2f	.	−36	+60	−68	+336	bd^2e	.	+96	−60	−280
bc^2e	.	+96	−60	−280	$bcde$	.	+148	−140	−140	−940	c^2f	.	.	−72	+240
bcd^2	.	.	+296	−560	bd^3	.	.	+128	.	−120	c^2de	.	.	+296	−360
c^3d	.	.	.	+512	c^3e	.	.	.	+128	−120	cd^3	.	.	.	+512
					c^2d^2	.	.	.	.	+592					
	±91	±236	±657	±971		±33	±350	±316	±316	±1285		±91	±236	±657	±971

	BDE^2	C^2E^2	CD^2E	D^4		BE^3	CDE^2	D^3E		CE^3	D^2E^2		DE^3		E^4
aef^2	+12	−6	∞	−4	adf^2	+6	+10	−2	aef^2	+2	+9	af^3	+1	bf^3	+1
$adef$	+28	+38	+48	+6	ae^2f	−5	+17	+29	bdf^2	+20	−22	bef^2	+7	cef^2	−4
ae^3	−28	−14	+60	+25	bcf^2	−6	−18	−10	$bcef$	−18	+31	cdf^2	−16	d^2f^2	∞
b^2f^2	−12	+10	−20	+5	$bdef$	+8	+96	−92	c^2f^2	−24	∞	ce^2f	−22	de^2f	+6
$bcef$	−36	−86	−8	−50	be^3	−3	−105	+75	$cdef$	+32	−56	d^2ef	+48	e^4	−3
bd^2f	+32	+100	+72	−64	c^2ef	.	−144	+40	ce^3	−12	−70	de^3	−18		
bde^2	+4	−70	−240	+50	cd^2f	.	+128	+128	d^2f	.	+96				
c^2df	.	−24	+144	+160	cde^2	.	+16	−360	d^2e^2	.	+12				
c^2e^2	.	+52	−440	∞	d^2e	.	.	+192							
cd^2e	.	.	+384	−320											
d^4	.	.	.	+192											
	±76	±200	±708	±438		±14	±267	±464		±54	±148		±56		±7

$\mathfrak{f} =$

	B^2		B^3C		B^4D	B^5C^2		B^6E	B^6CD	B^6C^3
a^4f	+ 1	a^3bf	+ 15	a^3cf	+ 30	− 10	a^2df	+20	∞	− 10
a^4be	− 5	a^3ce	− 20	a^3de	− 30	∞	a^2e^3	−20	∞	∞
a^3cd	∞	a^3d^2	∞	a^2b^2f	− 15	+100	a^2bcf	−30	+320	− 60
a^2b^2d	+10	a^2b^2e	− 55	a^2bce	−100	−190	a^2bde	+30	−360	+ 50
a^2bc^2	∞	a^2bcd	+ 80	a^2bd^2	+120	∞	a^2c^2e	∞	−400	+ 120
ab^3c	−10	a^2c^3	∞	a^2c^2d	∞	+160	a^2cd^2	∞	+480	∞
b^5	+ 4	ab^3d	+ 70	ab^3e	+ 55	−260	ab^2f	+15	−140	+ 340
		ab^2c^2	−120	ab^2cd	− 80	+540	ab^2ce	−20	−440	−1020
		b^4c	+ 30	abc^3	∞	−480	ab^2d^2	∞	+960	− 100
				b^4d	+ 20	+200	abc^2d	∞	−640	+ 960
				b^3c^2	.	− 60	ac^4	∞	∞	− 640
							b^4e	+ 5	+300	− 500
							b^3cd	.	− 80	+1900
							b^2c^3	.	.	−1040
	±15		±195		±225	±1000		±70	±2060	±3370

	B^3CE	B^3D^2	B^3C^2D	BC^4		B^3DE	B^3C^2E	B^3CD^2	BC^2D	C^3
a^3ef	∞	+ 10	− 20	+ 5	a^3f^2	+ 5	− 5	− 5	+ 5	− 1
a^2bdf	+240	− 40	+ 20	− 80	a^2bef	− 30	+ 30	+ 60	− 130	+ 25
a^2b^2e	−240	− 50	+ 100	− 25	a^2cdf	+220	− 40	+ 380	− 380	+ 80
a^2c^2f	−120	+300	+ 20	− 40	a^2ce^2	−400	+ 40	− 200	+ 400	− 100
a^2cde	+120	−600	+ 60	+ 200	a^2d^2e	+180	∞	− 180	+ 300	∞
a^2d^3	∞	+360	∞	∞	ab^2df	− 40	+1070	− 640	+ 380	− 250
ab^3cf	−140	−220	+1190	+ 20	ab^2e^2	+185	−1145	+ 25	+ 125	∞
ab^2de	+240	+540	−1980	+ 250	abc^2f	−300	− 940	+1640	+ 940	− 360
abc^2e	−160	−500	−3160	+ 320	$abcde$	+ 60	+1020	−4380	+ 1140	+1000
$abcd^2$	∞	+120	+4080	− 800	abd^2	∞	∞	+3960	− 1200	∞
ac^3d	∞	∞	−1280	− 320	ac^2e	∞	− 320	−2000	− 2080	+ 960
b^4f	+ 80	+ 40	− 400	+ 500	ac^2d^2	∞	∞	+ 480	+ 960	−1600
b^3ce	− 20	+200	+ 850	−2750	b^3cf	+120	+ 160	− 620	+ 800	+ 750
b^3d^2	.	−160	+2600	∞	b^3de	− 80	+ 650	+2900	− 3500	∞
b^2c^2d	.	.	−2080	+5600	b^2c^2e	.	− 520	− 100	− 2600	− 300
bc^4	.	.	.	−2880	b^2cd^2	.	.	−1320	+14800	∞
					bc^3d	.	.	.	− 7680	+4800
					c^5	.	.	.	.	−2304
	±680	±1570	±8920	±6895		±810	±2970	±9445	±18710	±7615

	B^4E^2	B^4CDE	B^4D^3	BC^3E	BC^3D	C^4D		B^4CE^2	B^4D^2E	B^4C^2DE	B^4CD^3	BC^4E	C^4D^2
a^3bf^2	+ 5	+ 15	− 10	− 25	∞	+ 5	a^3cf^2	− 50	+ 70	+ 10	− 50	− 10	+ 20
a^3cef	− 70	− 60	+ 200	+ 80	− 210	+ 30	a^3def	+ 110	+ 80	− 340	+ 160	+ 130	− 30
a^3d^2f	+100	+ 240	+ 100	− 200	− 180	+ 120	a^2e^3	− 80	− 200	+ 400	∞	− 100	∞
a^3de^2	− 40	− 240	− 300	+ 200	+ 600	− 150	ab^2f^2	+ 80	− 70	− 40	+ 20	− 25	∞
ab^2ef	+ 30	− 30	− 90	+ 70	∞	− 125	$abcef$	− 520	− 460	+ 620	+ 380	− 20	− 500
$abcdf$	−100	+1380	− 640	+ 260	+1920	−1180	abd^2f	+ 700	+ 460	− 1240	− 760	−1000	+ 540
$abce^2$	+ 70	−2820	− 500	− 360	+ 450	+ 500	$abde^2$	− 190	+ 620	− 2140	− 200	+1000	+ 750
abd^2e	∞	+1980	+1540	− 300	−3180	+1500	ac^2df	− 400	+ 200	− 1720	+ 2840	− 80	− 880
ac^3f	∞	−1200	+1000	− 560	+1080	− 240	ac^2e^2	+ 280	−2000	− 2240	− 2000	+ 880	+ 1400
ac^2de	∞	+ 240	−3000	+ 240	−8160	+ 3120	acd^2e	∞	+1080	+ 2520	− 6440	−1200	− 120
acd^3	∞	∞	+1440	∞	+5040	− 4800	ad^4	∞	∞	∞	+ 4320	∞	− 3600
b^3df	+ 40	+ 240	+ 120	+2000	−1800	+ 500	b^3ef	+ 160	+ 430	− 200	− 500	+ 125	∞
b^3e^3	− 35	+ 975	+ 500	−2125	∞	∞	b^2cdf	− 20	− 440	+ 5320	− 2240	+3000	− 700
b^2c^2f	.	− 60	− 400	−1060	+1620	+ 2100	b^2ce^2	− 70	+ 650	− 2950	+ 2750	−3500	∞
b^2cde	.	− 660	+ 600	+3700	+2700	− 6500	b^2d^2e	.	− 420	+ 4800	+ 7400	∞	− 3000
b^2d^3	.	.	− 560	∞	+9600	∞	bc^2f	.	.	− 3240	+ 1800	−1680	+ 3960
bc^3e	.	.	.	−1920	−5400	− 4800	bc^2de	.	.	− 2040	−10200	+4400	− 5600
bc^2d^2	.	.	.	.	−4080	+17600	bcd^3	.	.	.	+ 2720	∞	+20400
c^4d	.	.	.	.	.	− 7680	c^4e	.	.	.	.	−1920	− 7200
							c^3d^2	.	.	.	.	.	− 5440
	±245	±5070	±5500	±6550	±23010	±25475		±1330	±3590	±14910	±22390	±9535	±27070

	B^7DE^7	BC^2E^7	BCD^2E	BD^4	C^3DE	C^2D^3
a^3df^7	+ 50	− 70	− 10	+ 10	+ 50	− 20
a^2ef	− 80	+ 70	+ 40	+ 25	− 20	∞
$abcf^7$	− 110	− 80	+ 340	− 130	− 160	+ 30
$abdef$	+ 520	+ 460	− 620	+ 20	− 380	+ 500
abe^3	− 160	− 430	+ 200	− 125	+ 500	∞
ac^3ef	− 700	− 460	− 1240	+ 1000	+ 760	− 540
acd^2f	+ 400	− 200	+ 1720	+ 80	− 2840	+ 880
$acde^3$	+ 20	+ 440	− 5320	− 3000	+ 2240	+ 700
ad^3e	∞	∞	+ 3240	+ 1680	− 1800	− 3960
b^3f^7	+ 80	+ 200	− 400	+ 100	∞	∞
b^2cef	+ 190	− 620	+ 2140	− 1000	+ 200	− 730
b^2d^2f	− 280	+ 2000	+ 2240	− 880	+ 2000	− 1400
b^2de^2	+ 70	− 650	+ 2950	+ 3500	− 2750	∞
bc^2df	.	− 1080	− 2520	+ 1200	+ 6440	+ 120
bc^3e^2	.	+ 420	− 4800	∞	− 7400	+ 3000
bcd^2e	.	.	+ 2040	− 4400	+ 10200	+ 5600
bd^4	.	.	.	+ 1920	∞	+ 7200
c^4f	.	.	.	.	− 4320	+ 3600
c^3de	.	.	.	.	− 2720	− 20400
c^2d^3	.	.	.	.	.	+ 5440
	±1330	±3590	±14910	±9535	±22390	±27070

	B^5E^3	$BCDE^3$	BD^2E	C^2E^3	C^2D^2E	CD^4
a^3ef^3	− 5	− 15	+ 25	+ 10	∞	− 5
$abdf^3$	+ 70	+ 60	− 80	− 200	+ 210	− 30
abe^2f	− 30	+ 30	− 70	+ 90	∞	+ 125
ac^2f^3	− 100	− 240	+ 200	− 100	+ 180	− 120
$acdef$	+ 100	− 1380	− 260	+ 640	− 1920	+ 1180
ace^3	− 40	− 240	− 2000	− 120	+ 1800	− 500
ad^2f	∞	+ 1200	+ 560	− 1000	− 1080	+ 240
ad^2e^2	∞	+ 60	+ 1060	+ 400	− 1620	+ 2100
b^3ef^3	+ 40	+ 240	− 200	+ 300	− 600	− 150
b^2def	− 70	+ 2820	+ 360	+ 500	− 450	− 500
b^2e^3	+ 35	− 975	+ 2125	− 500	∞	∞
bc^2ef	.	− 1980	+ 300	− 1540	+ 3180	+ 1500
bcd^2f	.	− 240	− 240	+ 3000	+ 8160	− 3120
$bcde^2$	.	+ 660	− 3700	− 600	− 2700	+ 6500
bd^2e	.	.	+ 1920	∞	+ 5400	+ 4800
c^3df	.	.	.	− 1440	− 5040	+ 4800
c^2e^2	.	.	.	+ 560	− 9600	∞
c^2d^2e	.	.	.	.	+ 4080	− 17600
cd^4	.	.	.	.	.	+ 7680
	±245	±5070	±6550	±5500	±23010	±25475

	BCE^3	BD^2E^2	CD^2E^2	CD^3E	D^5
a^2f^3	− 5	+ 5	+ 5	− 5	+ 1
$abef^3$	+ 30	− 30	− 60	+ 130	− 25
$acdf^3$	−220	+ 40	− 380	+ 380	− 80
ace^2f	− 40	−1070	+ 640	− 380	+ 250
ad^2ef	+300	+ 940	−1640	− 940	+ 360
ade^3	−120	− 160	+ 620	− 800	− 750
b^2df^3	+400	− 40	+ 200	− 400	+ 100
b^2e^2f	−185	+1145	− 25	− 125	∞
bc^2f^3	−180	∞	+ 180	− 300	∞
$bcdef$	− 60	−1020	+4380	+ 1140	−1000
bce^3	+ 80	− 650	−2900	+ 3500	∞
bd^3f	.	+ 320	+2000	+ 2080	− 960
bd^2e^2	.	+ 520	+ 100	+ 2600	+3000
c^3ef	.	.	−3960	+ 1200	∞
c^2d^2f	.	.	− 480	− 960	+1600
c^2de^2	.	.	+1320	−14800	∞
cd^3e	.	.	.	+ 7680	−4800
d^5	.	.	.	.	+2304
	±810	±2970	±9445	±18710	±7615

	BDE^3	C^2E^3	CD^2E^2	D^4E
abf^3	∞	− 10	+ 20	− 5
$acef^3$	− 240	+ 40	− 20	+ 80
ad^2f^3	+ 120	− 300	− 20	+ 40
ade^2f	+ 140	+ 220	−1190	− 20
ae^4	− 80	− 40	+ 400	− 500
b^2ef^3	+ 240	+ 50	− 100	+ 25
$bcdf^3$	− 120	+ 600	− 60	− 200
bce^2f	− 240	− 540	+1980	− 250
bd^2ef	+ 160	+ 500	+3160	− 320
bde^3	+ 20	− 200	− 850	+2750
c^3f^3	.	− 360	∞	∞
c^2def	.	− 120	−4080	+ 800
c^2e^3	.	+ 160	−2600	∞
cd^3f	.	.	+1280	+ 320
cd^2e^2	.	.	+2080	−5600
d^4e	.	.	.	+2880
	± 680	±1570	±8920	±6895

	BE^4	CDE^3	D^3E^3		CE^4	D^2E^3
acf^3	−20	∞	+ 10	adf^3	− 30	+ 10
$adef^3$	+30	− 320	+ 60	ae^2f^3	+ 15	− 100
ae^3f	−15	+ 140	− 340	bcf^3	+ 30	∞
b^2f^3	+20	∞	∞	$bdef^3$	+100	+ 190
$bcef^3$	−30	+ 360	− 50	be^2f	− 55	+ 260
bd^2f^2	∞	+ 400	− 120	c^2ef^2	−120	∞
bde^2f	+20	+ 440	+1020	cd^2f^3	∞	− 160
be^4	− 5	− 300	+ 500	cde^2f	+ 80	− 540
c^2df^3	.	− 480	∞	ce^4	− 20	− 200
c^2e^2f	.	− 960	+ 100	d^3ef	.	+ 480
cd^2ef	.	+ 640	− 960	d^2e^3	.	+ 60
cde^3	.	+ 80	−1900			
d^4f	.	.	+ 640			
d^3e^2	.	.	+1040			
	±70	±2060	±3370		±225	±1000

	DE^4	E^5
aef^3	− 15	− 1
bdf^3	+ 20	+ 5
be^2f^2	+ 55	cdf^3 ∞
c^2f^3	∞	ce^2f^2 −10
$cdef^2$	− 80	d^2ef^2 ∞
ce^2f	− 70	de^3f +10
d^2f^3	∞	e^5 − 4
d^2e^2f	+120	
de^4	− 30	
	±195	±15

Upon writing $(B, C, D, E)=(x^3, xy^2, xy^2, y^3)$, the foregoing values of $\mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}$ become covariants of the quintic $(a, b, c, d, e, f)(x, y)^5$. In fact we have

$$\tfrac{1}{2}\mathbb{C}=2(\text{Tab. No. 15}),$$

$$\tfrac{1}{2}\mathbb{D}=2(\text{Tab. No. 18}),$$

$$\tfrac{1}{2}\mathbb{E}=(\text{Tab. No. 13})^2(\text{Tab. No. 14})-3(\text{Tab. No. 15})^3,$$

$$\mathbb{F}=(\text{Tab. No. 13})^3(\text{Tab. No. 17})-2(\text{Tab. No. 15})(\text{Tab. No. 18}),$$

where the Tables referred to are those of my Fifth Memoir on Quantics, Tab. No. 13 being the quintic itself. This is a further verification of the foregoing result.

I will conclude by showing how the formula may be applied to the reduction of the general quintic equation to Mr. JERRARD's form $x^5+ax+b=0$. It was long ago remarked by Professor SYLVESTER that TSCHIRNHAUSEN's Transformation, in its original form, gave the means of effecting this reduction. In fact, if the transforming equation be

$$y=\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4,$$

then the equation in y is of the form

$$(\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F})(y, 1)^5=0,$$

where $\mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}$ are, in regard to $\alpha, \beta, \gamma, \delta, \varepsilon$, of the degrees 1, 2, 3, 4, 5 respectively. And by assuming, say α a linear function of $\beta, \gamma, \delta, \varepsilon$, we may make $B=0$, and we have then $\mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}$ functions of the degrees 2, 3, 4, 5 respectively of the quantities $\beta, \gamma, \delta, \varepsilon$: and these can be determined by means of a quadric equation and a cubic equation in such manner that $\mathbb{C}=0$, $\mathbb{D}=0$, in which case the equation in y will be of the required form. For considering $\beta, \gamma, \delta, \varepsilon$ as the coordinates of a point in space, the equations $\mathbb{C}=0$, $\mathbb{D}=0$ will be the equations of a quadric surface and a cubic surface respectively, and if $\beta, \gamma, \delta, \varepsilon$ be the coordinates of a point on the curve of intersection, the required conditions will be satisfied. And by combining with the equation of this the quadric surface, the equation of any tangent plane thereto (or by the different process which is made use of in the sequel), we may, by means of a quadric equation, find a generating line of the quadric surface, and then, by means of a cubic equation, a point of intersection of this line with the cubic surface, *i. e.* a point the coordinates whereof give the required values $\beta, \gamma, \delta, \varepsilon$. And similarly for the new form of TSCHIRNHAUSEN's Transformation; the only difference being that, starting with an equation in y which contains the four arbitrary quantities B, C, D, E, we obtain in the first instance an equation $(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F})(y, 1)^5=0$ wanting the second term. And then B, C, D, E are to be so determined that $\mathbb{C}=0$, $\mathbb{D}=0$.

To proceed with the reduction, I write the foregoing value of $\mathbb{C}$ in the form

$$\tfrac{1}{4}\mathbb{C}=\left|\begin{matrix} 4ac-4b^2, & 6ad-6bc, & 4ae-4bd, & af-be \\ 6ad-6bc, & 4ae+20bd-24c^2, & af+15be-16cd, & 4bf-4ce \\ 4ae-4bd, & af+15be-16cd, & 4bf+20ce-24d^2, & 6cf-6de \\ af-be, & 4bf-4ce, & 6cf-6de, & 4df-4e^2 \end{matrix}\right|(B, C, D, E)^4,$$

which for shortness may be represented by

$$\tfrac{1}{3}\mathfrak{C}=\left(\begin{array}{cccc} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & p \end{array}\right)(B, C, D, E)^3,$$

or say

$$\tfrac{1}{3}\mathfrak{C}=(\Omega)(B, C, D, E)^3.$$

Now, by a formula in my memoir "On the Automorphic Linear Transformation of a Bipartite Quadric Function*," if Υ denote any skew symmetric matrix of the order 4, then if

$$(B, C, D, E)=(\Omega^{-1}(\Omega-\Upsilon)(\Omega+\Upsilon)^{-1}\Omega)(B', C', D', E'),$$

in which formula Ω^{-1}, $\Omega-\Upsilon$, $(\Omega+\Upsilon)^{-1}$, Ω are all matrices which are to be compounded together into a single matrix, we have identically

$$(\Omega)(B, C, D, E)^3=(\Omega)(B', C', D', E')^3.$$

Let Q denote the determinant $|\Omega+\Upsilon|$, then the terms of the matrix $(\Omega+\Upsilon)^{-1}$ are respectively divided by Q, and we may write

$$(\Omega+\Upsilon)^{-1}=\frac{1}{Q}.Q(\Omega+\Upsilon)^{-1},$$

where $Q(\Omega+\Upsilon)^{-1}$ is the matrix obtained from the matrix $(\Omega+\Upsilon)^{-1}$ by multiplying each term by Q, the terms of $Q(\Omega+\Upsilon)^{-1}$ being thus rational and integral functions of the terms of the matrix $(\Omega+\Upsilon)$. Hence if, instead of the before-mentioned relation between (B, C, D, E) and (B′, C′, D′, E′), we assume

$$(B, C, D, E)=(\Omega^{-1}(\Omega-\Upsilon)Q(\Omega+\Upsilon)^{-1}\Omega)(B', C', D', E'),$$

we find

$$(\Omega)(B, C, D, E)^3=Q^3(\Omega)(B', C', D', E')^3.$$

And if the matrix Υ is such that we have $Q=0$, *i. e.* Det. $(\Omega+\Upsilon)=0$ (which is a quadric relation between the terms of the skew matrix, that is, each term is contained therein in the first and second powers only), then the equation becomes

$$(\Omega)(B, C, D, E)^3=0.$$

It is clear that this can only be the case in consequence of the coefficients of transformation in the equation

$$(B, C, D, E)=(\Omega^{-1}(\Omega-\Upsilon)Q(\Omega+\Upsilon)^{-1}\Omega)(B', C', D', E')$$

being such that there shall exist at least two linear relations between the quantities (B, C, D, E), and I assume (without stopping to prove it) that they are such that the number of such linear relations is in fact two. That is, the last-mentioned equation establishes between the quantities (B, C, D, E) two linear relations, in virtue whereof

* Philosophical Transactions, vol. cxlviii. (1858), see p. 44.

$\mathfrak{C}=0$. And this being so, we may, without loss of generality, write $D'=0$, $E'=0$, or put

$$(B, C, D, E)=(\Omega^{-1}(\Omega-\Upsilon)Q(\Omega+\Upsilon)^{-1}\Omega)(B', C', 0, 0);$$

so that B, C, D, E are linear functions of B′, C′, such that $\mathfrak{C}=0$. And then substituting these values for (B, C, D, E), we find $\mathfrak{B}$ a cubic function of B′, C′; so that, putting $\mathfrak{B}=0$, we have a cubic equation to determine the ratio $B' : C'$.

The foregoing reasoning presents no real difficulty, but it is expressed by means of a very condensed notation, and it may be proper to illustrate it by the case of the quadric function $x^2+y^2+z^2$. Considering the equations

$$\begin{aligned} x&=(1+\lambda^2-\mu^2-\nu^2)x'+ \quad 2(\lambda\mu-\nu)y'+ \quad 2(\lambda\nu+\mu)z', \\ y&= \quad 2(\lambda\mu+\nu)x'+(1-\lambda^2+\mu^2-\nu^2)y'+ \quad 2(\mu\nu-\lambda)z', \\ z&= \quad 2(\nu\lambda-\mu)x'+ \quad 2(\mu\nu+\lambda)y'+(1-\lambda^2-\mu^2+\nu^2)z', \end{aligned}$$

these equations, if the expressions for x, y, z had been divided by $1+\lambda^2+\mu^2+\nu^2$, would have given

$$x^2+y^2+z^2=x'^2+y'^2+z'^2.$$

Hence they actually do give

$$x^2+y^2+z^2=(1+\lambda^2+\mu^2+\nu^2)^2(x'^2+y'^2+z'^2);$$

or if

$$1+\lambda^2+\mu^2+\nu^2=0,$$

they give

$$x^2+y^2+z^2=0.$$

But if

$$1+\lambda^2+\mu^2+\nu^2=0,$$

then

$$\begin{aligned} &1+\lambda^2-\mu^2-\nu^2 : 2(\lambda\mu-\nu) \quad : 2(\lambda\nu+\mu) \\ =&2(\lambda\mu+\nu) \quad : 1-\lambda^2+\mu^2-\nu^2 : 2(\mu\nu-\lambda) \\ =&2(\nu\lambda-\mu) \quad : 2(\mu\nu+\lambda) \quad : 1-\lambda^2-\mu^2+\nu^2; \end{aligned}$$

so that we have

$$x : y : z=1+\lambda^2-\mu^2-\nu^2 : 2(\lambda\mu+\nu) : 2(\nu\lambda-\mu),$$

which is the same result as would have been found by writing $y'=z'=0$, and which comes to saying that x, y, z are not independent, but are connected by two linear relations.

The equation Det. $(\Omega+\Upsilon)=0$, written at length, will be

$$\begin{vmatrix} a & , & h-\varpi, & g+\sigma, & l+\lambda \\ h+\varpi, & b & , & f-\varrho, & m+\mu \\ g-\sigma, & f+\varrho, & c & , & n+\nu \\ l-\lambda, & m-\mu, & n-\nu, & p & \end{vmatrix}=0,$$

where $\lambda, \mu, \nu, \varrho, \sigma, \varpi$ are the arbitrary constituents of the skew matrix; or developing, this is

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & p \end{vmatrix}$$

$$+\left|\begin{array}{cccccc} bc-f^2, & fg-ch, & hf-bg, & mg-nh, & fm-bn, & -fn+cm \\ fg-ch, & ca-g^2, & gh-af, & -gl+an, & nh-lf, & gn-cl \\ hf-bg, & gh-af, & ab-h^2, & hl-am, & -hm+bl, & lf-mg \\ mg-nh, & fm-bn, & -fn+cm, & ap-l^2, & ph-lm, & pg-ln \\ -gl+an, & nh-lf, & gn-cl, & ph-lm, & bp-m^2, & pf-mn \\ hl-am, & -hm+bl, & lf-mg, & pg-ln, & pf-mn, & cp-n^2 \end{array}\right| (\lambda, \mu, \nu, \varrho, \sigma, \tau)^2$$

$$+(\lambda\varrho+\mu\sigma+\nu\tau)^2=0,$$

the first term whereof, substituting for (a, b, c, f, g, h, l, m, n, p) their values, is in fact equal to the discriminant a^4f^4+ &c. of the quintic $(a, b, c, d, e, f)(X, Y)^5$. There is no loss of generality in putting all but two of the quantities $(\lambda, \mu, \nu, \varrho, \sigma, \tau)$ equal to zero; in fact this leaves in the formulæ a single arbitrary quantity, which is the right number, since the ratios B : C : D : E have to satisfy only the two conditions $\mathfrak{C}=0$, $\mathfrak{B}=0$.

Addition, Nov. 10, 1862.

I take the opportunity of remarking, with reference to my memoir "On a New Auxiliary Equation in the Theory of Equations of the Fifth Order*," that I recently discovered that the auxiliary equation there considered is in fact due to Jacobi, who, in his paper, "Observatiunculæ ad theoriam æquationum pertinentes†," under the heading "Observatio de æquatione sexti gradus ad quam æquationes quinti gradus revocari possunt," gives the theory, and observes that the equation is of the form

$$\varphi^6+a_4\varphi^4+a_3\varphi^3+a_2=32\sqrt{\square}\varphi,$$

and mentions that the value of a_2 is easily found to be (I adapt his notation to the denumerate form $(a, b, c, d, e, f)(v, 1)^5=0$)

$$=40ae-16bd+6c^2$$

(this ought, however, to be divided by -2), but that the values of a_4, a_3 "paullo ampliores calculos poscunt."

The value of the coefficient in question is correctly obtained (page 270 of my memoir) in the form

$$\begin{array}{l} -c^2 \\ +2(-16ae+4bd-c^2) \\ +12ae; \end{array}$$

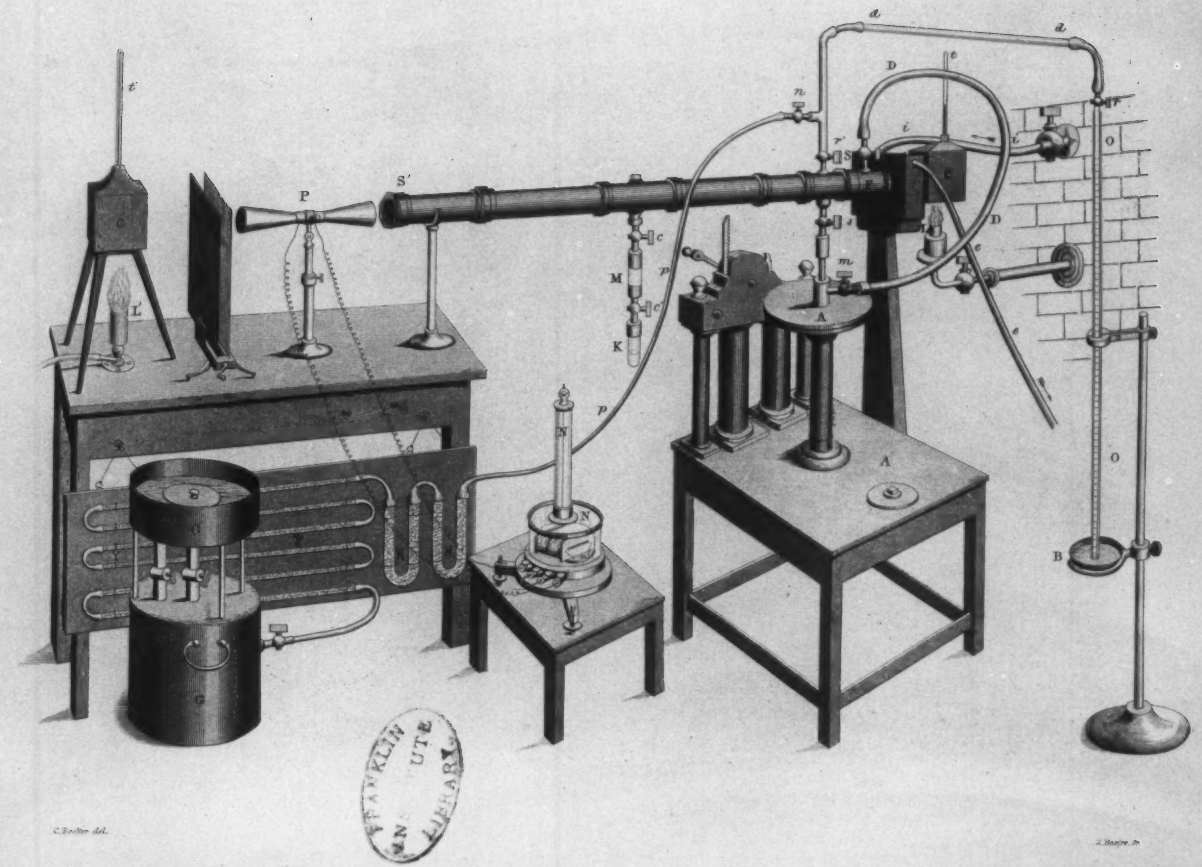
but the reduced value is given in two places (page 271) as equal to

$$\begin{array}{lll} -32\,ae, & \text{this should be} & -20\,ae, \\ +\ 8\,bd, & \text{,,} & +\ 8\,bd, \\ -\ 3c^2, & \text{,,} & -\ 3c^2. \end{array}$$

The last-mentioned correct value was used in obtaining the coefficient for the standard form, which coefficient is given correctly, page 274.

* Philosophical Transactions, vol. cli. (1861) pp. 263–276. † Crelle, t. xiii. (1835) pp. 340–352.

Phil. Trans. MDCCCLXII. Plate I.



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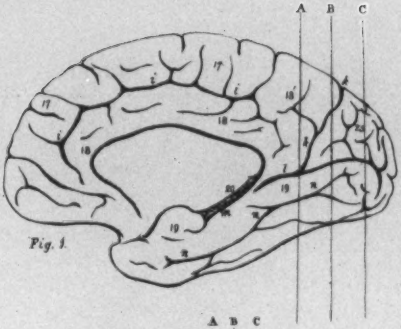

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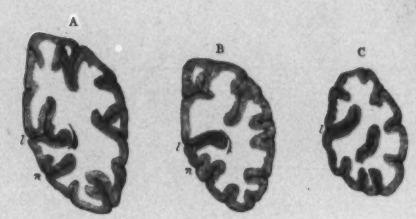

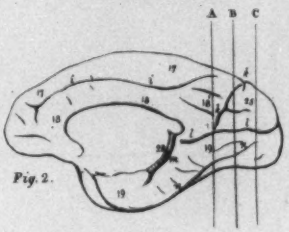

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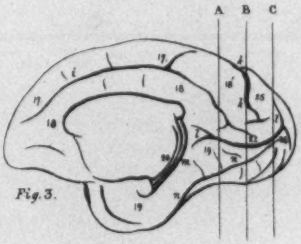

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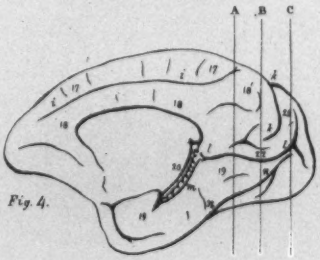


Fig. 4.

W. H. F. Del.

E. M. Williams Lith.

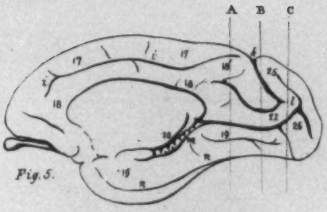
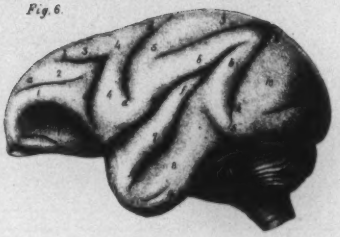


Fig. 5.

Fig. 7.

Fig. 6.



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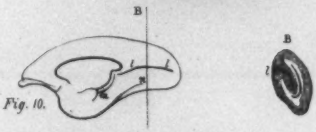


Fig. 10.

Fig. 5.[a]

Fig. 9.

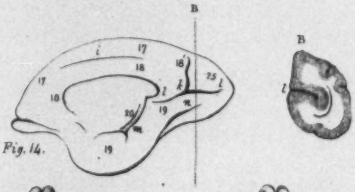


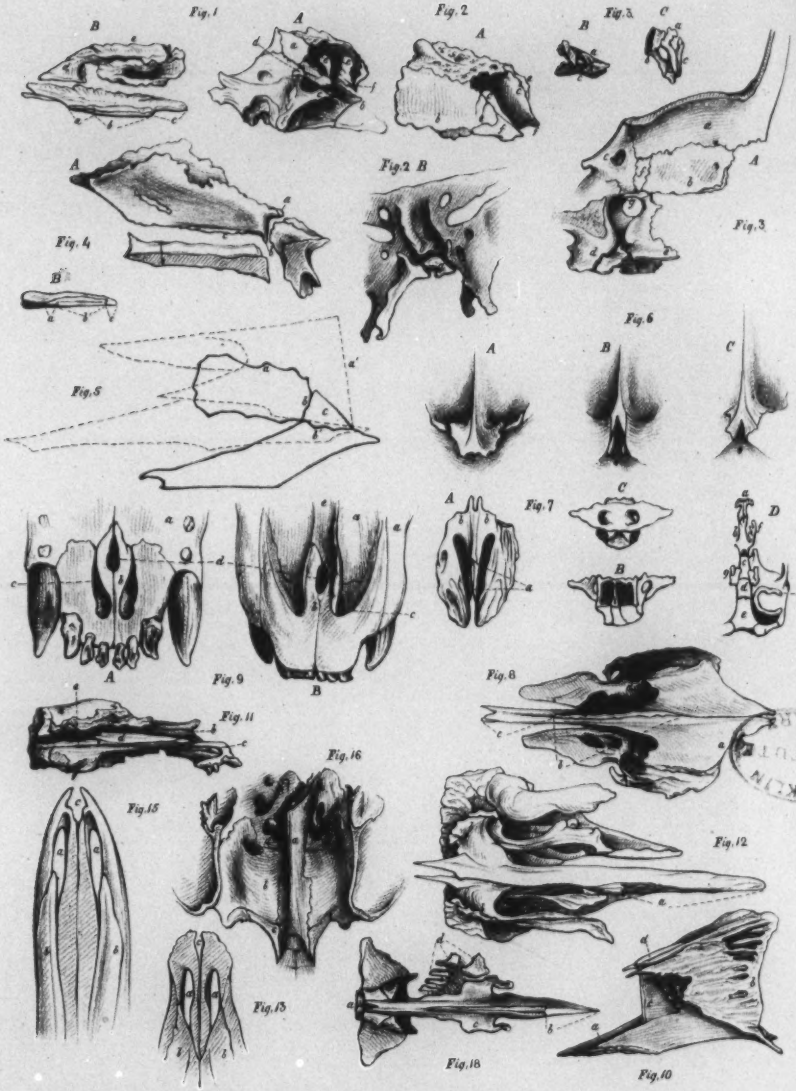
Fig. 14.

Fig. 11.

Fig. 12.

Fig. 13.

Fig. 15.



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J. Ollard del.

E. M. Williams, Lith.

Fig. 14

Fig. 17

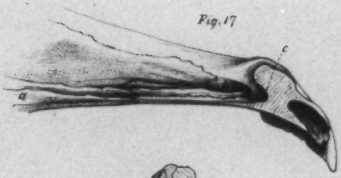


Fig. 21

Fig. 25

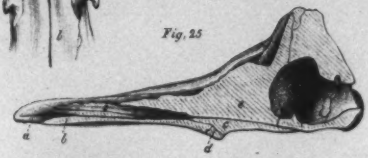


Fig. 19

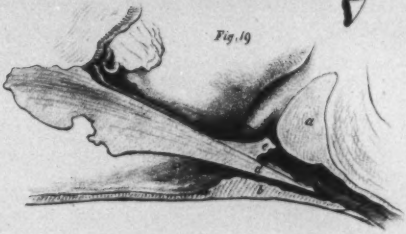


Fig. 20

Fig. 26

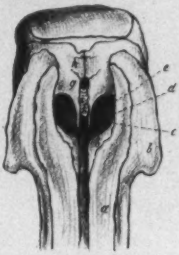


Fig. 22

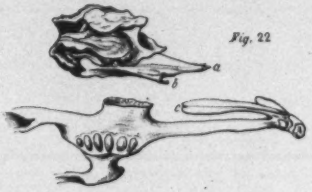


Fig. 28

Fig. 23

Fig. 24

Fig. 27

Fig. 30

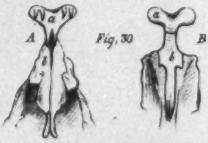
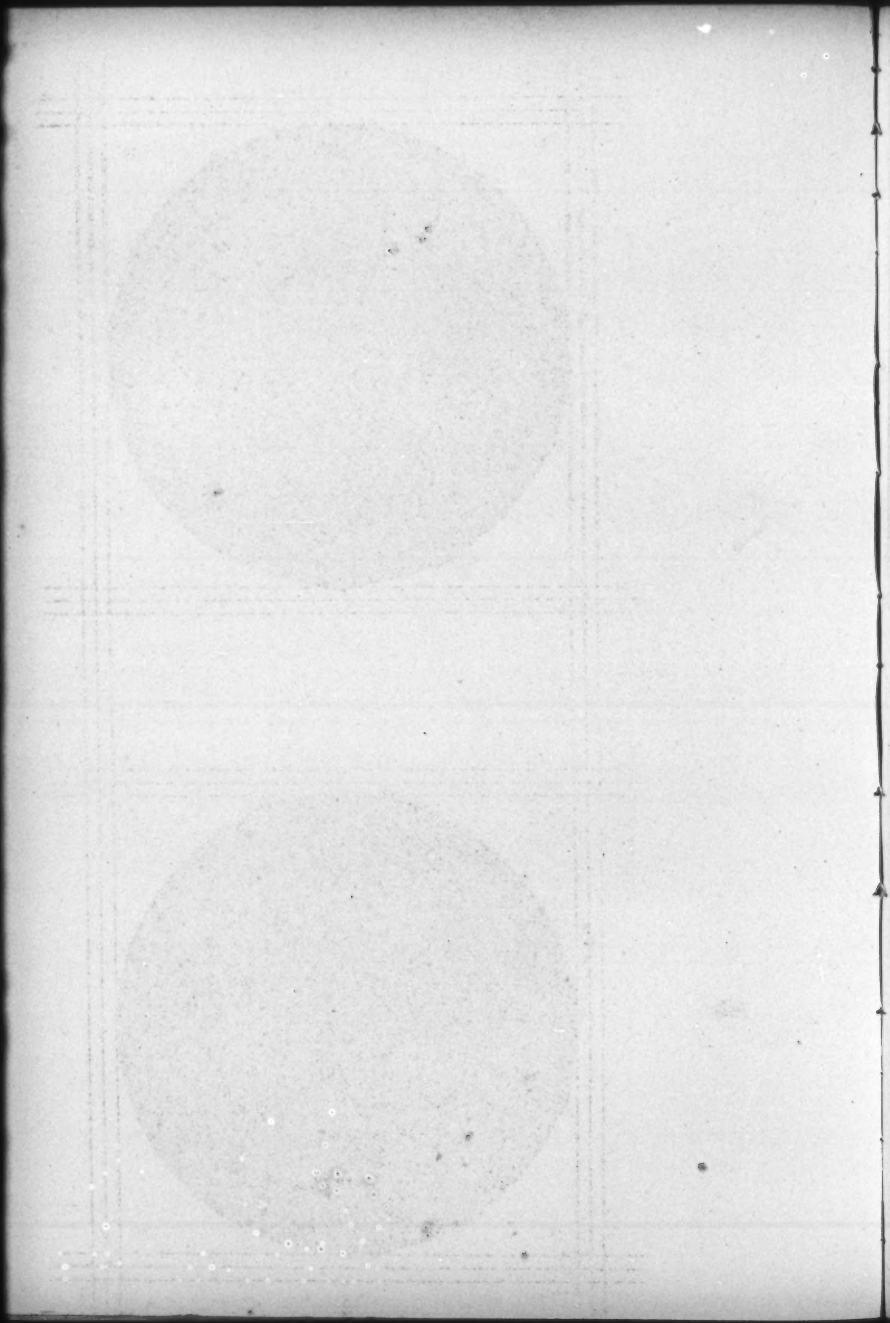


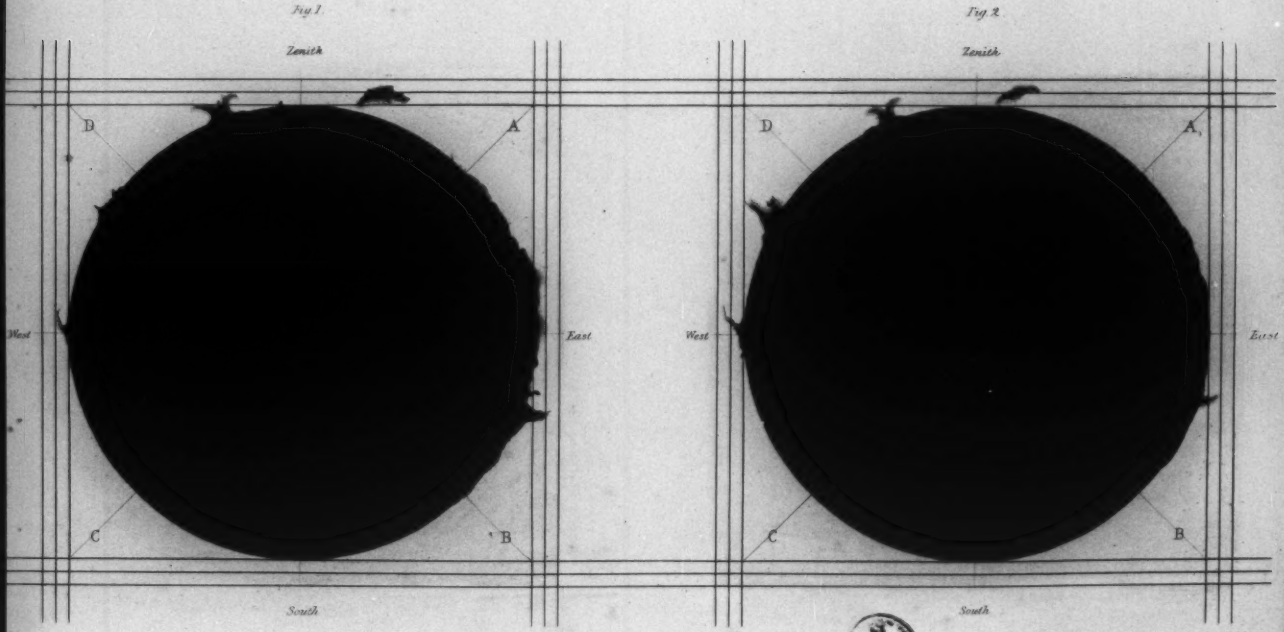
Fig. 29

Fig. 31

J. Clavard del.

E.M. Williams Lith.





Fac-simile of hand drawing

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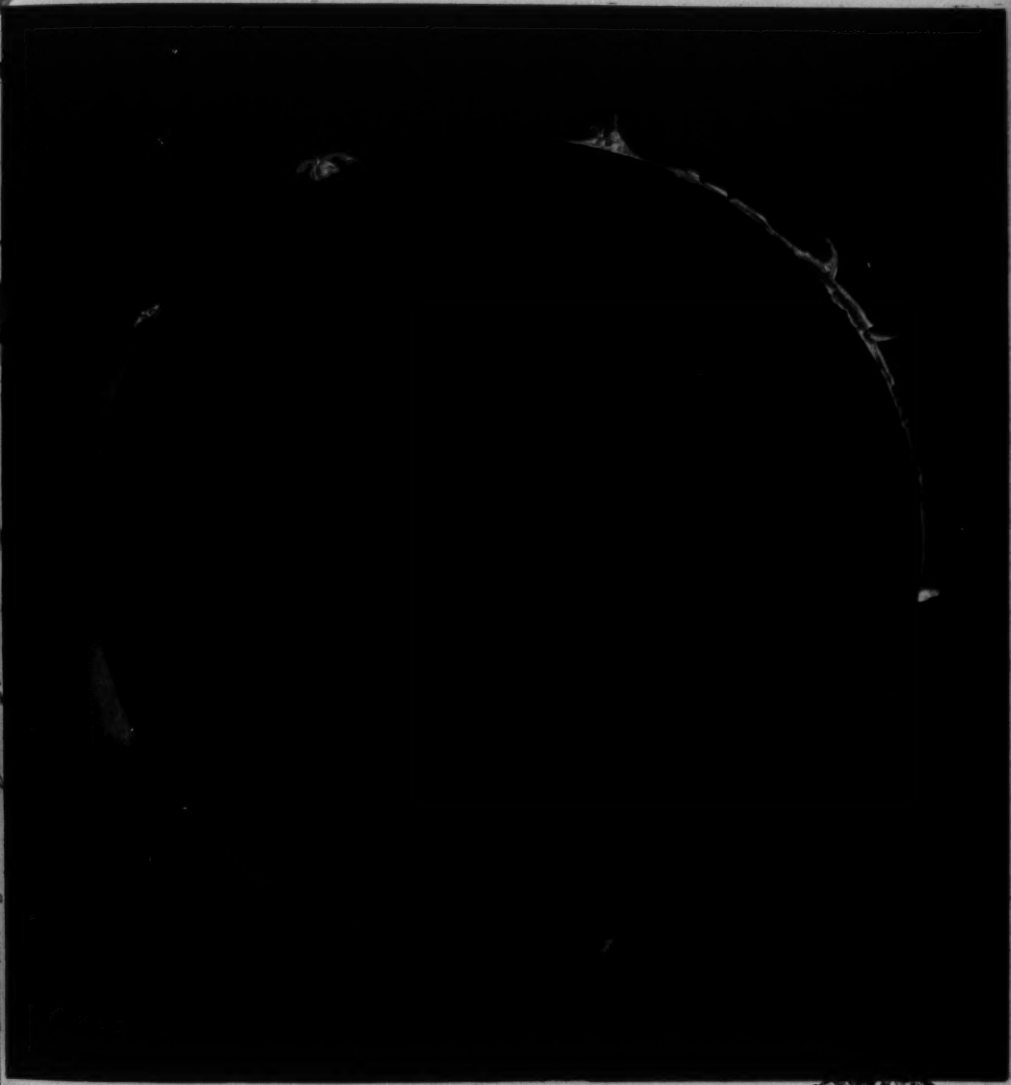
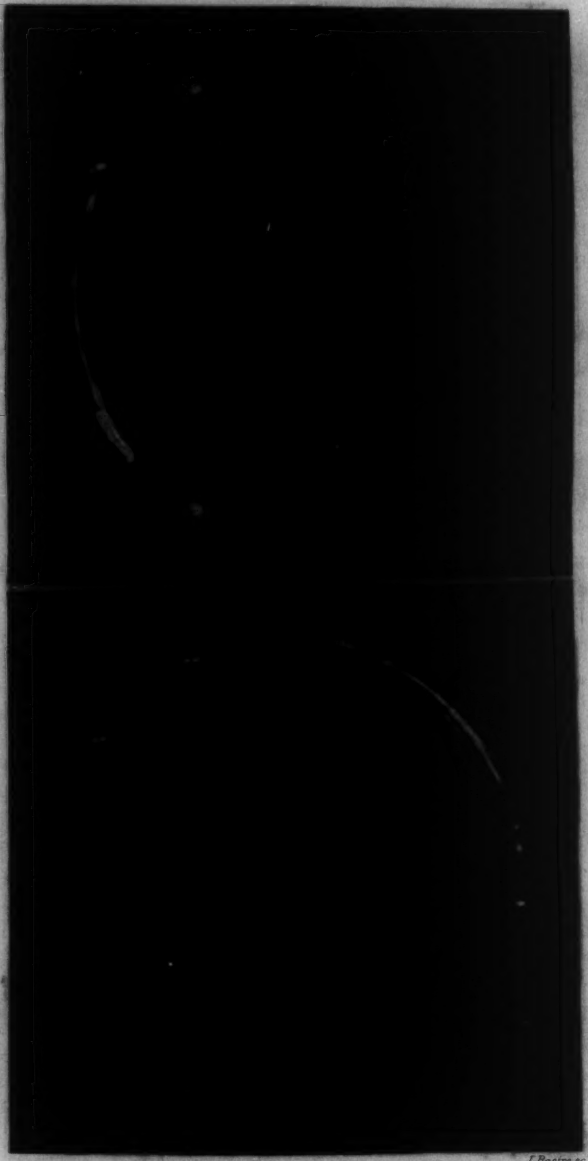
APPEARANCE OF PHENOMENA
IMMEDIATELY AFTER THE BEGINNING OF TOTALITY.

Vincent Brooks, lith.



APPEARANCE OF PHENOMENA
IMMEDIATELY PREVIOUS TO THE END OF TOTALITY.

FAC SIMILE OF Nº 25 PHOTOGRAPH - FIRST TOTALITY.



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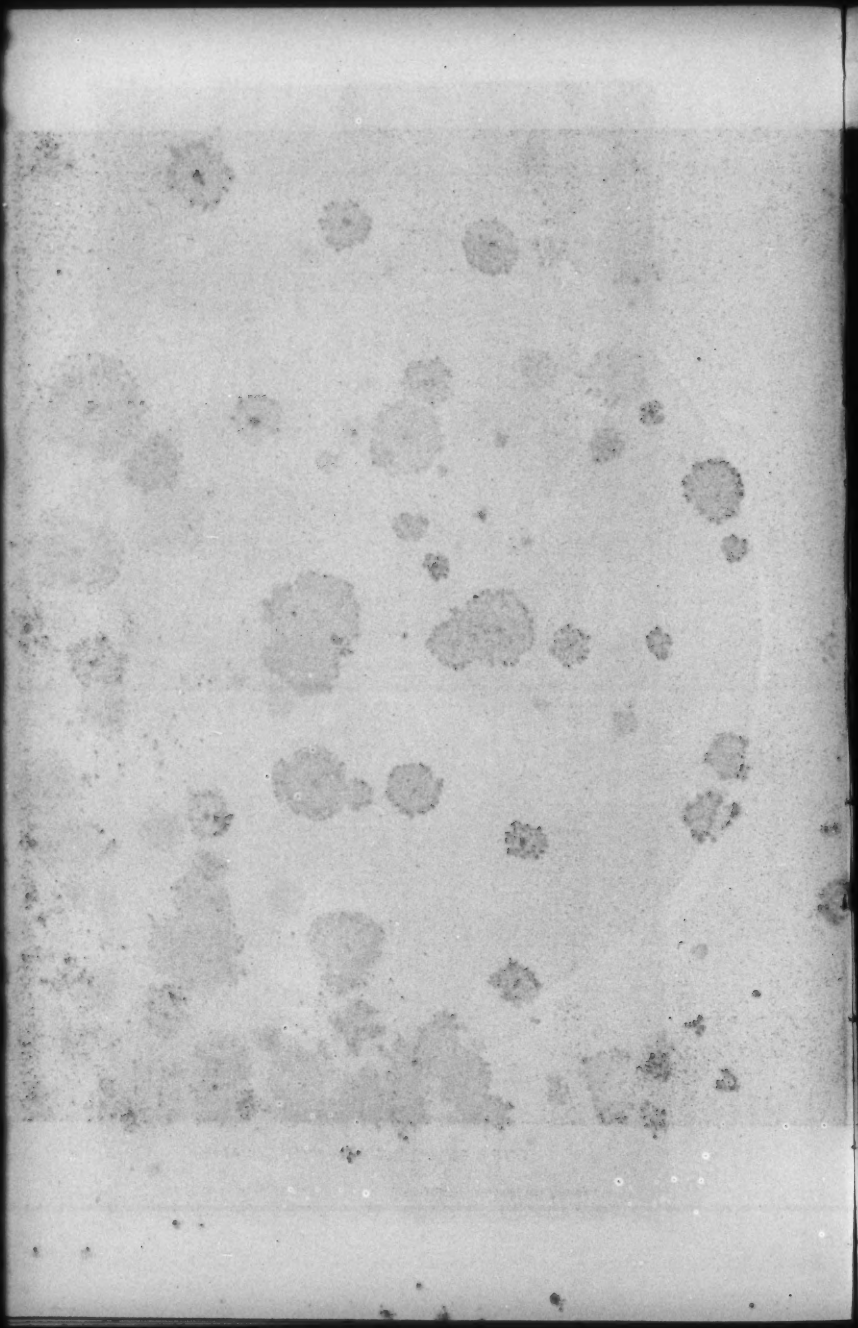
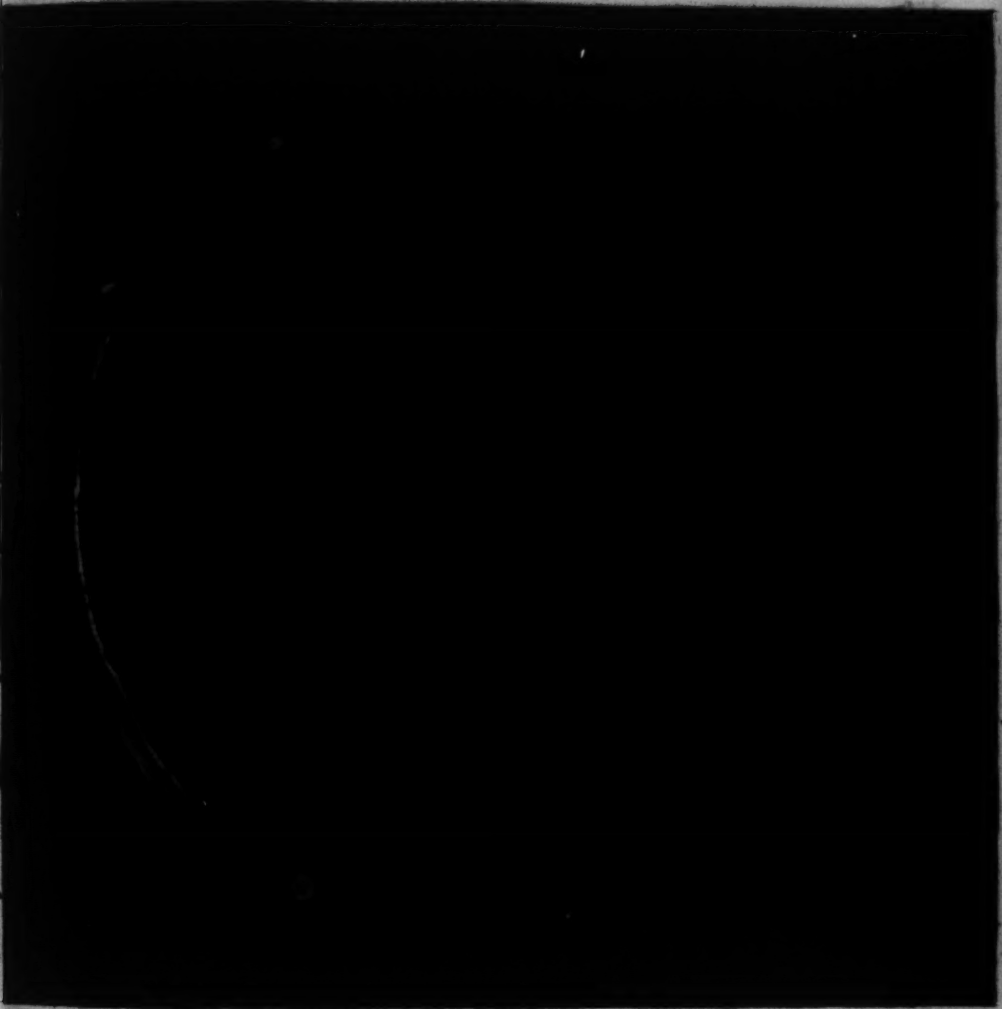
FAC SIMILE OF Nº 26 PHOTOGRAPH - SECOND TOTALITY.

J. Basire sc

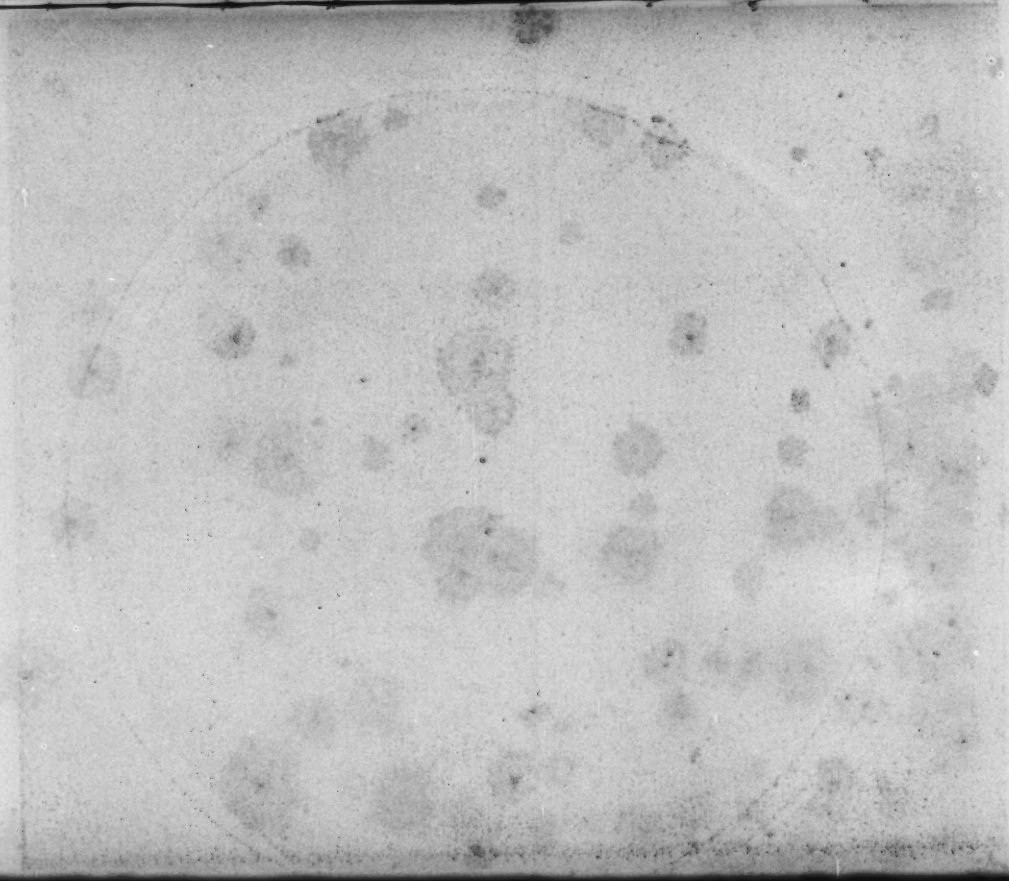




S. Russell sc.

COPY OF A TOUCHED PHOTOGRAPH

Shewing the phenomena of totality immediately after total obscuration.



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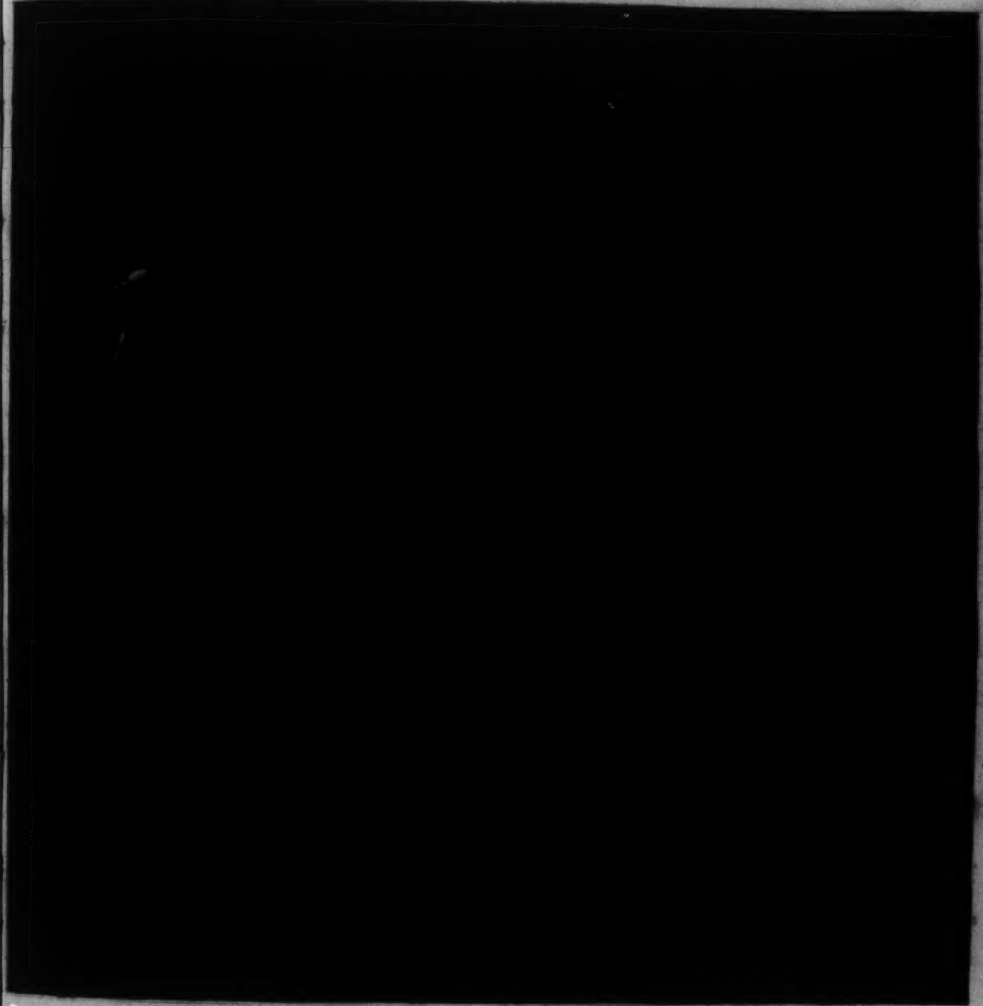

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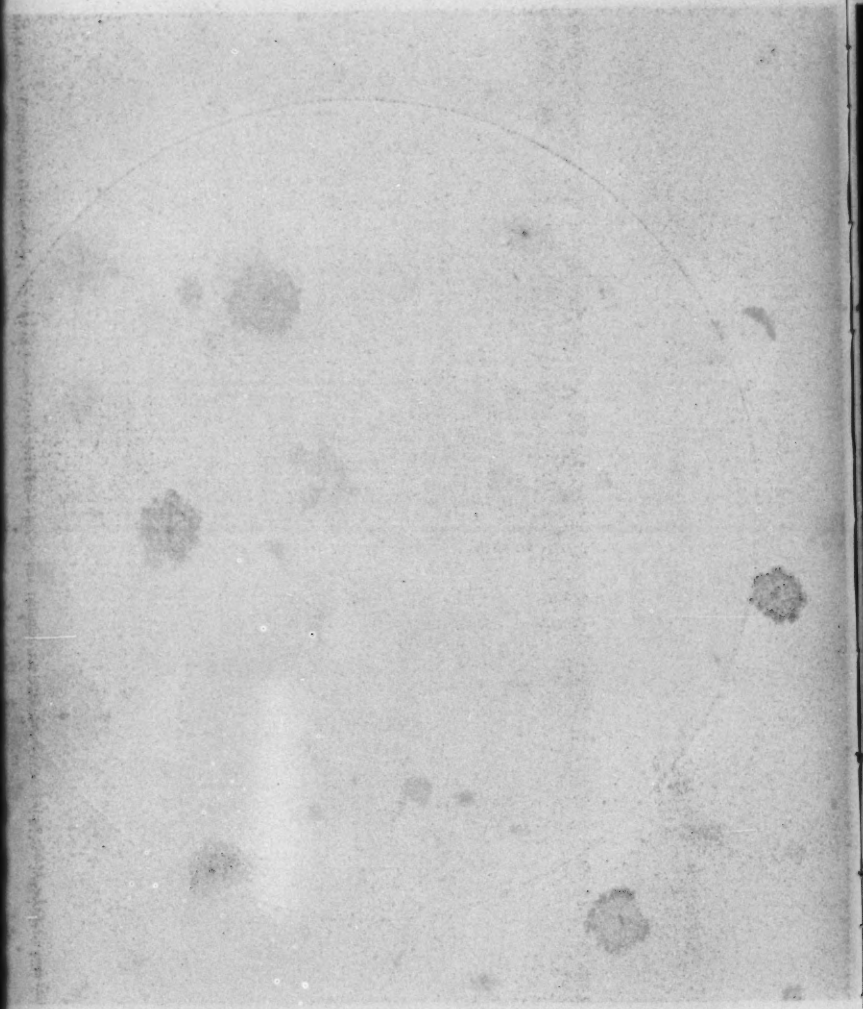
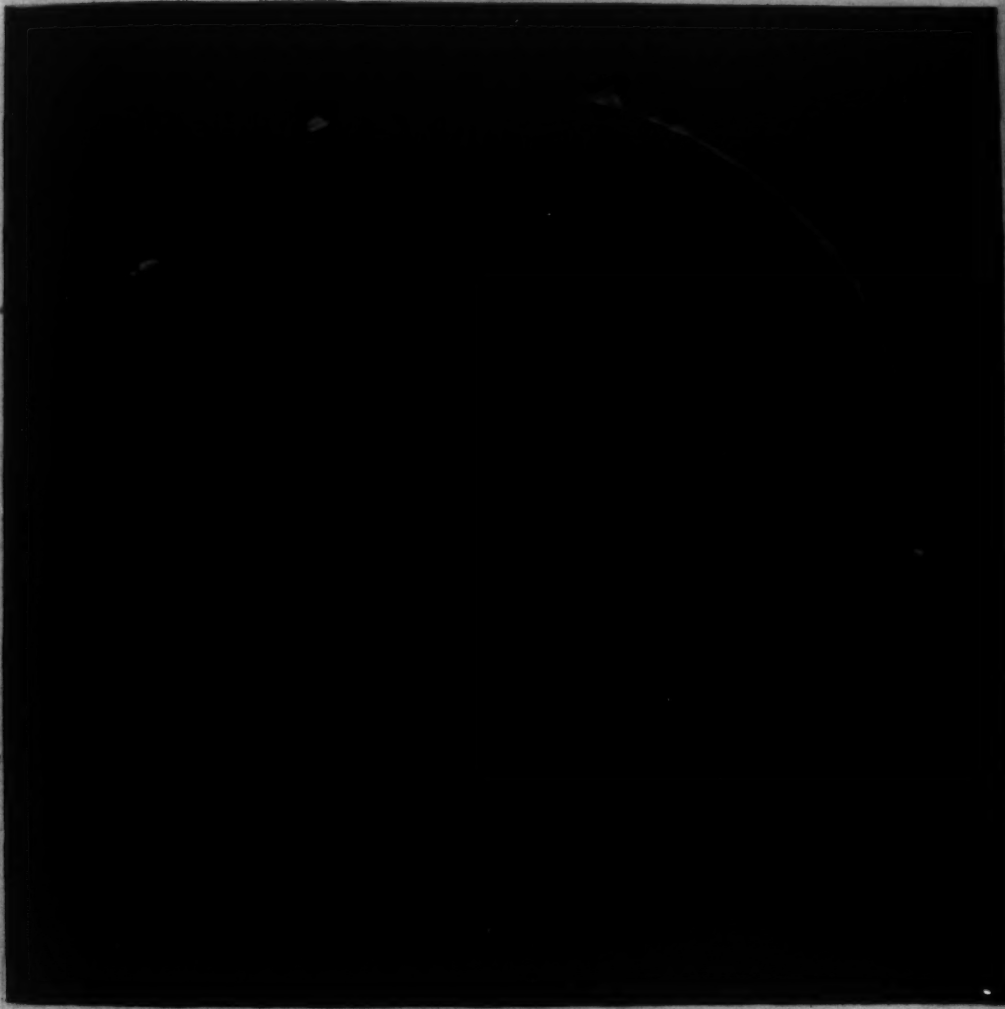
COPY OF A TOUCHED PHOTOGRAPH

Shewing the phenomena of totality one minute after total obscuration.



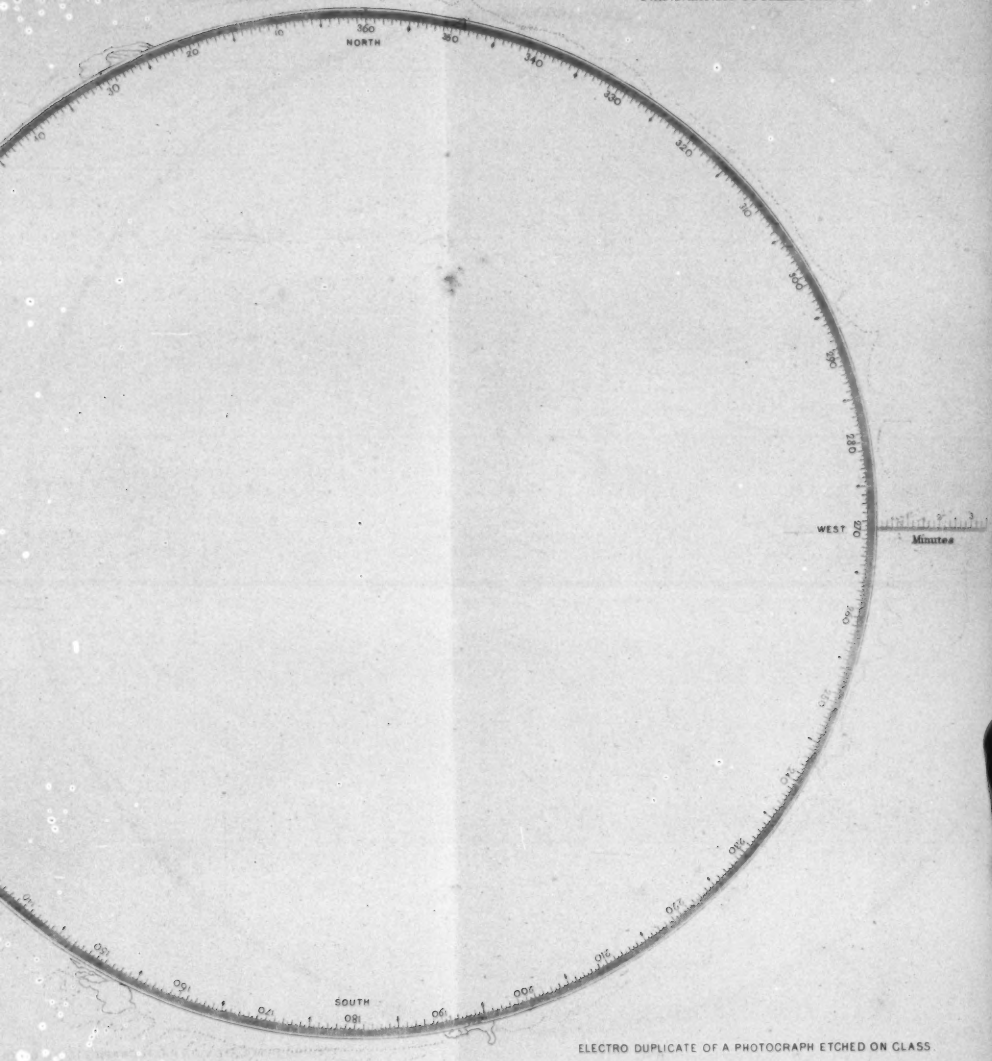


S. Russell. sc

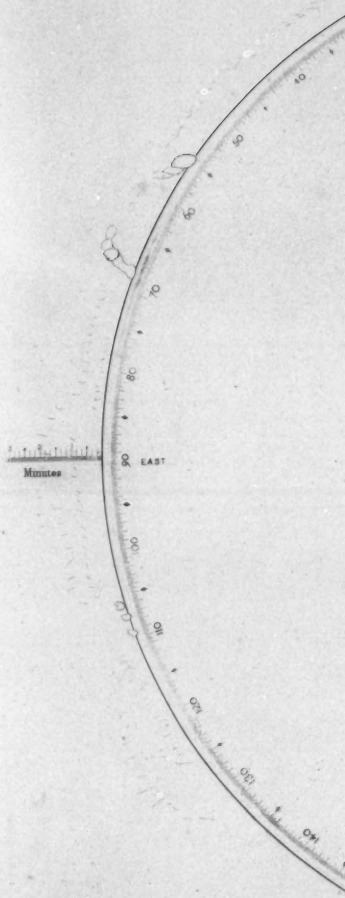
COPY OF A TOUCHED PHOTOGRAPH

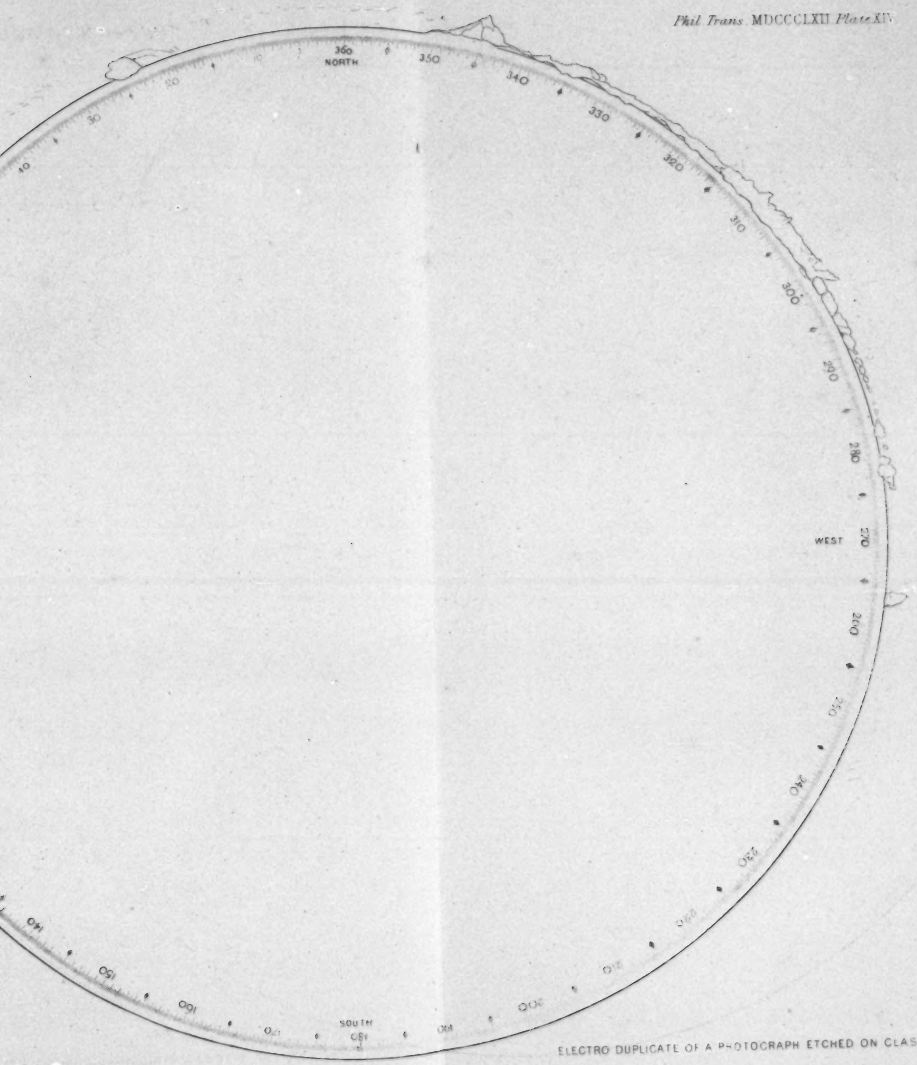
Shewing the phenomena of totality immediately previous to the reappearance of the Sun.

EAST

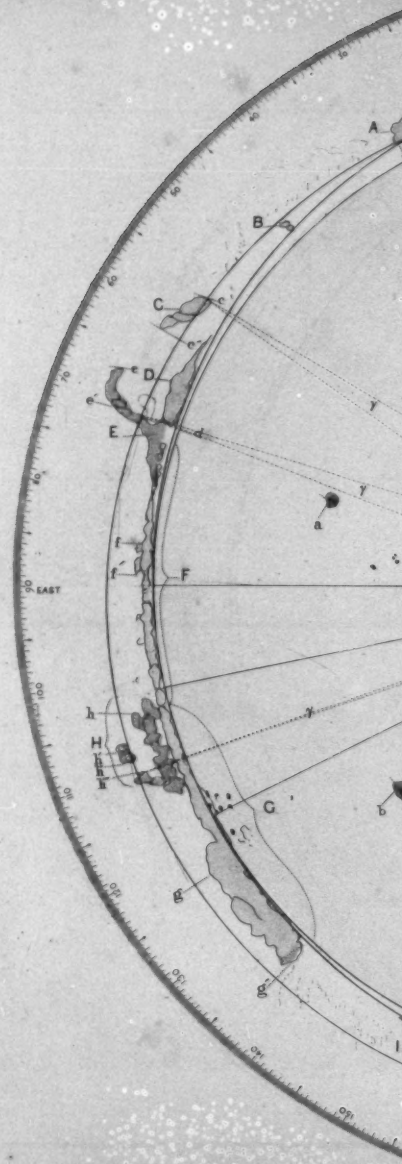


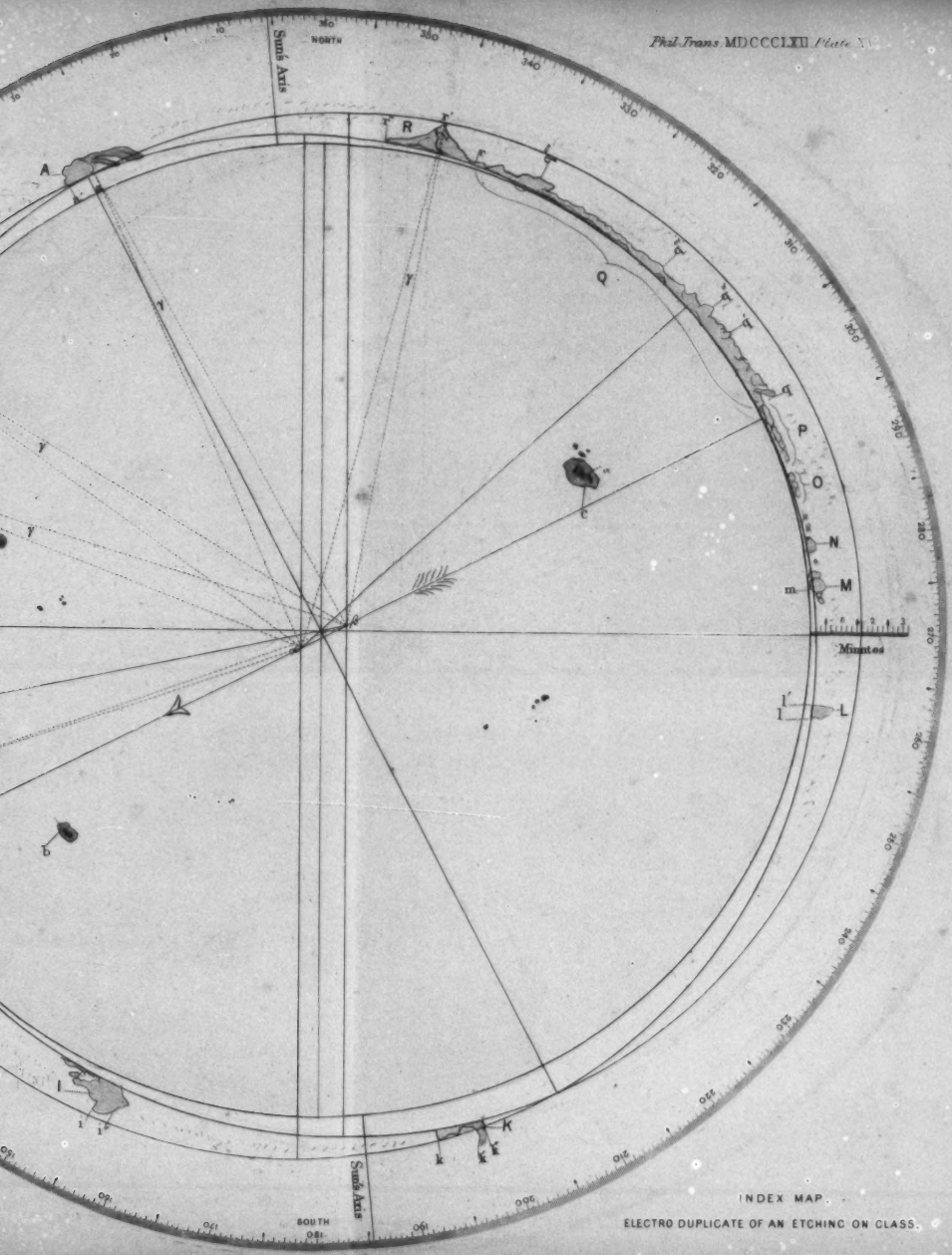
ELECTRO DUPLICATE OF A PHOTOGRAPH ETCHED ON GLASS.


Minutes
EAST



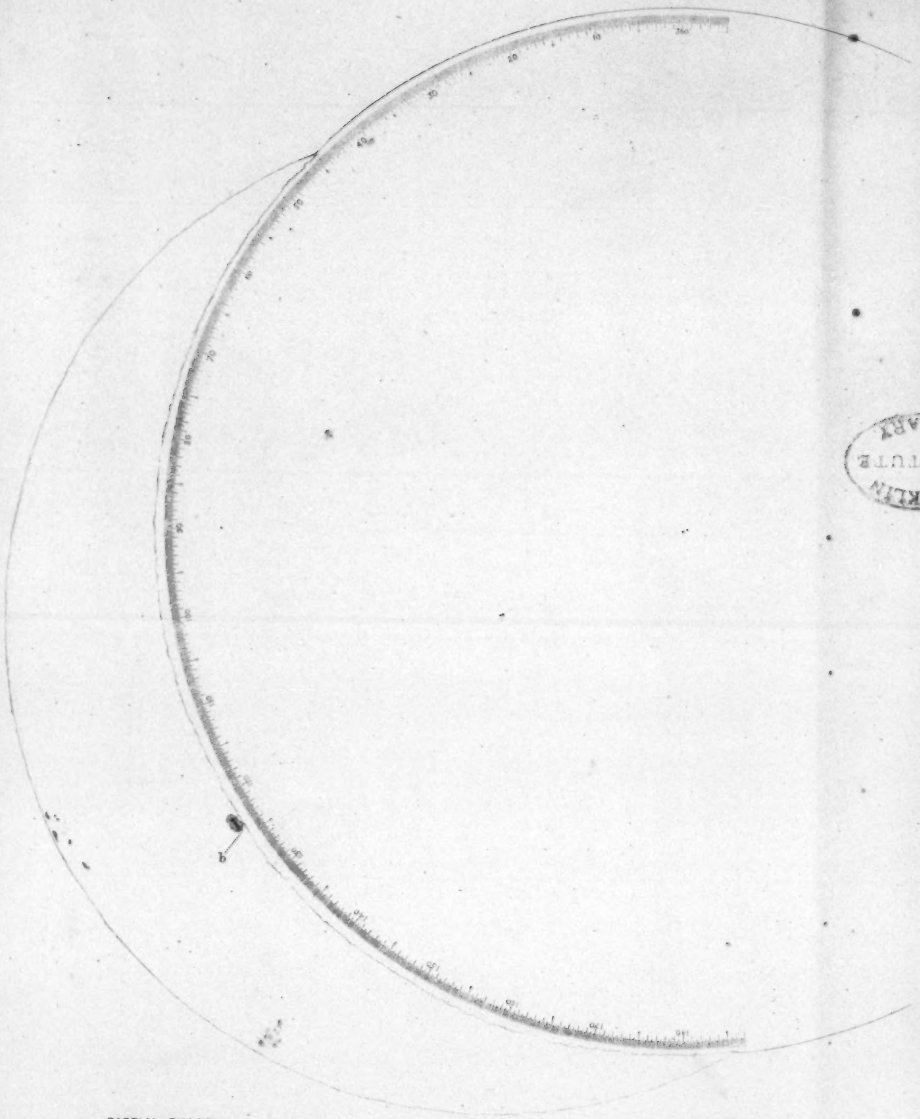
ELECTRO DUPLICATE OF A PHOTOGRAPH ETCHED ON GLASS


A
B
C
D
E
F
G
H
a
b
EAST



INDEX MAP.

ELECTRO DUPLICATE OF AN ETCHING ON GLASS.



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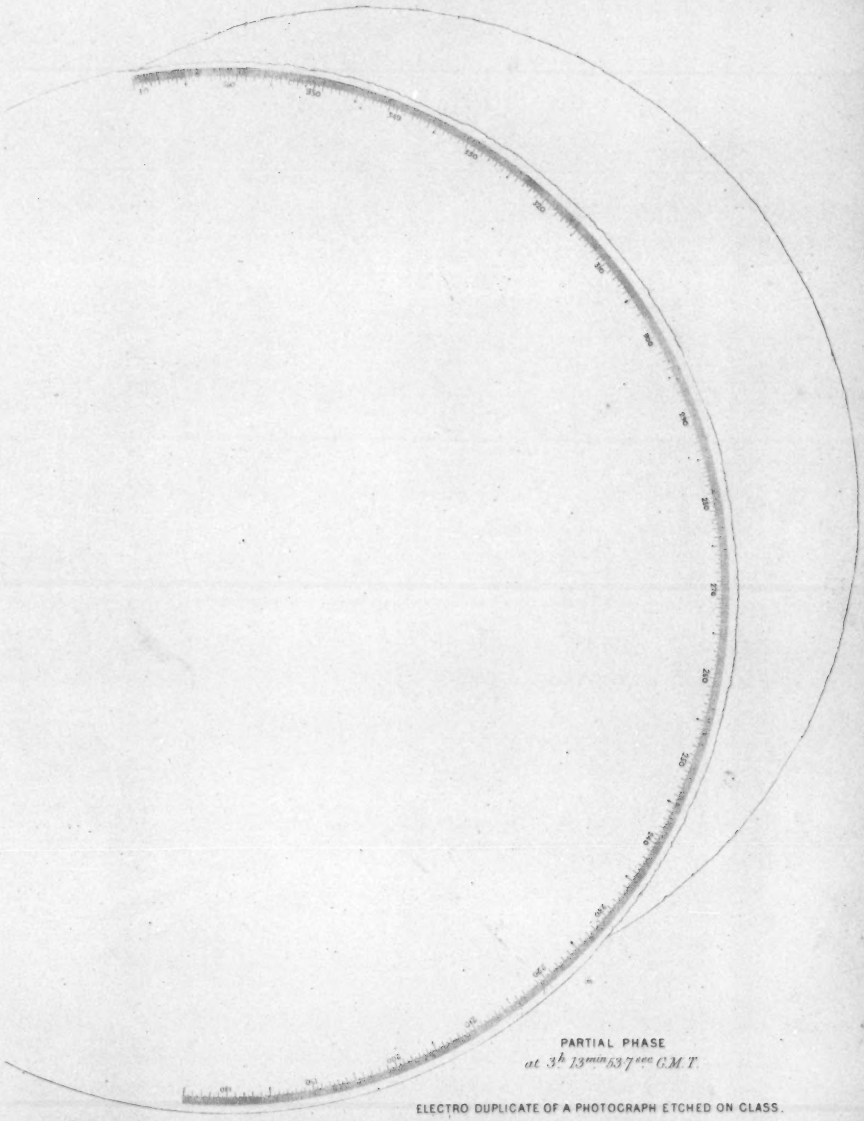
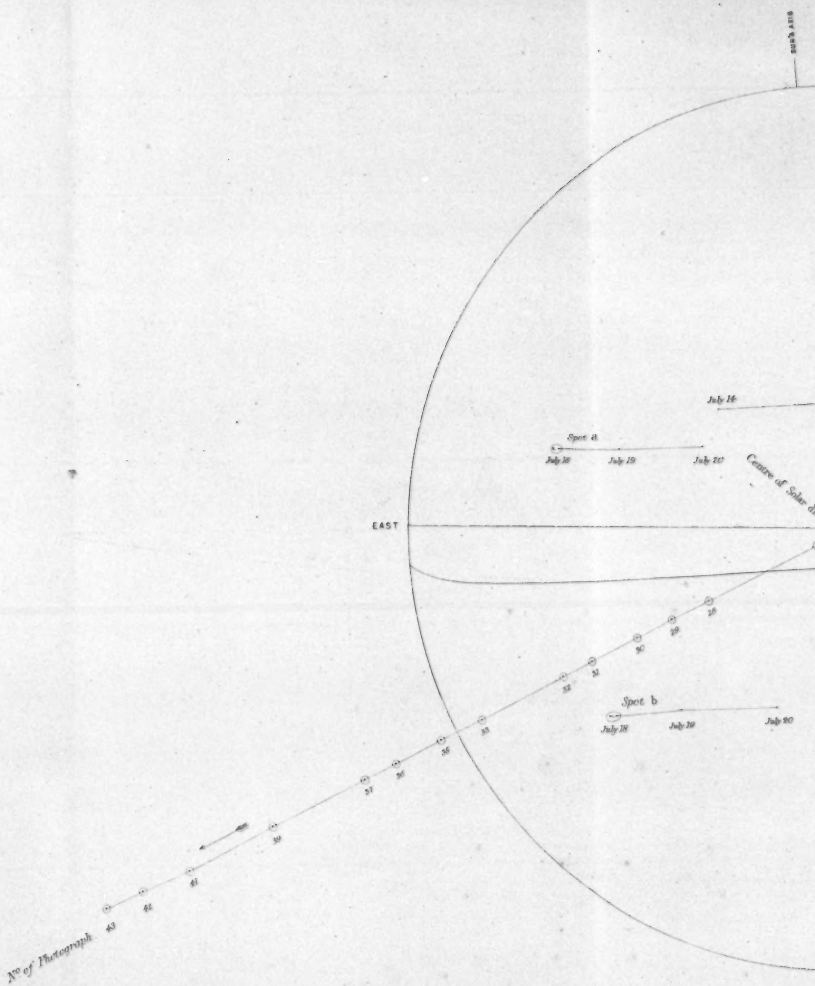
PARTIAL PHASE
at $2^{h}\ 48^{min}\ 25{\cdot}6^{sec}$ G.M.T.

ELECTRO DUPLICATE OF A PHOTOGRAPH ETCHED ON GLASS.



PARTIAL PHASE
at $3^h\ 13^{min}\ 53^{sec}$ G.M.T.

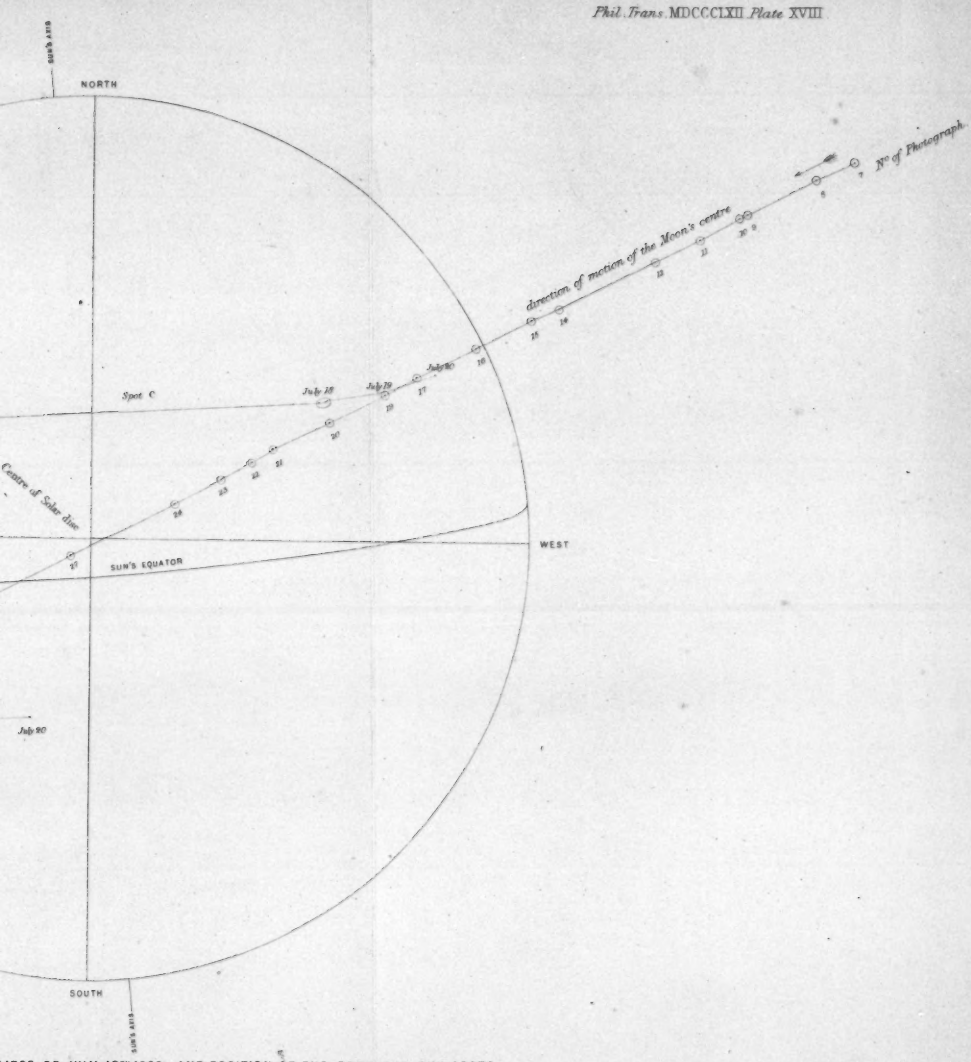
ELECTRO DUPLICATE OF A PHOTOGRAPH ETCHED ON GLASS.



PATH OF THE MOON'S CENTRE DURING THE ECLIPSE OF

set off in accordance with MEASUREMENTS OF PHOTOGRAP

SCALE



LIPSE OF JULY 18TH 1862, AND POSITION OF THE PRINCIPAL SUN-SPOTS,

PHOTOGRAPHS with an instrument designed specially for such operations.

SCALE 2 inches = 1 inch

Fig. 1.

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Fig. 2.

Fig. 3.

Fig. 4.

Fig. 5.

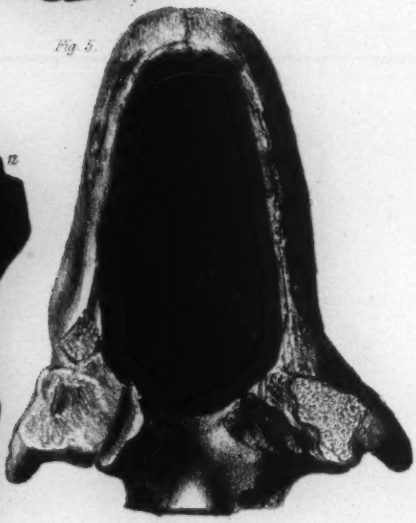

Fig. 1.

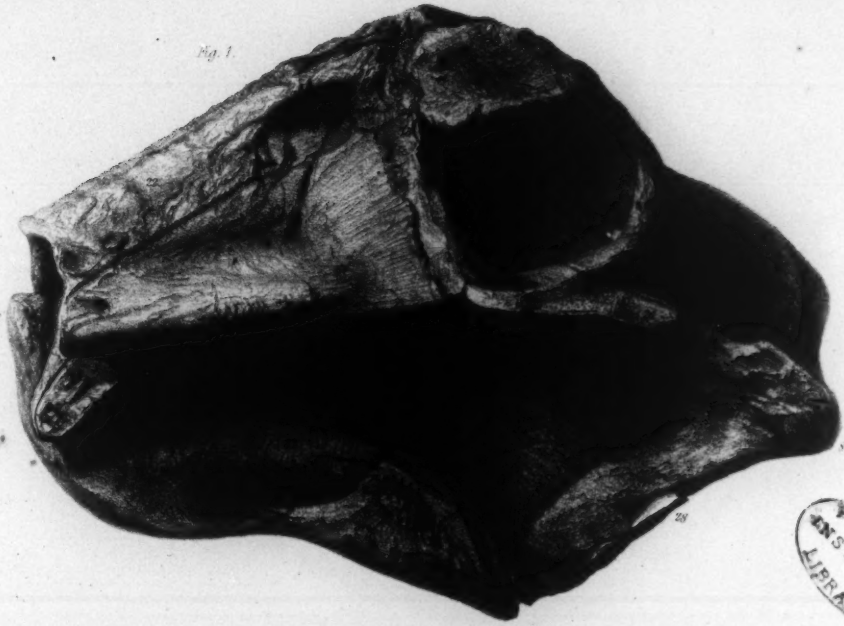

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Fig. 2.

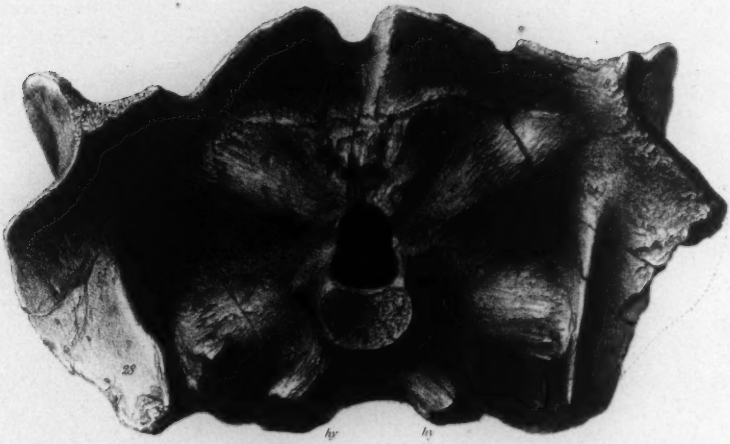

Fig. 1.

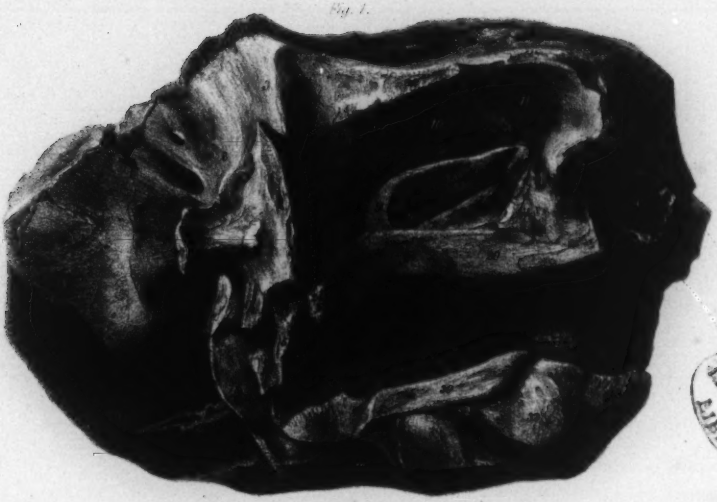

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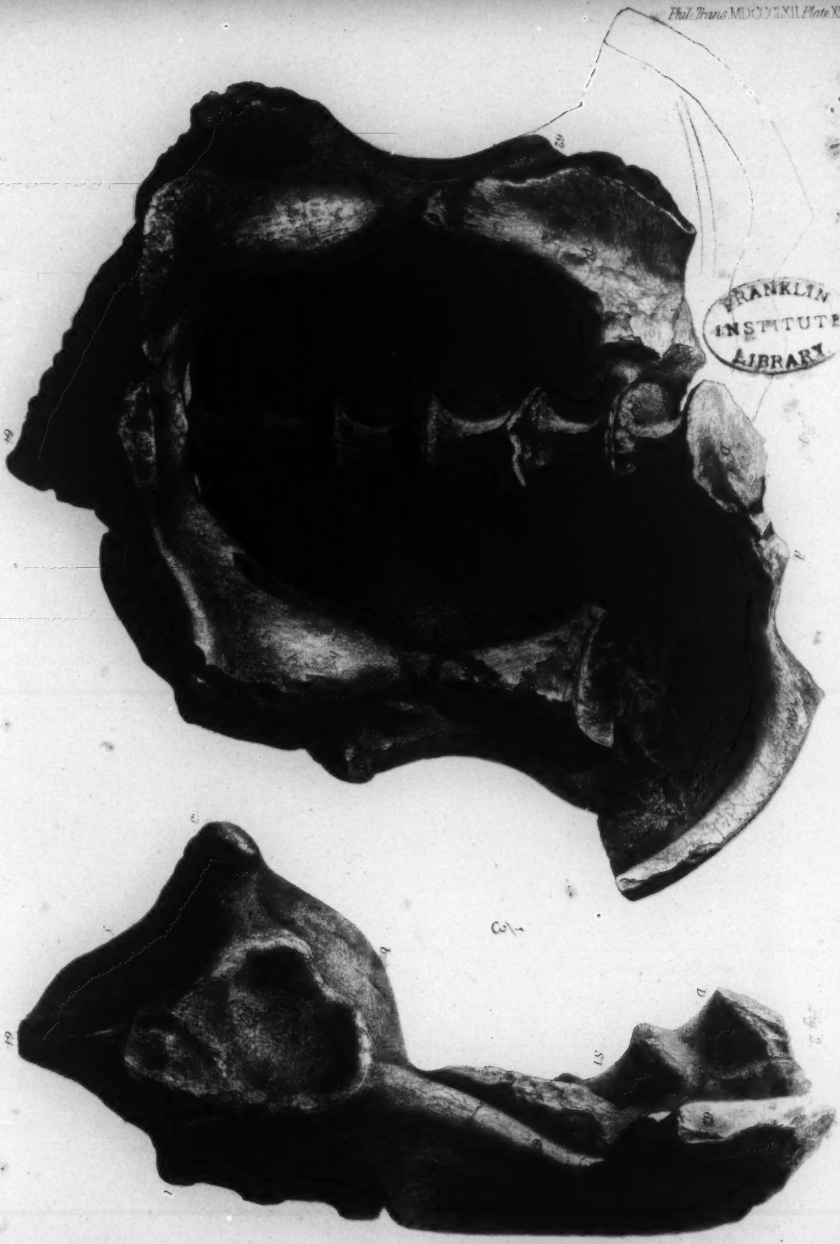
Fig. 2.



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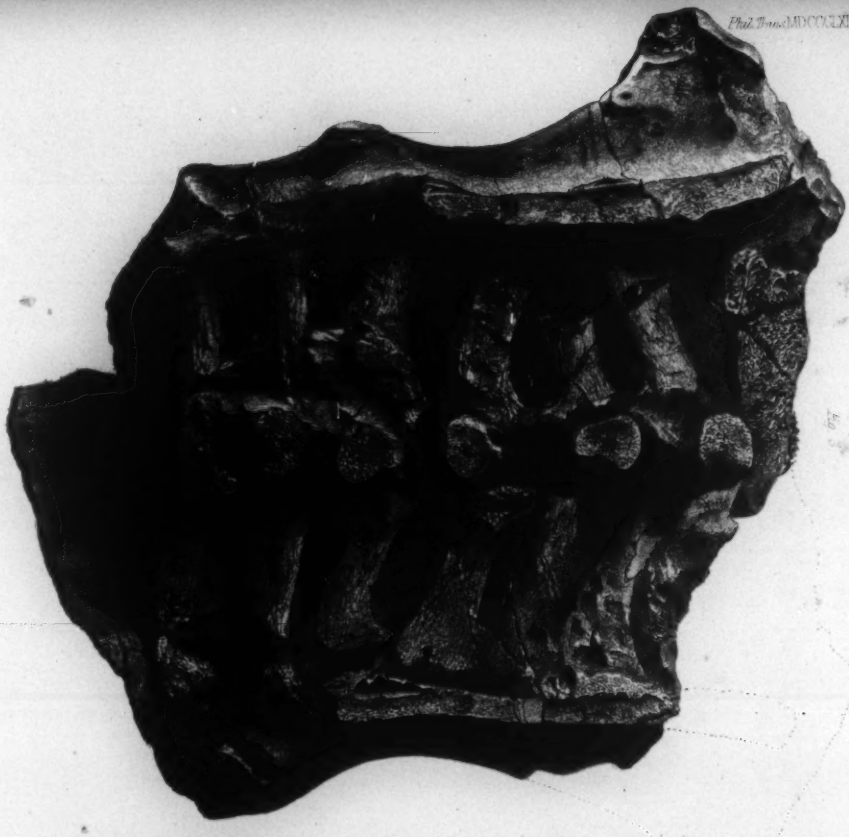

Fig. 1.

Fig. 3.

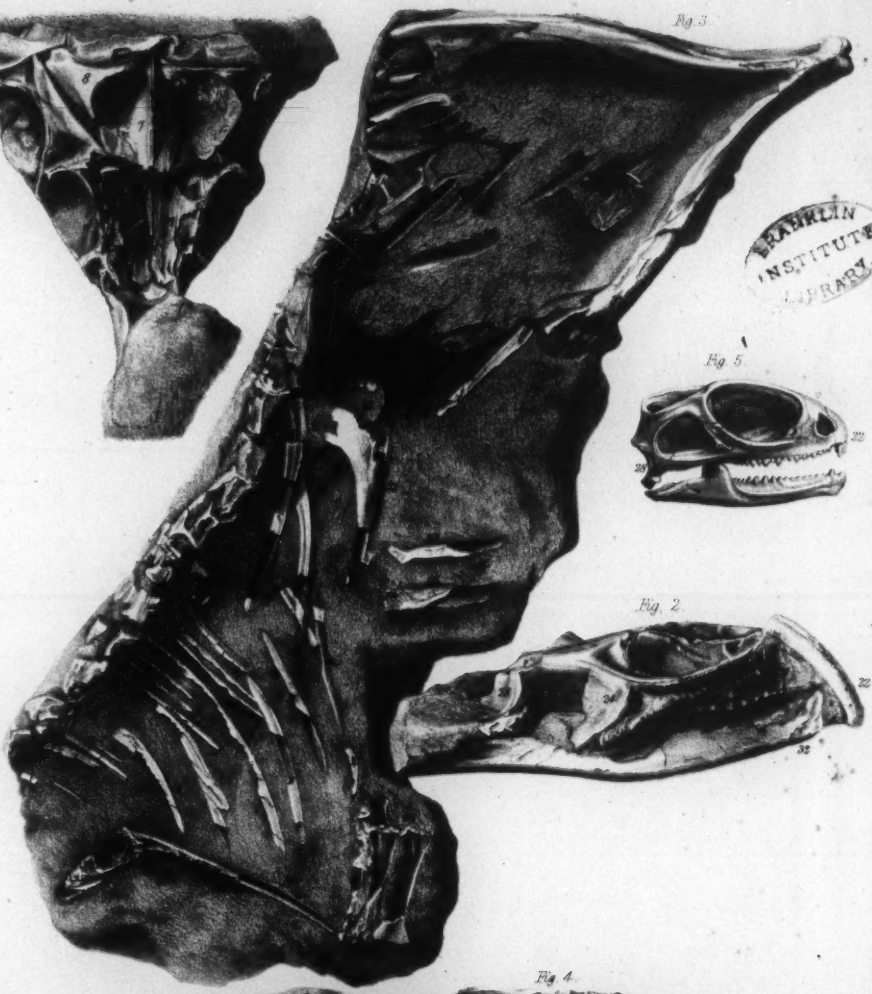

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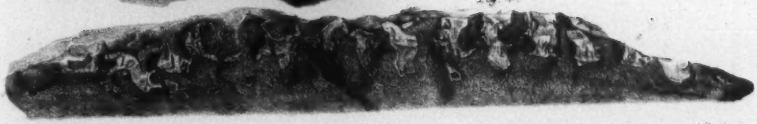
Fig. 5.

Fig. 2.

Fig. 4.



J. Dinkel del. et lith.

J. Basire imp.